MAT 401: Undergraduate Seminar
Introduction to Enumerative Geometry
Fall 2008

Homework Assignment VI

Written Assignment due on Thursday, 12/11, at 11:20am in Physics P-117

Problem K

Let $F = F(X, Y, Z)$ be a homogeneous polynomial of degree 2 which is not a product of linear factors. Thus,

$$C \equiv Z(F) \equiv \big\{[X, Y, Z] \in \mathbb{P}^2 : F(X, Y, Z) = 0\big\}$$

is a smooth curve of degree 2 in $\mathbb{P}^2$. Show that there are homogeneous polynomials $P_i = P_i(u, v)$ of degree 2 so that the image of the map

$$f : \mathbb{P}^1 \longrightarrow \mathbb{P}^2, \qquad [u, v] \longrightarrow \big[P_0(u, v), P_1(u, v), P_2(u, v)\big]$$

is the curve C.

There are at least two ways of going about this. The homogeneous polynomials P_0, P_1, P_2 are determined by 3 coefficients each; the homogeneous polynomial F is given by 6 coefficients. The requirement $f(\mathbb{P}^1) \subset Z(F)$ is equivalent to

$$F\big(P_0(u, v), P_1(u, v), P_2(u, v)\big) = 0 \qquad \forall\, u, v.$$

The left-hand side of this equation is a homogeneous polynomial of degree $2 \cdot 2$ in u and v. Collecting the coefficients of the various terms $u^a v^{4-a}$, one obtains 5 equations in 9 unknowns. The extra $9-5$ degrees of freedom correspond to the fact that if P_0, P_1, P_2 work, so do the polynomials $P_i(au+bv, cu+dv)$ for any fixed $a, b, c, d \in \mathbb{C}$. This approach is direct, but would be very messy.

Here is another approach. It is based on the following observation. Let $M \in \mathrm{GL}_3\mathbb{C}$ be an invertible 3×3-matrix; it determines a bijective linear map $M : \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ and induces a bijective map

$$\bar{M} : \mathbb{P}^2 \longrightarrow \mathbb{P}^2, \qquad [v] \longrightarrow [Mv].$$

If $F = F(X, Y, Z)$ is homogeneous polynomial of degree 2, then so is $F \circ M$. Furthermore, F does not split into linear factors if and only if $F \circ M$ does not (you can prove this either directly or using the approach of Chapter 2, #8). If $Z(F) = f(\mathbb{P}^1)$, then $Z(F \circ M) = \{\bar{M}^{-1} \circ f\}(\mathbb{P}^1)$, and $\bar{M}^{-1} \circ f$ is given by the polynomials $M^{-1}(P_0\, P_1\, P_2)^{tr}$. Thus, it is sufficient to prove the statement with F replaced by $F \circ M$ for some $M \in \mathrm{GL}_3\mathbb{C}$, perhaps repeating the replacement process several times.

For example, if $F(X, Y, Z) = X^2 + Y^2 + Z^2$, we could take

$$M = \begin{pmatrix} 1 & \mathfrak{i} & 0 \\ 1 & -\mathfrak{i} & 0 \\ 0 & 0 & \mathfrak{i} \end{pmatrix}^{-1}$$

This replaces $X+\mathrm{i}Y$ with X, $X-\mathrm{i}Y$ with Y, and Z with $\mathrm{i}Z$, so that F is replaced with $XY-Z^2$. So it is sufficient to do the following steps.
(a) Find f as above that works for $F(X,Y,Z)=XY-Z^2$.
(b) If F does not split into linear factors, show that there exists $M \in GL_n\mathbb{C}$ so that $F \circ M$ is $X^2+Y^2+Z^2$ or $XY-Z^2$.

Discussion Problems for 12/11

Counting plane rational curves

Please read the attached note, *even if you are not presenting*, and make sure to actively participate in the discussion, with questions or comments.

If you are presenting,
(1) State formula (1), recalling what n_d is.
(2) Describe how you are going to prove it; this is essentially Sections 1 and 2.
(3) Prove the formula; this is Section 3 plus you need to derive formula (1) from (6). If time permits, use (1) to compute a few of the numbers n_d. What is the analogue of this for $\mathbb{P}^3$?
Please draw pictures, more of them than in the note, and do not just copy the formulas!

