Chapter 2

Smooth Vector Bundles

7 Definitions and Examples

A *(smooth) real vector bundle* V of rank k over a smooth manifold M is a smoothly varying family of k-dimensional real vector spaces which is locally trivial. Formally, it is a triple (M, V, π), where M and V are smooth manifolds and

$$\pi\colon V \longrightarrow M$$

is a surjective submersion. For each $p \in M$, the fiber $V_p \equiv \pi^{-1}(p)$ of V over p is a real k-dimensional vector space; see Figure 2.1. The vector-space structures in V_p vary smoothly with $p \in M$. This means that the scalar multiplication map

$$\mathbb{R} \times V \longrightarrow V, \qquad (c, v) \longrightarrow c \cdot v, \tag{7.1}$$

and the addition map

$$V \times_M V \equiv \big\{(v_1, v_2) \in V \times V : \pi(v_1) = \pi(v_2) \in M\big\} \longrightarrow V, \qquad (v_1, v_2) \longrightarrow v_1 + v_2, \tag{7.2}$$

are smooth. Note that we can add $v_1, v_2 \in V$ only if they lie in the same fiber over M, i.e.

$$\pi(v_1) = \pi(v_2) \qquad \Longleftrightarrow \qquad (v_1, v_2) \in V \times_M V.$$

The space $V \times_M V$ is a smooth submanifold of $V \times V$ by the *Implicit Function Theorem for Maps* (Corollary 6.7). The local triviality condition means that for every point $p \in M$ there exist a neighborhood U of p in M and a diffeomorphism

$$h\colon V|_U \equiv \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$$

such that h takes every fiber of π to the corresponding fiber of the projection map $\pi_1 : U \times \mathbb{R}^k \longrightarrow U$, i.e. $\pi_1 \circ h = \pi$ on $V|_U$ so that the diagram

$$\begin{array}{ccc} V|_U \equiv \pi^{-1}(U) & \xrightarrow[\approx]{h} & U \times \mathbb{R}^k \\ & \searrow^{\pi} \quad {}^{\pi_1}\swarrow & \\ & U & \end{array}$$

commutes, and the restriction of h to each fiber is linear:

$$h(c_1 v_1 + c_2 v_2) = c_1 h(v_1) + c_2 h(v_2) \in x \times \mathbb{R}^k \qquad \forall\ c_1, c_2 \in \mathbb{R},\ v_1, v_2 \in V_x,\ x \in U.$$

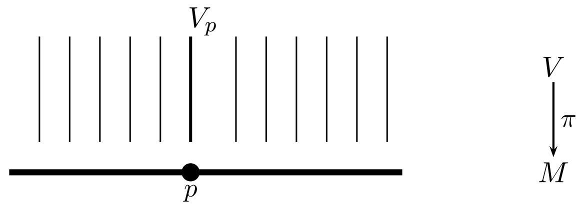


Figure 2.1: Fibers of a vector-bundle projection map are vector spaces of the same rank.

These conditions imply that the restriction of h to each fiber V_x of π is an isomorphism of vector spaces. In summary, V locally (and not just pointwise) looks like bundles of $\mathbb{R}^k$'s over open sets in M glued together. This is in a sense analogous to an m-manifold being open subsets of $\mathbb{R}^m$ glued together in a nice way. Here is a formal definition.

Definition 7.1. *A real vector bundle of rank k is a tuple $(M, V, \pi, \cdot, +)$ such that*

($\mathbb{R}$VB1) *M and V are smooth manifolds and $\pi\colon V \longrightarrow M$ is a smooth map;*

($\mathbb{R}$VB2) *$\cdot\colon \mathbb{R}\times V \longrightarrow V$ is a map s.t. $\pi(c\cdot v)=\pi(v)$ for all $(c,v)\in\mathbb{R}\times V$;*

($\mathbb{R}$VB3) *$+\colon V\times_M V \longrightarrow V$ is a map s.t. $\pi(v_1+v_2)=\pi(v_1)=\pi(v_2)$ for all $(v_1,v_2)\in V\times_M V$;*

($\mathbb{R}$VB4) *for every point $p\in M$ there exist a neighborhood U of p in M and a diffeomorphism $h\colon V|_U \longrightarrow U\times\mathbb{R}^k$ such that*

- ($\mathbb{R}$VB4-a) *$\pi_1\circ h=\pi$ on $V|_U$ and*
- ($\mathbb{R}$VB4-b) *the map $h|_{V_x}\colon V_x \longrightarrow x\times\mathbb{R}^k$ is an isomorphism of vector spaces for all $x\in U$.*

The spaces M and V are called the base and the total space of the vector bundle (M, V, π). It is customary to call $\pi : V \longrightarrow M$ a vector bundle and V a vector bundle over M. If M is an m-manifold and $V\longrightarrow M$ is a real vector bundle of rank k, then V is an $(m+k)$-manifold. Its smooth charts are obtained by restricting the trivialization maps h for V, as above, to small coordinate charts in M.

