Topology MAT 530 Homework 2

1. Prove that an open connected subset of $\mathbb{R}^n$ is path connected.

2. What are the connected components of the *Cantor set*, consisting of all numbers from the segment $[0,1]$ such that the digit 1 does not appear in their base-3 expansions?

3. Prove that the Cantor set is compact.

4. Define an equivalence relation on nonzero vectors from $\mathbb{R}^n$ as "being parallel". The corresponding quotient space $\mathbb{R}P^n$ is called the real *projective space*. Define a metric on $\mathbb{R}P^n$ that gives the quotient topology.

5. Prove that real projective spaces are compact. *Hint:* use that the spheres are compact.

6. A *topological group* is a group G equipped with a topology such that the maps

$$G \times G \to G, \quad (g,h) \mapsto gh,$$
$$G \to G, \quad g \mapsto g^{-1}$$

are continuous. Prove that the connected component of the identity in a topological group is a normal subgroup (a subgroup $H \subset G$ is *normal* if $gHg^{-1} = H$ for every $g \in G$).

7*. A *topological n-manifold* is a topological space such that every its point has a neighborhood homeomorphic to $\mathbb{R}^n$. Prove that any compact connected topological 1-manifold is homeomorphic to a circle.