Some of this material is related to some of the material in Chapter 3 of the book.

Please prepare your presentation ahead of time so that it fits in 1 hour and 10 minutes. You should come to my office hours on Tuesday 5-7 or Wednesday 9-10 with any questions you might have.

Counting Plane Rational Curves: a modern approach

Aleksey Zinger

December 3, 2008

Enumerative geometry of algebraic varieties is a field of mathematics that dates back to the nineteenth century. The general goal of this subject is to determine the number of geometric objects that satisfy pre-specified geometric conditions. The objects are typically (complex) curves in a smooth algebraic manifold. Such curves are usually required to represent the given homology class, to have certain singularities, and to satisfy various contact conditions with respect to a collection of subvarieties. One of the most well-known examples of an enumerative problem is

Question 1 *If d is a positive integer, what is the number n_d of degree d rational curves that pass through $3d-1$ points in general position in the complex projective plane $\mathbb{P}^2$?*

Since the number of (complex) lines through any two distinct points is one, $n_1=1$. A little bit of algebraic geometry and topology gives $n_2=1$ and $n_3=12$. It is far harder to find that $n_4=620$, but this number was computed as early as the middle of the nineteenth century; see [5, p378].

The higher-degree numbers n_d remained unknown until the early 1990s, when a recursive formula for the numbers n_d was announced in [2] and [4]:

$$n_d = \frac{1}{6(d-1)} \sum_{d_1+d_2=d} \left(d_1 d_2 - 2\frac{(d_1-d_2)^2}{3d-2} \right) \binom{3d-2}{3d_1-1} d_1 d_2 n_{d_1} n_{d_2}. \tag{1}$$

The argument of the latter paper is described below. It can also be used to solve the natural generalization of Question 1 to the higher-dimensional projective spaces; see [4, Section 10].

We will define an invariant that counts holomorphic maps into $\mathbb{P}^2$. A priori, the number we describe depends on the cross ratio of the chosen four points on a sphere. However, it turns out that this number is well-defined. We use its independence to express this invariant in terms of the numbers n_d in two different ways. By comparing the two expressions, we obtain (1).

1 The moduli space of four marked points on a sphere

Let x_0, x_1, x_2 and x_3 be the four points in $\mathbb{P}^2$ given by

$$x_0 = [1,0,0], \quad x_1 = [0,1,0], \quad x_2 = [0,0,1], \quad x_3 = [1,1,1].$$

We denote by $H^0(\mathbb{P}^2;\gamma^{*\otimes 2})$ the space of holomorphic sections of the holomorphic line bundle $\gamma^{*\otimes 2}\longrightarrow\mathbb{P}^2$, or equivalently of the degree 2 homogeneous polynomials in three variables. Let

$$\begin{aligned}\mathcal{U} &= \big\{([s],x)\in\mathbb{P}H^0(\mathbb{P}^2;\gamma^{*\otimes 2})\times\mathbb{P}^2 : s(x_i)=0\ \forall i=0,1,2,3;\ s(x)=0\big\}\\ &\approx \big\{\big([A,B];[z_0,z_1,z_2]\big)\in\mathbb{P}^1\times\mathbb{P}^2 : (A-B)z_0z_1-Az_1z_2+Bz_0z_2=0\big\}.\end{aligned}$$

The space $\mathcal{U}$ is a compact complex manifold of dimension 2.