Example 7.2. If M is a smooth manifold and k is a nonnegative integer, then

$$\pi_1\colon M\times\mathbb{R}^k \longrightarrow M$$

is a real vector bundle of rank k over M. It is called the trivial rank k real vector bundle over M and denoted $\pi\colon \tau_k^{\mathbb{R}} \longrightarrow M$ or simply $\pi\colon \tau_k \longrightarrow M$ if there is no ambiguity.

Example 7.3. Let $M=S^1$ be the unit circle and $V=\mathrm{MB}$ the infinite Mobius band of Example 1.8. With notation as in Example 1.8, the map

$$\pi\colon V \longrightarrow M, \qquad [s,t] \longrightarrow \mathrm{e}^{2\pi\mathrm{i}s},$$

defines a real line bundle (i.e. rank 1 bundle) over S^1. Trivializations of this vector bundle can be constructed as follows. With $U_\pm=S^1-\{\pm 1\}$, let

$$h_+\colon V|_{U_+} \longrightarrow U_+\times\mathbb{R}, \qquad [s,t] \longrightarrow \big(\mathrm{e}^{2\pi\mathrm{i}s}, t\big);$$
$$h_-\colon V|_{U_-} \longrightarrow U_-\times\mathbb{R}, \qquad [s,t] \longrightarrow \begin{cases} (\mathrm{e}^{2\pi\mathrm{i}s}, t), & \text{if } s\in(1/2,1]; \\ (\mathrm{e}^{2\pi\mathrm{i}s}, -t), & \text{if } s\in[0,1/2). \end{cases}$$

Both maps are diffeomorphisms, with respect to the smooth structures of Example 1.8 on MB and of Example 1.7 on S^1. Furthermore, $\pi_1\circ h_\pm=\pi$ and the restriction of $h_\pm$ to each fiber of π is a linear map to $\mathbb{R}$.

Example 7.4. Let $\mathbb{R}P^n$ be the real projective space of dimension n described in Example 1.9 and

$$\gamma_n = \big\{(\ell,v)\in\mathbb{R}P^n\times\mathbb{R}^{n+1}: v\in\ell\big\},$$

where $\ell\subset\mathbb{R}^{n+1}$ denotes a one-dimensional linear subspace. If $U_i\subset\mathbb{R}P^n$ is as in Example 1.9, the map

$$h_i\colon \gamma_n\cap U_i\times\mathbb{R}^{n+1}\longrightarrow U_i\times\mathbb{R}, \qquad \big(\ell,(v_0,\ldots,v_n)\big)\longrightarrow(\ell,v_i),$$

is a homeomorphism. The overlap maps,

$$h_i\circ h_j^{-1}\colon U_i\cap U_j\times\mathbb{R}\longrightarrow U_i\cap U_j\times\mathbb{R}, \qquad (\ell,c)\longrightarrow\big(\ell,(X_i/X_j)c\big),$$

are smooth. By Lemma 2.6, the collection $\{(\gamma_n\cap U_i\times\mathbb{R}^{n+1},h_i)\}$ of generalized smooth charts then induces a smooth structure on the topological subspace $\gamma_n\subset\mathbb{R}P^n\times\mathbb{R}^{n+1}$. With this smooth structure, γ_n is an embedded submanifold of $\mathbb{R}P^n\times\mathbb{R}^{n+1}$ and the projection on the first component,

$$\pi=\pi_1 : \gamma_n\longrightarrow\mathbb{R}P^n\,,$$

defines a smooth real line bundle. The fiber over a point $\ell\in\mathbb{R}P^n$ is the *one-dimensional subspace ℓ of* $\mathbb{R}^{n+1}$! For this reason, γ_n is called the tautological line bundle over $\mathbb{R}P^n$. Note that $\gamma_1\longrightarrow S^1$ is the infinite Mobius band of Example 7.3.

