MAT 401: Undergraduate Seminar

Introduction to Enumerative Geometry

Fall 2008

Homework Assignment V

Written Assignment due on Thursday, 12/2, at 11:20am in Physics P-117

Please write up solutions to all of the following:

- Chapter 6, #6, 7 or Problem J below (one of the three);
- Problem I below, Chapter 7, #2,6 (all three);
- Chapter 7, #5,9, or 10 (one of the three).

Problem I

Recall that an element ℓ of $\mathbb{P}^n$ is a complex one-dimensional linear subspace of $\mathbb{C}^{n+1}$, i.e. a complex line through the origin in $\mathbb{C}^{n+1}$. Let

$$\gamma = \big\{(\ell,v)\in\mathbb{P}^n\times\mathbb{C}^{n+1}:\ v\in\ell\subset\mathbb{C}^{n+1}\big\}.$$

Thus, γ is a subset of the complex manifold $\mathbb{P}^n\times\mathbb{C}^{n+1}$. Show that the projection to the first component

$$\pi_1\colon\gamma\longrightarrow\mathbb{P}^n, \qquad (\ell,v)\longrightarrow\ell,$$

defines a holomorphic line bundle. In particular, describe trivializations of γ over the open subsets

$$\mathcal{U}_i\equiv\big\{[X_0,\ldots,X_n]:\ X_i\neq0\big\}$$

and compute all the transition maps $g_{ij}\colon\mathcal{U}_i\cap\mathcal{U}_j\longrightarrow\mathbb{C}-\{0\}$; these maps should be analytic. The line bundle $\gamma\longrightarrow\mathbb{P}^n$ is called *the tautological line bundle over* $\mathbb{P}^n$.

Problem J (roughly Chapter 7, #8)

Recall that an element P of $G(2,n)$ is a complex two-dimensional linear subspace of $\mathbb{C}^n$, i.e. a complex plane through the origin in $\mathbb{C}^n$. Let

$$\gamma = \big\{(P,v)\in G(2,n)\times\mathbb{C}^n:\ v\in P\subset\mathbb{C}^n\big\}.$$

Thus, γ is a subset of the complex manifold $G(2,n)\times\mathbb{C}^n$. Show that the projection to the first component

$$\pi_1\colon\gamma\longrightarrow G(2,n), \qquad (P,v)\longrightarrow P,$$

defines a vector bundle of rank 2. The vector bundle $\gamma\longrightarrow G(2,n)$ is called *the tautological two-plane bundle over* $G(2,n)$.

Discussion Problems for 12/2,4

Lines in projective spaces

Day 1: Review the definition of the Schubert cycles σ_{ab} in $G(2,n)$. Verify Pieri's formula:

$$\sigma_{a_1} \cdot \sigma_{a_2} = \sum_{c \geq a_1, a_2} a_{c, a_1+a_2-c} \, .$$

Recall the other formulas from the previous discussion (state them without re-proving). Describe a solution to Exercise 5 in Chapter 7.

Day 2: Review the definition of the Schubert cycles σ_{ab} in $G(2,n)$ and the intersection formulas for them. Describe a solution to Exercise 10 in Chapter 7. Find the number of lines that lie on a general quintic hypersurface in $\mathbb{P}^4$ (this is mostly done in the book, but not completely).

We should finish all of the relevant material for this homework assignment by Thursday, 11/20. Please try to complete the written assignment and study the discussion part before Tuesday, 11/25, and come to the office hours then with any questions.