Let $\pi:\mathcal{U}\longrightarrow\overline{\mathfrak{M}}_{0,4}\equiv\mathbb{P}^1$ denote the projection onto the first component. If $[A,B]\in\overline{\mathfrak{M}}_{0,4}$, the fiber $\pi^{-1}([A,B])$ is the conic

$$\mathcal{C}_{A,B}=\big\{[z_0,z_1,z_2]\in\mathbb{P}^2 : (A-B)z_0z_1-Az_1z_2+Bz_0z_2=0\big\}.$$

If $[A,B]\neq[1,0],[0,1],[1,1]$, $\mathcal{C}_{A,B}$ is a smooth complex curve of genus zero; it is a sphere with four distinct marked points. If $[A,B]=[1,0],[0,1],[1,1]$, $\mathcal{C}_{A,B}$ is a union of two lines. One of the lines contains two of the four points $x_0,\ldots,x_3$, and the other line passes through the remaining two points. The two lines intersect in a single point. Figure 1 shows the three singular fibers of the projection map $\pi:\mathcal{U}\longrightarrow\overline{\mathfrak{M}}_{0,4}$. The other fibers are smooth conics. The fibers should be viewed as lying in planes orthogonal to the horizontal line in the figure.

The following remarks concerning the family $\mathcal{U}\longrightarrow\overline{\mathfrak{M}}_{0,4}$ are not directly relevant for the purposes of the next two sections and can be omitted. If $[A,B]\in\overline{\mathfrak{M}}_{0,4}-\{[1,0],[0,1],[1,1]\}$, $\mathcal{C}_{A,B}$ is a smooth complex curve of genus zero, i.e. it is a sphere holomorphically embedded in $\mathbb{P}^2$. Thus, there exists a one-to-one holomorphic map $f:\mathbb{P}^1\longrightarrow\mathcal{C}_{A,B}$. It can be shown directly that if $[u_i,v_i]=f^{-1}(x_i)$,

$$\frac{v_0/u_0-v_2/u_2}{v_0/u_0-v_3/u_3}:\frac{v_1/u_1-v_2/u_2}{v_1/u_1-v_3/u_3}=\frac{B}{A}.$$

The cross-ratio is the only invariant of four distinct points on $\mathbb{P}^1$; see [1], for example. Thus,

$$\begin{aligned}\mathbb{P}^1-\{[1,0],[0,1],[1,1]\} &= \mathfrak{M}_{0,4}\equiv\big\{(x_0,x_1,x_2,x_3)\in(\mathbb{P}^1)^4 : x_i\neq x_j \text{ if } i\neq j\big\}/\sim,\\ \text{where}\qquad (x_0,x_1,x_2,x_3) &\sim \big(\tau(x_0),\tau(x_1),\tau(x_2),\tau(x_3)\big)\quad\text{if}\quad \tau\in\mathrm{PSL}_2\equiv\mathrm{Aut}(\mathbb{P}^1).\end{aligned}$$

Furthermore, the restriction of the projection map $\pi:\mathcal{U}|_{\mathfrak{M}_{0,4}}\longrightarrow\mathfrak{M}_{0,4}$ to each fiber $\mathcal{C}_{[A,B]}$ is the cross ratio of the points $x_0,\ldots,x_3$ on $\mathcal{C}_{[A,B]}$, viewed as an element of $\mathbb{P}^1\supset\mathbb{C}$.

2 Counts of holomorphic maps

If d is an integer and $\mathcal{C}$ is a complex curve, which may be a wedge of spheres, let

$$\mathcal{H}_d(\mathcal{C})=\big\{f\in C^\infty(\mathcal{C};\mathbb{P}^2) : f \textit{ is holomorphic},\ \deg f=d\big\}. \tag{2}$$

We give a more explicit description of the space $\mathcal{H}_d(\mathcal{C})$ in the relevant cases below.

Suppose ℓ_0,ℓ_1 and $p_2,\ldots,p_{3d-1}$ are two lines and $3d$–2 points in general position in $\mathbb{P}^2$. If $\sigma\in\overline{\mathfrak{M}}_{0,4}$, let $N_d^\sigma(l_0,l_1,p_2,\ldots,p_{3d-1})$ denote the cardinality of the set

$$\big\{f\in\mathcal{H}_d(\mathcal{C}_\sigma) : f(x_0)\in\ell_0,\ f(x_1)\in\ell_1,\ f(x_2)=p_2,\ f(x_3)=p_3,\ p_i\in\mathrm{Im}\,f\ \forall i\big\}. \tag{3}$$

