MAT 401: Undergraduate Seminar
Introduction to Enumerative Geometry
Fall 2008

More on Grassmannians

A flag $\mathbf{V}$ in $\mathbb{C}^n$ is an increasing sequence of $n+1$ linear subspaces of $\mathbb{C}^n$:

$$\mathbf{V} = (V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq \ldots V_{n-1} \subsetneq V_n). \tag{1}$$

Thus, V_i is a linear subspace of $\mathbb{C}^n$ of dimension i containing V_{i-1}; in particular, $V_0 = \{0\}$ and $V_n = \mathbb{C}^n$. The standard flag $\mathbf{V}^{\mathrm{std}}$ on $\mathbb{C}^n$ is the sequence as above with $V_i^{\mathrm{std}} = \mathbb{C}^i \times \{0\}^{n-i} \equiv \mathbb{C}^i$.

Let $G(2,n)$ denote the Grassmannian of two-dimensional linear subspaces of $\mathbb{C}^n$. For any non-negative integers a, b and a flag $\mathbf{V}$ as in (1), let

$$\sigma_{ab}(\mathbf{V}) = \big\{ P \in G(2,n) \colon\ P \subset V_{n-b},\ P \cap V_{n-1-a} \neq \{0\} \big\}. \tag{2}$$

The second condition implies that $n-1-a \geq 1$ if $\sigma_{ab}(\mathbf{V}) \neq \emptyset$, so that $n-2 \geq a$. Since P is a linear subspace of $\mathbb{C}^n$ of dimension 2 and V_{n-1-b} is a linear subspace of V_{n-b}, the first condition in (2) and linear algebra imply that for any $P \in \sigma_{ab}(\mathbf{V})$

$$\dim P \cap V_{n-1-b} \geq \dim P + \dim V_{n-1-b} - \dim V_{n-b} = 1 \qquad \Longrightarrow \qquad P \cap V_{n-1-b} \neq \{0\}.$$

Thus, if $n-1-a > n-1-b$, the second condition in (2) is meaningless since then $V_{n-1-b} \subset V_{n-1-a}$ and $P \cap V_{n-1-a} \neq \{0\}$. Therefore, one always requires that $a \geq b$. Similarly, since V_{n-b} is defined only for $b \geq 0$, we assume that $b \geq 0$. In summary, $\sigma_{ab}(\mathbf{V}) \subset G(2,n)$ is defined by (2) whenever $n-2 \geq a \geq b \geq 0$; otherwise, it is defined to be empty.

If $P \in G(2,n)$, then $P \subset V_{n-0}$ and by linear algebra

$$\dim P \cap V_{n-1} \geq \dim P + \dim V_{n-1} - \dim V_n = 1 \qquad \Longrightarrow \qquad P \cap V_{n-1-0} \neq \emptyset.$$

Thus, $\sigma_{00}(\mathbf{V}) = G(2,n)$. In general, the integers a and b in (2) indicate who much earlier a typical element P of $\sigma_{ab}(\mathbf{V})$ satisfies the containment and intersection conditions with respect to the given flag. The subspace $\sigma_{ab}(\mathbf{V})$ is an analytic subvariety of the complex manifold $G(2,n)$ of (complex) codimension $a+b$. Since the dimension of $G(2,n)$ is $2(n-2)$, this means that $\sigma_{ab}(\mathbf{V})$ can be written as a disjoint union of finitely many complex manifolds with the largest dimension equal $2(n-2) - (a+b)$.

If $\mathbf{V}$ and $\mathbf{V}'$ are two different flags in $\mathbb{C}^n$, there is a smooth path of flags $\mathbf{V}^{(t)}$, with $t \in [0,1]$, so that $\mathbf{V}^{(0)} = \mathbf{V}$ and $\mathbf{V}^{(1)} = \mathbf{V}'$. The smooth manifold with boundary

$$M = \big\{ (t, P) \in [0,1] \times G(2,n) \colon P \in \sigma_{ab}(\mathbf{V}^{(t)}) \big\}$$

is then an equivalence between the cycles $\sigma_{ab}(\mathbf{V})$ and $\sigma_{ab}(\mathbf{V}')$. Thus, the equivalence class of $\sigma_{ab}(\mathbf{V})$ as a cycle in $G(2,n)$ is independent of the flag $\mathbf{V}$ and is denoted σ_{ab}. It is customary to abbreviate σ_{a0} as σ_a.