Example 7.5. If M is a smooth m-manifold, let

$$TM=\bigsqcup_{p\in M}T_pM, \qquad \pi\colon TM\longrightarrow M, \quad \pi(v)=p \ \text{ if } v\in T_pM.$$

If $\varphi_\alpha\colon U_\alpha\longrightarrow\mathbb{R}^m$ is a smooth chart on M, let

$$\tilde\varphi_\alpha\colon TM|_{U_\alpha}\equiv\pi^{-1}(U_\alpha)\longrightarrow U_\alpha\times\mathbb{R}^m, \qquad \tilde\varphi_\alpha(v)=\big(\pi(v),\mathrm{d}_{\pi(v)}\varphi_\alpha v\big). \tag{7.3}$$

If $\varphi_\beta\colon U_\beta\longrightarrow\mathbb{R}^m$ is another smooth chart, the overlap map

$$\tilde\varphi_\alpha\circ\tilde\varphi_\beta^{-1}\colon U_\alpha\cap U_\beta\times\mathbb{R}^m\longrightarrow U_\alpha\cap U_\beta\times\mathbb{R}^m$$

is a smooth map between open subsets of $\mathbb{R}^{2m}$. By Corollary 2.7, the collection of generalized smooth charts

$$\big\{(\pi^{-1}(U_\alpha),\tilde\varphi_\alpha)\colon(U_\alpha,\varphi_\alpha)\in\mathcal{F}_M\big\},$$

where $\mathcal{F}_M$ is the smooth structure of M, then induces a manifold structure on the set TM. With this smooth structure on TM, the projection $\pi\colon TM\longrightarrow M$ defines a smooth real vector bundle of rank m, called the tangent bundle of M.

Definition 7.6. *A complex vector bundle of rank k is a tuple $(M,V,\pi,\cdot,+)$ such that*

($\mathbb{C}$VB1) *M and V are smooth manifolds and $\pi\colon V\longrightarrow M$ is a smooth map;*

($\mathbb{C}$VB2) *$\cdot\colon\mathbb{C}\times V\longrightarrow V$ is a map s.t. $\pi(c\cdot v)=\pi(v)$ for all $(c,v)\in\mathbb{C}\times V$;*

(ℂVB3) $+\colon V\times_M V \longrightarrow V$ *is a map s.t.* $\pi(v_1+v_2)=\pi(v_1)=\pi(v_2)$ *for all* $(v_1,v_2)\in V\times_M V$;

(ℂVB4) *for every point* $p\in M$ *there exists a neighborhood* U *of* p *in* M *and a diffeomorphism* $h\colon V|_U \longrightarrow U\times\mathbb{C}^k$ *such that*

(ℂVB4-a) $\pi_1\circ h=\pi$ *on* $V|_U$ *and*

(ℂVB4-b) *the map* $h|_{V_x}\colon V_x\longrightarrow x\times\mathbb{C}^k$ *is an isomorphism of complex vector spaces for all* $x\in U$.

Similarly to a real vector bundle, a complex vector bundle over M locally looks like bundles of $\mathbb{C}^k$'s over open sets in M glued together. If M is an m-manifold and $V\longrightarrow M$ is a complex vector bundle of rank k, then V is an $(m+2k)$-manifold. A complex vector bundle of rank k is also a real vector bundle of rank $2k$, but a real vector bundle of rank $2k$ need not in general admit a complex structure.

Example 7.7. If M is a smooth manifold and k is a nonnegative integer, then

$$\pi_1\colon M\times\mathbb{C}^k \longrightarrow M$$

is a complex vector bundle of rank k over M. It is called the trivial rank-k complex vector bundle over M and denoted $\pi\colon\tau_k^{\mathbb{C}}\longrightarrow M$ or simply $\pi\colon\tau_k\longrightarrow M$ if there is no ambiguity.

Example 7.8. Let $\mathbb{C}P^n$ be the complex projective space of dimension n described in Example 1.10 and

$$\gamma_n=\big\{(\ell,v)\in\mathbb{C}P^n\times\mathbb{C}^{n+1}\colon v\in\ell\big\}.$$

The projection $\pi\colon\gamma_n\longrightarrow\mathbb{C}P^n$ defines a smooth complex line bundle. The fiber over a point $\ell\in\mathbb{C}P^n$ is the *one-dimensional complex subspace* ℓ *of* $\mathbb{C}^{n+1}$. For this reason, γ_n is called the tautological line bundle over $\mathbb{C}P^n$.