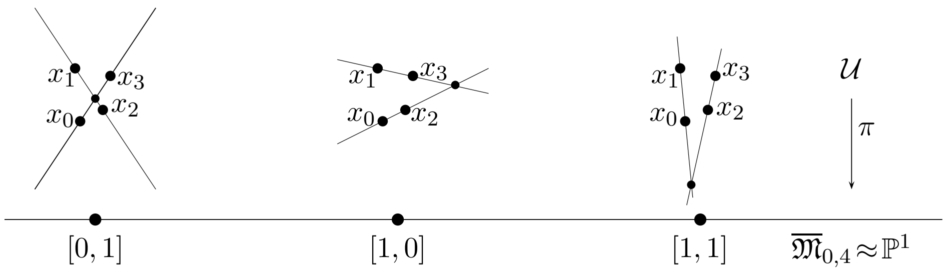

Figure 1: The Family $\mathcal{U} \longrightarrow \overline{\mathfrak{M}}_{0,4}$

Here $\mathcal{C}_\sigma$ denotes the rational curve with four marked points, x_0, x_1, x_2, and x_3, whose cross ratio is σ; see Section 1. If $\sigma \neq [1,0], [0,1], [1,1]$, $\mathcal{C}_\sigma$ is a sphere with four, distinct, marked points. In this case, the condition $f \in \mathcal{H}_d(\mathcal{C}_\sigma)$ means that f has the form

$$f([u,v]) = \big[P_0(u,v), P_1(u,v), P_2(u,v)\big] \qquad \forall [u,v] \in \mathbb{P}^1,$$

for some degree d homogeneous polynomials P_0, P_1, P_2 that have no common factor. If $\sigma = [1,0], [0,1], [1,1]$, $\mathcal{C}_\sigma$ is a wedge of two spheres, $\mathcal{C}_{\sigma,1}$ and $\mathcal{C}_{\sigma,2}$, with two marked points each. In this case, the first condition in (2) means that f is continuous and $f|_{\mathcal{C}_{\sigma,1}}$ and $f|_{\mathcal{C}_{\sigma,2}}$ are holomorphic. The second condition in (2) means that $d = d_1 + d_2$ if the degrees of $f|_{\mathcal{C}_{\sigma,1}}$ and $f|_{\mathcal{C}_{\sigma,2}}$ are d_1 and d_2, respectively.

The requirement that the two lines, ℓ_0 and ℓ_1, and the $3d-2$ points, $p_2, \ldots, p_{3d-1}$, are in general position means that they lie in a dense open subset $\mathcal{U}_\sigma$ of the space of all possible tuples $(\ell_0, \ell_1, p_2, \ldots, p_{3d-1})$:

$$\mathfrak{X} \equiv \mathrm{Gr}_2\mathbb{C}^3 \times \mathrm{Gr}_2\mathbb{C}^3 \times \big(\mathbb{P}^2\big)^{3d-2}.$$

Here $\mathrm{Gr}_2\mathbb{C}^3$ denotes the Grassmanian manifold of two-planes through the origin in $\mathbb{C}^3$, or equivalently of lines in $\mathbb{P}^2$. The dense open subset $\mathcal{U}_\sigma$ of $\mathfrak{X}$ consists of tuples $(\ell_0, \ell_1, p_2, \ldots, p_{3d-1})$ that satisfy a number of geometric conditions. In particular, $\ell_0 \neq \ell_1$, none of the points $p_2, \ldots, p_{3d-1}$ lies on either ℓ_0 or ℓ_1, the $3d-1$ points $\ell_0 \cap \ell_1, p_2, \ldots, p_{3d-1}$ are distinct, no three of them lie on the same line, and so on. In addition, we need to impose certain cross-ratio conditions on the rational curves that pass through ℓ_0, ℓ_1, p_2, p_3, and a subset of the remaining $3d-4$ points. These conditions can be stated more formally. Define

$$\mathrm{ev}_\sigma \colon \mathcal{H}_d(\mathcal{C}_\sigma) \times \big(\mathcal{C}_\sigma\big)^{3d-4} \longrightarrow \big(\mathbb{P}^2\big)^{3d} \quad \text{by} \quad \mathrm{ev}_\sigma\big(f; x_4, \ldots, x_{3d-1}\big) = \big(f(x_0), f(x_1), \ldots, f(x_{3d-1})\big).$$