Given equivalence classes σ_{ab} and $\sigma_{a'b'}$, their intersection product $\sigma_{ab}\cdot\sigma_{a'b'}$ is the equivalence class of the cycle $\sigma_{ab}(\mathbf{V})\cap\sigma_{a'b'}(\mathbf{V}')$ for a generic pair of flags $\mathbf{V}$ and $\mathbf{V}'$ on $\mathbb{C}^n$. Similarly to the previous paragraph, any two pairs of such flags are homotopic, so that the equivalence class of the cycle $\sigma_{ab}(\mathbf{V})\cap\sigma_{a'b'}(\mathbf{V}')$ is independent of the generic pair $(\mathbf{V},\mathbf{V}')$. The codimension of the cycle $\sigma_{ab}\cdot\sigma_{a'b'}$ is given by

$$\operatorname{codim}\sigma_{ab}\cdot\sigma_{a'b'} = \operatorname{codim}\sigma_{ab} + \operatorname{codim}\sigma_{a'b'} = (a+b) + (a'+b').$$

If $\sigma_{a_1b_1},\ldots,\sigma_{a_kb_k}$ are k cycles, the codimension of the cycle $\sigma_{a_1b_1}\cdot\ldots\cdot\sigma_{a_kb_k}$ is thus

$$\operatorname{codim}\big(\sigma_{a_1b_1}\cdot\ldots\cdot\sigma_{a_kb_k}\big) = \sum_{i=1}^{i=k}(a_i+b_i).$$

If this number equals $2(n-2)$, then the dimension of this cycle in $G(2,n)$ is

$$\dim\big(\sigma_{a_1b_1}\cdot\ldots\cdot\sigma_{a_kb_k}\big) = \dim G(2,n) - \operatorname{codim}\big(\sigma_{a_1b_1}\cdot\ldots\cdot\sigma_{a_kb_k}\big) = 0,$$

i.e. $\sigma_{a_1b_1}\cdot\ldots\cdot\sigma_{a_kb_k}$ is simply a finite collection of points. The number of these points is denoted by

$$\big\langle\sigma_{a_1b_1}\cdot\ldots\cdot\sigma_{a_kb_k}, G(2,n)\big\rangle \in \mathbb{Z}.$$

These numbers are called intersection numbers of Schubert cycles.

The above intersection numbers satisfy the following identities:

$$\big\langle\sigma_{a_1b_1}\cdot\ldots\cdot\sigma_{a_kb_k}, G(2,n)\big\rangle = \big\langle\sigma_{a_1-b_1}\cdot\ldots\cdot\sigma_{a_k-b_k}, G(2,n-b_1-\ldots-b_k)\big\rangle; \tag{S1}$$
$$\big\langle\sigma_{n-2}\cdot\sigma_{n-2}, G(2,n-2)\big\rangle = 1; \tag{S2}$$
$$\big\langle\sigma_{a_1}\sigma_{a_2}\sigma_{a_3}, G(2,n)\big\rangle = 1 \quad \text{if} \quad n-2\ge a_1,a_2,a_3\ge 0,\ a_1+a_2+a_3 = 2n-4; \tag{S3}$$
$$\sigma_{a_1}\cdot\sigma_{a_2} = \sum_{c\ge a_1,a_2}\sigma_{c,a_1+a_2-c}\,. \tag{S4}$$

We verified (S1)-(S3) directly from the definition during one of the discussion sessions. The second identity is the $a_1,a_2=n-2$, $a_3=0$ case of (S3).

The relation (S4) is known as Pieri's formula. It is actually an immediate consequence of (S1), (S3), and the structure of the cohomology of $G(2,n)$. The latter implies that two cycles A and B in $G(2,n)$ are equivalent if and only if

$$\big\langle A\cdot\sigma_{de}, G(2,n)\big\rangle = \big\langle B\cdot\sigma_{de}, G(2,n)\big\rangle \qquad \forall\, d,e\in\mathbb{Z}.$$

Thus, in order to verify (S4), it is sufficient to show that

$$\big\langle\sigma_{a_1}\cdot\sigma_{a_2}\cdot\sigma_{de}, G(2,n)\big\rangle = \sum_{c\ge a_1,a_2}\big\langle\sigma_{c,a_1+a_2-c}\cdot\sigma_{de}\,, G(2,n)\big\rangle \qquad \forall\, d,e\in\mathbb{Z}. \tag{3}$$