Example 7.9. If M is a complex m-manifold, the tangent bundle TM of M is a complex vector bundle of rank m over M.

8 Sections and Homomorphisms

Definition 8.1. (1) *A (smooth)* section *of a (real or complex) vector bundle* $\pi: V\longrightarrow M$ *is a (smooth) map* $s\colon M\longrightarrow V$ *such that* $\pi\circ s=\mathrm{id}_M$, *i.e.* $s(x)\in V_x$ *for all* $x\in M$.

(2) *A* vector field *on a smooth manifold is a section of the tangent bundle* $TM\longrightarrow M$.

If $\pi: V=M\times\mathbb{R}^k\longrightarrow M$ is the trivial bundle of rank k, a section of π is a map $s: M\longrightarrow V$ of the form

$$s=(\mathrm{id}_M,f)\colon M\longrightarrow M\times\mathbb{R}^k$$

for some map $f: M\longrightarrow\mathbb{R}^k$. This section is smooth if and only if f is a smooth map. Thus, a (smooth) section of the trivial vector bundle of rank k over M is essentially a (smooth) map $M\longrightarrow\mathbb{R}^k$.

If s is a smooth section, then $s(M)$ is an embedded submanifold of V: the injectivity of s and $\mathrm{d}s$ is immediate from $\pi\circ s=\mathrm{id}_M$, while the embedding property follows from the continuity of π. Every fiber V_x of V is a vector space and thus has a distinguished element, the zero vector in V_x, which

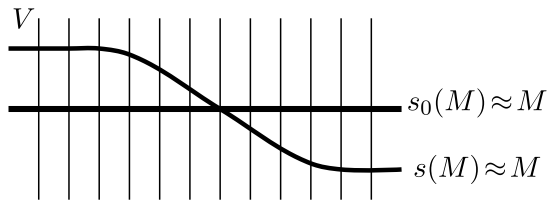


Figure 2.2: The image of a vector-bundle section is an embedded submanifold of the total space.

we denote by 0_x. It follows that every vector bundle admits a canonical section, called the zero section,

$$s_0(x) = (x, 0_x) \in V_x.$$

This section is smooth, since on a trivialization of V over an open subset U of M it is given by the inclusion of U as $U\times 0$ into $U\times\mathbb{R}^k$ or $U\times\mathbb{C}^k$. Thus, M can be thought of as sitting inside of V as the zero section, which is a deformation retract of V; see Figure 2.2.

If $s\colon M\longrightarrow V$ is a section of a vector bundle $V\longrightarrow M$ and $h\colon V|_U\longrightarrow U\times\mathbb{R}^k$ is a trivialization of V over an open subset $U\subset M$, then

$$h\circ s = (\mathrm{id}_U, s_h)\colon U \longrightarrow U\times\mathbb{R}^k \tag{8.1}$$

for some $s_h\colon U\longrightarrow\mathbb{R}^k$. Since the trivializations h cover V and each trivialization h is a diffeomorphism, a section $s\colon M\longrightarrow V$ is smooth if and only if the induced functions $s_h\colon U\longrightarrow\mathbb{R}^k$ are smooth in all trivializations $h\colon V|_U\longrightarrow U\times\mathbb{R}^k$ of V.

Every trivialization $h: V|_U \longrightarrow U\times\mathbb{R}^k$ of a vector bundle $V\longrightarrow M$ over an open subset $U\subset M$ corresponds to a k-tuple $(s_1,\ldots,s_k)$ of smooth sections of V over U such that the set $\{s_i(x)\}_i$ forms a basis for $V_x\equiv\pi^{-1}(x)$ for all $x\in U$. Let $e_1,\ldots,e_k$ be the standard coordinate vectors in $\mathbb{R}^k$. If $h\colon V|_U\longrightarrow U\times\mathbb{R}^k$ is a trivialization of V, then each section

$$s_i = h^{-1}\circ(\mathrm{id}_U, e_i)\colon U \longrightarrow V|_U, \qquad s_i(x) = h^{-1}(x, e_i),$$

is smooth. Since $\{e_i\}$ is a basis for $\mathbb{R}^k$ and $h\colon V_x\longrightarrow x\times\mathbb{R}^k$ is a vector-space isomorphism, $\{s_i(x)\}_i$ is a basis for V_x for all $x\in U$. Conversely, if $s_1,\ldots,s_k: U\longrightarrow V|_U$ are smooth sections such that $\{s_i(x)\}_i$ is a basis for V_x for all $x\in U$, then the map