The space $\mathcal{H}_d(\mathcal{C}_\sigma)$ is a dense open subset of $\mathbb{P}^{3d+2}$ and the evaluation map ev_σ is holomorphic. There is a natural compactification $\overline{\mathfrak{M}}_\sigma(\mathbb{P}^2, d)$ of $\mathcal{H}_d(\mathcal{C}_\sigma)$, which consists spaces of holomorphic maps from various wedges of spheres into $\mathbb{P}^2$. The complex dimension of each such boundary stratum is less than that of $\mathcal{H}_d(\mathcal{C}_\sigma)$. The evaluation map ev_σ admits a continuous extension over $\partial\overline{\mathfrak{M}}_\sigma(\mathbb{P}^2, d)$, whose restriction to each stratum is holomorphic. The elements $(\ell_0, \ell_1, p_2, \ldots, p_{3d-1})$ of the subspace $\mathcal{U}_\sigma$ of $\mathfrak{X}$ are characterized by the condition that the restriction of the evaluation map to each stratum of $\overline{\mathfrak{M}}_\sigma(\mathbb{P}^2, d)$ is transversal to the submanifold

$$\ell_0 \times \ell_1 \times p_2 \times \ldots \times p_{3d-1} \subset (\mathbb{P}^2)^{3d}.$$

This condition implies that

$$\mathrm{ev}_\sigma^{-1}\big(\ell_0\times\ell_1\times p_2\times\ldots\times p_{3d-1}\big)\cap\partial\overline{\mathfrak{M}}_\sigma(\mathbb{P}^2,d)=\emptyset$$

and the set in (3) is a finite subset of $\mathcal{H}_d(\mathcal{C}_\sigma)$.

The set $\mathcal{U}_\sigma$ of "general" tuples $(\ell_0,\ell_1,p_2,\ldots,p_{3d-1})$ is path-connected. Indeed, it is the complement of a finite number of proper complex submanifolds in $\mathfrak{X}$. It follows that the number in (3) is independent of the choice of two lines and $3d-2$ points in general position in $\mathbb{P}^2$. We thus may simply denote it by N_d^σ. If $\sigma\neq[1,0],[0,1],[1,1]$, $\mathcal{C}_\sigma$ is a sphere with four distinct points. In such a case, it is fairly easy to show that the number N_d^σ does not change under small variations of σ, or equivalently of the four points on the sphere. Thus, N_d^σ is independent of

$$\sigma\in\mathfrak{M}_{0,4}=\mathbb{P}^1-\big\{[1,0],[0,1],[1,1]\big\}=\overline{\mathfrak{M}}_{0,4}-\big\{[1,0],[0,1],[1,1]\big\}.$$

It is far harder to prove

Proposition 2 *The function $\sigma\longrightarrow N_d^\sigma$ is constant on $\overline{\mathfrak{M}}_{0,4}$.*

This proposition is a special case of the gluing theorems first proved in [3] and [4].

3 Holomorphic maps vs. complex curves

In this subsection, we express the numbers $N_d^{[0,1]}$ and $N_d^{[1,1]}$ of Subsections 2 in terms of the numbers $n_{d'}$, with $d'\le d$, of Question 1. By Proposition 2, $N_d^{[0,1]}=N_d^{[1,1]}$. We obtain a recursion for the numbers of Question 1 by comparing the expressions for $N_d^{[0,1]}$ and $N_d^{[1,1]}$.