We can assume that $n-2\geq a_1, a_2\geq 0$, $a_1+a_2+d+e=2(n-2)$, and $n-2\geq d\geq e\geq 0$; otherwise, both sides of (3) are zero. By (S1) and then (S3),

$$\begin{aligned}\big\langle \sigma_{a_1}\cdot\sigma_{a_2}\cdot\sigma_{de}, G(2,n)\big\rangle &= \big\langle \sigma_{a_1}\cdot\sigma_{a_2}\cdot\sigma_{d-e}, G(2,n-e)\big\rangle\\ &= \begin{cases}1, & \text{if } \min(n-2, 2(n-2)-(a_1+a_2)) \geq d \geq (n-2)-\min(a_1,a_2),\\ 0, & \text{otherwise,}\end{cases}\end{aligned}\tag{4}$$

with the above assumptions on d and e. Similarly, if $\min(n-2, a_1+a_2)\geq c\geq \max(a_1,a_2)$,

$$\begin{aligned}\big\langle \sigma_{c,a_1+a_2-c}\cdot\sigma_{de}, G(2,n)\big\rangle &= \big\langle \sigma_{2c-a_1-a_2}\cdot\sigma_{d-e}, G(2,n+c-a_1-a_2-e)\big\rangle\\ &= \begin{cases}1, & \text{if } d=n-2+c-a_1-a_2;\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

This gives

$$\begin{aligned}&\sum_{c\geq a_1,a_2}\big\langle \sigma_{c,a_1+a_2-c}\cdot\sigma_{de}, G(2,n)\big\rangle\\ &\qquad= \begin{cases}1, & \text{if } \min(n-2, 2(n-2)-(a_1+a_2)) \geq d\geq (n-2)-\min(a_1,a_2);\\ 0, & \text{otherwise.}\end{cases}\end{aligned}\tag{5}$$

Comparing (4) with (5), we obtain (3) and thus (S4).

Unfortunately, this argument requires deep facts about cycles in $G(2,n)$. There is a direct argument as well. A special case of this argument is used in the book to show that

$$\sigma_1\cdot\sigma_1=\sigma_2+\sigma_{11}\tag{6}$$

in $G(2,4)$. The argument in the book views the elements of $G(2,4)$ as lines in $\mathbb{P}^3$ by taking their projectivizations. Here is the argument by considering them as 2-planes in $\mathbb{C}^4$. We need to intersect

$$\begin{aligned}\sigma_1(\mathbf{V}^{\text{std}}) &\equiv \big\{P\in G(2,4):\ P\cap\mathbb{C}^2\neq\{0\}\big\},\\ \sigma_1(\mathbf{V}) &\equiv \big\{P\in G(2,4):\ P\cap V_2\neq\{0\}\big\},\end{aligned}$$

for a generic flag $\mathbf{V}$. For such a flag $\mathbb{C}^2\cap V_2=\{0\}$. However, we can move $\mathbf{V}$ so that $\mathbb{C}^2\cap V_2=\mathbb{C}$ and thus $\mathbb{C}^2+V_2$ is a three-dimensional linear subspace of $\mathbb{C}^4$, which we can assume to be $\mathbb{C}^3$; it is spanned by the one-dimensional linear subspace $L_0=\mathbb{C}$, a one-dimensional linear subspace L_1 in $\mathbb{C}^2$ different from L_0, and a one-dimensional linear subspace L_2 in V_2 different from L_0. An element of

$$\sigma_1(\mathbf{V}^{\text{std}})\cap\sigma_1(\mathbf{V}) \equiv \big\{P\in G(2,4):\ P\cap\mathbb{C}^2\neq\{0\},\ P\cap V_2\neq\{0\}\big\}$$

must either contain the line $L_0=\mathbb{C}$ common to $\mathbb{C}^2$ and V_2 or intersect $\mathbb{C}^2$ and $\mathbf{V}_2$ along some one-dimensional linear subspace $L_1\subset\mathbb{C}^2$ and $L_2\subset V_2$. In the latter case, P must lie in $\mathbb{C}^2+V_2$. Conversely, if P is a two-dimensional linear subspace of $\mathbb{C}^2+V_2=\mathbb{C}^3$, then P must intersect $\mathbb{C}^2$ and V_2 at least in a one-dimensional linear subspace, since by linear algebra

$$\begin{aligned}\dim P\cap\mathbb{C}^2 &\geq \dim P+\dim\mathbb{C}^2-\dim\mathbb{C}^3\geq 1;\\ \dim P\cap V_2 &\geq \dim P+\dim V_2-\dim\mathbb{C}^3\geq 1.\end{aligned}$$