$$\psi\colon U\times\mathbb{R}^k \longrightarrow V|_U, \qquad (x, c_1,\ldots,c_k) \longrightarrow c_1s_1(x)+\ldots+c_ks_k(x), \tag{8.2}$$

is a diffeomorphism commuting with the projection maps; its inverse, $h=\psi^{-1}$, is thus a trivialization of V over U. If in addition $s\colon M\longrightarrow V$ is any bundle section and

$$s_h \equiv \big(s_{h,1},\ldots,s_{h,k}\big)\colon U \longrightarrow \mathbb{R}^k$$

is as in (8.1), then

$$s(x) = h^{-1}\big(x, s_{h,1}(x),\ldots,s_{h,k}(x)\big) = s_{h,1}(x)s_1(x)+\ldots+s_{h,k}(x)s_k(x) \quad \forall\, x\in U.$$

Thus, a bundle section $s: M\longrightarrow V$ is smooth if and only if for every open subset $U\subset M$ and a k-tuple of smooth sections $s_1,\ldots,s_k: U\longrightarrow V|_U$ such that $\{s_i(x)\}_i$ is a basis for V_x for all $x\in U$ the coefficient functions

$$c_1,\ldots,c_k\colon U \longrightarrow \mathbb{R}, \qquad s(x) \equiv c_1(x)s_1(x)+\ldots+c_k(x)s_k(x) \quad \forall\, x\in U,$$

are smooth.

For example, let $\pi\colon V=TM\longrightarrow M$ be the tangent bundle of a smooth m-manifold M. If $\tilde\varphi_\alpha$ is a trivialization of TM over $U_\alpha\subset M$ as in (7.3),

$$s_i(x)\equiv\tilde\varphi_\alpha^{-1}(x,e_i)=\frac{\partial}{\partial x_i}\bigg|_x\qquad\forall\,x\in U_\alpha$$

is the i-th coordinate vector field. Thus, a vector field $X\colon M\longrightarrow TM$ is smooth if and only if for every smooth chart $\varphi_\alpha=(x_1,\ldots,x_m)\colon U_\alpha\longrightarrow\mathbb{R}^m$ the coefficient functions

$$c_1,\ldots,c_m\colon U\longrightarrow\mathbb{R},\qquad X(p)\equiv c_1(p)\frac{\partial}{\partial x_1}\bigg|_p+\ldots+c_m(p)\frac{\partial}{\partial x_m}\bigg|_p\quad\forall\,p\in U,$$

are smooth. If $X\colon M\longrightarrow TM$ is a vector field on M and $p\in M$, sometimes it will be convenient to denote the value $X(p)\in T_pM$ of X at p by X_p. If in addition $f\in C^\infty(M)$, define

$$Xf\colon M\longrightarrow\mathbb{R}\qquad\text{by}\qquad\{Xf\}(p)=X_p(f)\qquad\forall\,p\in M.$$

A vector field X on M is smooth if and only if $Xf\in C^\infty(M)$ for every $f\in C^\infty(M)$.

The set of all smooth sections of a vector bundle $\pi\colon V\longrightarrow M$ is denoted by $\Gamma(M;V)$. This is naturally a module over the ring $C^\infty(M)$ of smooth functions on M, since $fs\in\Gamma(M;V)$ whenever $f\in C^\infty(M)$ and $s\in\Gamma(M;V)$. We will denote the set $\Gamma(M;TM)$ of smooth vector fields on M by $\mathrm{VF}(M)$. It carries a canonical structure of Lie algebra over $\mathbb{R}$, with the Lie bracket defined by

$$\begin{gathered}[\cdot,\cdot]\colon\mathrm{VF}(M)\times\mathrm{VF}(M)\longrightarrow\mathrm{VF}(M),\\ [X,Y]_p(f)=X_p(Yf)-Y_p(Xf)\qquad\forall\,p\in M,\ f\in C^\infty(U),\ U\subset M\text{ open},\ p\in U;\end{gathered}\tag{8.3}$$

see Exercise 5.