Let $\mathcal{C}_1$ denote the component of $\mathcal{C}_{[0,1]}$ containing the marked points x_0 and x_3; see Figure 1. We denote by $\mathcal{C}_2$ the other component of $\mathcal{C}_{[0,1]}$. By definition,

$$N_d^{[0,1]}=\sum_{d_1+d_2=d}N_{d_1,d_2}^{[0,1]}\qquad\text{where}$$

$$N_{d_1,d_2}^{[0,1]}=\big|\big\{f\in\mathcal{H}_d(\mathcal{C}_{[0,1]};\mathbb{P}^2):\ \deg f|_{\mathcal{C}_1}=d_1,\ \deg f|_{\mathcal{C}_2}=d_2;\ p_i\in\operatorname{Im} f\ \forall i;\\ f(x_0)\in\ell_0,\ f(x_1)\in\ell_1,\ f(x_2)=p_2,\ f(x_3)=p_3\big\}\big|.$$

Since the group PSL_2 of holomorphic automorphisms acts transitively on triples of distinct points on the sphere,

$$N_{d_1,d_2}^{[0,1]}=\big|\big\{(f_1,f_2)\in\mathcal{H}_{d_1}(S^2)\times\mathcal{H}_{d_2}(S^2):\ f_1(\infty)=f_2(\infty),\ p_i\in f_1(S^2)\cup f_2(S^2)\ \forall i;\\ f_1(0)\in\ell_0,\ f_1(1)=p_3,\ f_2(0)\in\ell_1,\ f_2(1)=p_2\big\}\big|.$$

Since the maps f_1 and f_2 above are holomorphic, $d_1,d_2\ge0$ if $N_{d_1,d_2}^{[0,1]}\neq0$. Since every degree 0 holomorphic map is constant and $p_3\notin\ell_0$, $N_{0,d}^{[0,1]}=0$. Similarly, $N_{d,0}^{[0,1]}=0$. Thus, we assume that $d_1,d_2>0$. Since the points $p_3,\ldots,p_{3d-1}$ are in general position, $f_1(S^2)$ contains at most $3d_1-2$ of the points $p_4,\ldots,p_{3d-1}$. Similarly, the curve $f_2(S^2)$ passes through at most $3d_2-2$ of the points $p_4,\ldots,p_{3d-1}$. Thus, if $I=\{4,\ldots,3d-1\}$,

$$N^{[0,1]}_{d_1,d_2}=\sum_{I=I_1\sqcup I_2,|I_1|=3d_1-2}N^{[0,1]}_{d_1,d_2}(I_1,I_2),$$

where $N^{[0,1]}_{d_1,d_2}(I_1,I_2)$ is the cardinality of the set

$$\begin{aligned}\mathcal{S}^{[0,1]}_{d_1,d_2}(I_1,I_2)=\big\{(f_1,f_2)\in\mathcal{H}_{d_1}(S^2)\times\mathcal{H}_{d_2}(S^2):\ &p_i\in f_1(S^2)\ \forall i\in I_1,\ p_i\in f_2(S^2)\ \forall i\in I_2;\\ &f_1(\infty)=f_2(\infty),\ f_1(0)\in\ell_0,\ f_1(1)=p_3,\ f_2(0)\in\ell_1,\ f_2(1)=p_2\big\}.\end{aligned}$$

If $(f_1,f_2)\in\mathcal{S}^{[1,0]}_{d_1,d_2}(I_1,I_2)$, $f_1(S^2)$ is one of the n_{d_1} curves passing through the points $\{p_i: i\in\{3\}\sqcup I_1\}$. Similarly, $f_2(S^2)$ is one of the n_{d_2} curves passing through the points $\{p_i: i\in\{2\}\sqcup I_2\}$. The point $f_1(\infty)=f_2(\infty)$ must be one of the points of $f_1(S^2)\cap f_2(S^2)$; by Bezoit's theorem there are d_1d_2 such points. Finally, $f_1(0)$ must be one of the d_1 points of $f_1(S^2)\cap\ell_0$, while $f_2(0)$ must be one of the d_2 points of $f_2(S^2)\cap\ell_1$. Thus, we conclude that

$$\begin{aligned}N^{[0,1]}_d=\sum_{d_1+d_2=d}N^{[0,1]}_{d_1,d_2}&=\sum_{d_1+d_2=d}\ \sum_{I=I_1\sqcup I_2,|I_1|=3d_1-2}N^{[0,1]}_{d_1,d_2}(I_1,I_2)\\ &=\sum_{d_1+d_2=d}\ \sum_{I_1\subset I,|I_1|=3d_1-2}(d_1d_2)(d_1n_{d_1})(d_2n_{d_2})\\ &=\sum_{d_1+d_2=d}\binom{3d-4}{3d_1-2}d_1^2d_2^2n_{d_1}n_{d_2};\end{aligned}\tag{4}$$

where $I=\{4,\ldots,3d-1\}$.