From this, we obtain

$$\begin{aligned}\sigma_1(\mathbf{V}^{\rm std})\cap\sigma_1(\mathbf{V}) &= \big\{P\in G(2,n):\ P\cap\mathbb{C}^{4-1-2}\neq\{0\}\big\}\\ &\qquad \cup\big\{P\in G(2,n):\ P\subset\mathbb{C}^{4-1},\ P\cap\mathbb{C}^{4-1-1}\neq\{0\}\big\}\\ &= \sigma_2(\mathbf{V}^{\rm std})\cup\sigma_{11}(\mathbf{V}^{\rm std}).\end{aligned}$$

This implies (6) in the case of $G(2,4)$.

A similar argument extends to the general case of (S4) with some care. The formula (S4) holds for $G(2,2)$, with the only choices for a_1,a_2 being $a_1,a_2=0$; otherwise, both sides of (S4) vanish by definition. Suppose $n\geq 3$ and the formula holds for all $G(2,m)$ with $m<n$. We need to determine the intersection of

$$\begin{aligned}\sigma_a(\mathbf{V}^{\rm std}) &\equiv \big\{P\in G(2,n):\ P\cap\mathbb{C}^{n-1-a}\neq\{0\}\big\},\\ \sigma_b(\mathbf{V}) &\equiv \big\{P\in G(2,n):\ P\cap V_{n-1-b}\neq\{0\}\big\},\end{aligned}$$

for a generic flag $\mathbf{V}$ as in (1).

Case 1: $a+b>n-2$. In this case,

$$\dim(\mathbb{C}^{n-1-a}+V_{n-1-b})\leq \dim\mathbb{C}^{n-1-a}+\dim V_{n-1-b}=n+(n-2)-(a+b)<n.$$

Thus, $\mathbb{C}^{n-1-a}\cap V_{n-1-b}=\{0\}$ if V_{n-1-b} is generic. We can also assume that $\mathbb{C}^{n-1-a}+\mathbf{V}_{n-1-b}\subset\mathbb{C}^{n-1}$. Then,

$$\begin{aligned}\sigma_a(\mathbf{V}^{\rm std})\cap\sigma_b(\mathbf{V}) &\equiv \big\{P\in G(2,n):\ P\cap\mathbb{C}^{n-1-a}\neq\{0\},\ P\cap V_{n-1-b}\neq\{0\}\big\}\\ &= \big\{P\in G(2,n-1):\ P\cap\mathbb{C}^{(n-1)-1-(a-1)}\neq\{0\},\ P\cap V_{(n-1)-1-(b-1)}\neq\{0\}\big\}.\end{aligned}$$

The last set is precisely the intersection of $\sigma_{a-1}(\mathbf{V}^{\rm std})$ with $\sigma_{b-1}(\mathbf{V})$ in $G(2,n-1)$. By the inductive assumption,

$$\sigma_{a-1}\cdot\sigma_{b-1}=\sum_{c=\max(a,b)}^{\min(n-2,a+b-1)}\sigma_{c-1,a+b-c-1}\qquad\text{in}\ \ G(2,n-1).$$

Thus, as a cycle in $G(2,n-1)\subset G(2,n)$, this intersection is equivalent to

$$\begin{aligned}&\sum_{c=\max(a,b)}^{\min(n-2,a+b-1)}\big\{P\in G(2,n-1): P\subset\mathbb{C}^{(n-1)-(a+b-c-1)},\ P\cap C^{(n-1)-1-(c-1)}\neq\{0\}\big\}\\ &\quad\sum_{c=\max(a,b)}^{\min(n-2,a+b-1)}\big\{P\in G(2,n): P\subset\mathbb{C}^{n-(a+b-c)},\ P\cap\mathbb{C}^{n-1-c}\neq\{0\}\big\}\\ &\quad=\sum_{c=\max(a,b)}^{\min(n-2,a+b-1)}\sigma_{c,a+b-c}(\mathbf{V}_{\rm std}).\end{aligned}$$

This gives (S4).

