MAT 401: Undergraduate Seminar
Introduction to Enumerative Geometry
Fall 2018

Homework Assignment II

Written Assignment due on Thursday, 9/13, at 1pm in ESS 181
(or by 9/13, noon, in Math 3-111)

Chapter 2, #1,5,6; optional bonus problem on the back

Please aim to make your solutions as concise and to the point as possible.

Discussion Problem for 9/13
Duality for Conics

Let $n_2(i)$ be the number of plane conics that are tangent to i general lines and pass through $5-i$ points, with $i=0,1,\ldots,5$. It is stated at the end of Chapter 2 that

i	0	1	2	3	4	5
$n_2(i)$	1	2	4	4	2	1

An argument for the numbers $n_2(i)$ for $i=0,1,2$ is given in the book. The aim of this discussion problem is to obtain the remaining numbers by showing that

$$n_2(i) = n_2(5-i). \tag{*}$$

Part I: Chapter 2, #8 ($\sim$30 mins)

Part II: Recall from the first discussion session and Chapter 2 that a line in $\mathbb{C}P^2$ is also a point in the dual projective plane, $(\mathbb{C}P^2)^* \approx \mathbb{C}P^2$. If $C \subset \mathbb{C}P^2$ is a smooth conic, there is a well-defined tangent line $\tau_C(z) \in (\mathbb{C}P^2)^*$ at each point $z \in C$. Show that the map

$$\tau_C : C \longrightarrow (\mathbb{C}P^2)^*, \qquad z \longrightarrow \tau_C(z),$$

is injective and is a homeomorphism (or at least a bijection) onto a smooth conic C^* in $(\mathbb{C}P^2)^*$. Furthermore, the image of the map

$$\tau_{C^*} : C^* \longrightarrow ((\mathbb{C}P^2)^*)^* = \mathbb{C}P^2$$

is the original conic C (i.e. dualizing twice gets us back to where we started). ($\sim$30 mins)

Part III: Prove the identity ($*$) ($\sim$10 mins)

On Tuesday, 9/11, please volunteer to discuss one of the three parts on Thursday, 9/13.

Problem A (bonus)

There are two natural coordinate charts on $\mathbb{C}P^1$:

$$\varphi_0 : U_0 \equiv \{[Z_0, Z_1] \in \mathbb{C}P^1 : Z_0 \neq 0\} \longrightarrow \mathbb{C}, \qquad \varphi_0([Z_0, Z_1]) = Z_1/Z_0;$$
$$\varphi_1 : U_1 \equiv \{[Z_0, Z_1] \in \mathbb{C}P^1 : Z_1 \neq 0\} \longrightarrow \mathbb{C}, \qquad \varphi_1([Z_0, Z_1]) = Z_0/Z_1.$$

A map $F : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ is called holomorphic if F is continuous and the four maps

$$F_{ij} \equiv \varphi_i \circ F \circ \varphi_j^{-1} : \varphi_j\big(F^{-1}(U_i) \cap U_j\big) \longrightarrow \mathbb{C}, \qquad i, j = 0, 1,$$

are holomorphic (as maps between open subsets of $\mathbb{C}$).

(1) Show that the overlap maps between the coordinate charts,

$$\varphi_{10} \equiv \varphi_1 \circ \varphi_0^{-1} : \varphi_0(U_0 \cap U_1) \longrightarrow \varphi_1(U_0 \cap U_1), \quad \varphi_{01} \equiv \varphi_0 \circ \varphi_1^{-1} : \varphi_1(U_0 \cap U_1) \longrightarrow \varphi_0(U_0 \cap U_1)$$

are holomorphic (as maps between open subsets of $\mathbb{C}$).

(2) Let $p_0, p_1 : \mathbb{C}^2 \longrightarrow \mathbb{C}$ be homogeneous polynomials of the same degree without a common linear factor. Show that the map

$$F : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1, \qquad F\big([Z_0, Z_1]\big) = \big[p_0(Z_0, Z_1), p_1(Z_0, Z_1)\big],$$

is well-defined and holomorphic (you can assume continuity).

(3) Let $F : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ be a non-constant holomorphic map. Show that

(a) $\mathbb{C} - \varphi_j(F^{-1}(U_i) \cap U_j)$ is a finite set of points for all $i, j = 0, 1$;

(b) there exist homogeneous polynomials $p_0, p_1 : \mathbb{C}^2 \longrightarrow \mathbb{C}$ of the same degree without a common linear factor such that

$$F\big([Z_0, Z_1]\big) = \big[p_0(Z_0, Z_1), p_1(Z_0, Z_1)\big] \qquad \forall\ [Z_0, Z_1] \in \mathbb{C}P^1.$$

Note: This problem requires some relatively basic stuff from MAT 536, which is probably also done in MAT 342.