We compute the number $N^{[1,1]}_d$ similarly. We denote by $\mathcal{C}_1$ the component of $\mathcal{C}_{[1,1]}$ containing the points x_0 and x_1 and by $\mathcal{C}_2$ the other component of $\mathcal{C}_{[1,1]}$. By definition,

$$N^{[1,1]}_d=\sum_{d_1+d_2=d}N^{[1,1]}_{d_1,d_2},\qquad\text{where}$$

$$\begin{aligned}N^{[1,1]}_{d_1,d_2}=\big|\big\{(f_1,f_2)\in\mathcal{H}_{d_1}(S^2)\times\mathcal{H}_{d_2}(S^2):\ &f_1(\infty)=f_2(\infty),\ p_i\in f_1(S^2)\cup f_2(S^2)\ \forall i;\\ &f_1(0)\in\ell_0,\ f_1(1)\in\ell_1,\ f_2(0)=p_2,\ f_2(1)=p_3\big\}\big|.\end{aligned}$$

Since every degree-zero holomorphic map is constant, $N^{[1,1]}_{d,0}=0$ as before. However,

$$\begin{aligned}N^{[1,1]}_{0,d}=\big|\big\{f_2\in\mathcal{H}_d(S^2): f_2(\infty)\in\ell_0\cap\ell_1,\ f_2(0)=p_2,\ f_2(1)=p_3;\\ p_i\in f_2(S^2)\ \forall i=4,\ldots,3d-1\big\}\big|.\end{aligned}$$

Thus, $N^{[1,1]}_{0,d}=n_d$. If $d_1,d_2>0$,

$$N^{[1,1]}_{d_1,d_2}=\sum_{I=I_1\sqcup I_2,|I_1|=3d_1-1}N^{[1,1]}_{d_1,d_2}(I_1,I_2),$$

where $N^{[1,1]}_{d_1,d_2}(I_1,I_2)$ is the cardinality of the set

$$\mathcal{S}^{[1,1]}_{d_1,d_2}(I_1,I_2)=\big\{(f_1,f_2)\in\mathcal{H}_{d_1}(S^2)\times\mathcal{H}_{d_2}(S^2):\ p_i\in f_1(S^2)\ \forall i\in I_1,\ p_i\in f_2(S^2)\ \forall i\in I_2;\\ f_1(\infty)=f_2(\infty),\ f_1(0)\in\ell_0,\ f_1(1)\in\ell_1,\ f_2(0)=p_2,\ f_2(1)=p_3\big\}.$$

Proceeding as in the previous paragraph, we conclude that

$$\begin{aligned} N^{[1,1]}_d=\sum_{d_1+d_2=d}N^{[1,1]}_{d_1,d_2}&=n_d+\sum_{d_1+d_2=d}\ \sum_{I=I_1\sqcup I_2,|I_1|=3d_1-1}N^{[1,1]}_{d_1,d_2}(I_1,I_2)\\ &=n_d+\sum_{d_1+d_2=d}\ \sum_{I_1\subset I,|I_1|=3d_1-1}(d_1d_2)(d_1^2n_{d_1})(n_{d_2}) \\ &=n_d+\sum_{d_1+d_2=d}\binom{3d-4}{3d_1-1}d_1^3d_2n_{d_1}n_{d_2}; \end{aligned} \tag{5}$$

Comparing equations (4) and (5), we obtain

$$n_d=\sum_{d_1+d_2=d}\left(\binom{3d-4}{3d_1-2}d_1d_2-\binom{3d-4}{3d_1-1}d_1^2\right)d_1d_2n_{d_1}n_{d_2}. \tag{6}$$

The recursive formula (1) is the symmetrized version of (6).

References

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