Case 2: $a+b \le n-2$. If $\mathbf{V}$ is a generic flag,

$$\dim(\mathbb{C}^{n-1-a} \cap V_{n-1-b}) = \dim \mathbb{C}^{n-1-a} + \dim V_{n-1-b} - \dim \mathbb{C}^n = n-2-(a+b) \ge 0.$$

However, we can move $\mathbf{V}$ so that $\mathbb{C}^{n-1-a} \cap V_{n-1-b} = \mathbb{C}^{n-1-(a+b)}$, and thus by linear algebra

$$\begin{aligned}\dim(\mathbb{C}^{n-1-a}+V_{n-1-b}) &= \dim \mathbb{C}^{n-1-a} + \dim V_{n-1-b} - \dim(\mathbb{C}^{n-1-a} \cap V_{n-1-b}) \\ &= (n-1-a)+(n-1-b)-(n-1-(a+b)) \\ &= n-1.\end{aligned}$$

Thus, we can assume that $\mathbb{C}^{n-1-a}+V_{n-1-b}=\mathbb{C}^{n-1}$. An element of

$$\sigma_a(\mathbf{V}^{\mathrm{std}}) \cap \sigma_b(\mathbf{V}) \equiv \big\{P \in G(2,n):\ P \cap \mathbb{C}^{n-1-a} \neq \{0\},\ P \cap V_{n-1-b} \neq \{0\}\big\}$$

must either have a one-dimensional linear subspace in common with $\mathbb{C}^{n-1-a} \cap V_{n-1-b} = \mathbb{C}^{n-1-(a+b)}$ or intersect $\mathbb{C}^{n-1-a}$ and V_{n-1-b} along some one-dimensional linear subspace $L_1 \subset \mathbb{C}^{n-1-a}$ and $L_2 \subset V_{n-1-b}$. In the latter case, P must lie in $\mathbb{C}^{n-1-a}+V_{n-1-b}=\mathbb{C}^{n-1}$, since P is a two-dimensional linear subspace of $\mathbb{C}^n$. From this, we obtain

$$\begin{aligned}\sigma_a(\mathbf{V}^{\mathrm{std}}) \cap \sigma_b(\mathbf{V}) = \big\{P \in G(2,n):\ & P \cap \mathbb{C}^{n-1-(a+b)} \neq \{0\}\big\} \\ \cup \big\{P \in G(2,n):\ & P \subset \mathbb{C}^{n-1},\ P \cap \mathbb{C}^{n-1-a} \neq \{0\}, \\ & P \cap V_{n-1-b} \neq \{0\}\big\}.\end{aligned} \tag{7}$$

The first set on the right-hand side of (7) is precisely the cycle $\sigma_{a+b}(\mathbf{V}^{\mathrm{std}})$ in $G(2,n)$; this set is empty if $a+b > n-2$. The second set is the intersection of the cycles

$$\begin{aligned}\sigma_{a-1}(\mathbf{V}^{\mathrm{std}}) &\equiv \big\{P \in G(2,n-1):\ P \cap \mathbb{C}^{(n-1)-1-(a-1)} \neq \{0\}\big\}, \\ \sigma_{b-1}(\mathbf{V}) &\equiv \big\{P \in G(2,n-1):\ P \cap V_{(n-1)-1-(b-1)} \neq \{0\}\big\},\end{aligned}$$

in $G(2,n-1)$. By the inductive assumption,

$$\sigma_{a-1} \cdot \sigma_{b-1} = \sum_{c=\max(a,b)}^{\min(n-2,a+b-1)} \sigma_{c-1,a+b-c-1} \qquad \text{in} \quad G(2,n-1).$$

Thus, as a cycle in $G(2,n-1) \subset G(2,n)$, the second set in (7) is equivalent to

$$\begin{aligned}&\sum_{c=\max(a,b)}^{\min(n-2,a+b-1)} \big\{P \in G(2,n-1): P \subset \mathbb{C}^{(n-1)-(a+b-c-1)},\ P \cap C^{(n-1)-1-(c-1)} \neq \{0\}\big\} \\ &\sum_{c=\max(a,b)}^{\min(n-2,a+b-1)} \big\{P \in G(2,n): P \subset \mathbb{C}^{n-(a+b-c)},\ P \cap \mathbb{C}^{n-1-c} \neq \{0\}\big\} \\ &= \sum_{c=\max(a,b)}^{\min(n-2,a+b-1)} \sigma_{c,a+b-c}(\mathbf{V}_{\mathrm{std}}).\end{aligned} \tag{8}$$

Combining (7) and (8), we obtain (S4).