Definition 8.2. (1) *Suppose $\pi\colon V\longrightarrow M$ and $\pi'\colon V'\longrightarrow N$ are real (or complex) vector bundles. A (smooth) map $\tilde f\colon V\longrightarrow V'$ is a (smooth) vector-bundle homomorphism if $\tilde f$ descends to a map $f\colon M\longrightarrow N$, i.e. the diagram*

$$\begin{array}{ccc}V&\xrightarrow{\ \tilde f\ }&V'\\ {\scriptstyle\pi}\big\downarrow&&\big\downarrow{\scriptstyle\pi'}\\ M&\xrightarrow{\ f\ }&N\end{array}\tag{8.4}$$

commutes, and the restriction $\tilde f\colon V_x\longrightarrow V_{f(x)}$ is linear (or $\mathbb{C}$-linear, respectively) for all $x\in M$.

(2) *If $\pi\colon V\longrightarrow M$ and $\pi'\colon V'\longrightarrow M$ are vector bundles, a smooth vector-bundle homomorphism $\tilde f\colon V\longrightarrow V'$ is an isomorphism of vector bundles if $\pi'\circ\tilde f=\pi$, i.e. the diagram*

$$\begin{array}{ccc}V&\xrightarrow{\qquad\tilde f\qquad}&V'\\ &{\scriptstyle\pi}\searrow\quad\swarrow{\scriptstyle\pi'}&\\ &M&\end{array}\tag{8.5}$$

commutes, and $\tilde f$ is a diffeomorphism (or equivalently, its restriction to each fiber is an isomorphism of vector spaces). If such an isomorphism exists, then V and V' are said to be isomorphic vector bundles.

Let $\tilde{f}\colon V \longrightarrow V'$ be a vector-bundle homomorphism between vector bundles over the same space M that covers id_M as in (8.5). If

$$h\colon V|_U \longrightarrow U\times\mathbb{R}^k \qquad \text{and} \qquad h'\colon V'|_U \longrightarrow U\times\mathbb{R}^{k'}$$

are trivializations of V and V' over the same open subset $U\subset M$, then there exists

$$\tilde{f}_{h'h}\colon U \longrightarrow \mathrm{Mat}_{k'\times k}\mathbb{R} \qquad \text{s.t.} \qquad h'\circ\tilde{f}\circ h^{-1}(x,v) = \big(x, \tilde{f}_{h'h}(x)v\big) \quad \forall\, x\in U,\, v\in\mathbb{R}^k. \tag{8.6}$$

Since the trivializations h and h' are diffeomorphisms that cover V and V', respectively, a vector-bundle homomorphism as in (8.5) is smooth if and only if the induced function

$$\tilde{f}_{hh'}\colon U \longrightarrow \mathrm{Mat}_{k'\times k}\mathbb{R}$$

is smooth for every pair, $h\colon V|_U \longrightarrow U\times\mathbb{R}^k$ and $h'\colon V'|_U \longrightarrow U\times\mathbb{R}^{k'}$, of trivializations of V and V' over U.

Example 8.3. The tangent bundle $\pi\colon T\mathbb{R}^n \longrightarrow \mathbb{R}^n$ of $\mathbb{R}^n$ is canonically trivial. The map

$$T\mathbb{R}^n \longrightarrow \mathbb{R}^n\times\mathbb{R}^n, \qquad v \longrightarrow \big(\pi(v); v(\pi_1),\ldots,v(\pi_n)\big),$$

where $\pi_i\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ are the component projection maps, is a vector-bundle isomorphism.

Lemma 8.4. *The real line bundle $V \longrightarrow S^1$ given by the infinite Mobius band of Example 7.3 is not isomorphic to the trivial line bundle $S^1\times\mathbb{R} \longrightarrow S^1$.*

Proof. In fact, (V,S^1) is not even homeomorphic to $(S^1\times\mathbb{R}, S^1)$. Since

$$S^1\times\mathbb{R} - s_0(S^1) \equiv S^1\times\mathbb{R} - S^1\times 0 = S^1\times\mathbb{R}^- \sqcup S^1\times\mathbb{R}^+,$$

the space $S^1\times\mathbb{R} - S^1$ is not connected. On the other hand, $V - s_0(S^1)$ is connected. If MB is the standard Mobius Band and $S^1\subset MB$ is the central circle, $MB - S^1$ is a deformation retract of $V - S^1$. On the other hand, the boundary of MB has only one connected component (this is the primary feature of MB) and is a deformation retract of $MB - S^1$. Thus, $V - S^1$ is connected as well. □

Lemma 8.5. *If $\pi\colon V \longrightarrow M$ is a real (or complex) vector bundle of rank k, V is isomorphic to the trivial real (or complex) vector bundle of rank k over M if and only if V admits k sections $s_1,\ldots,s_k$ such that the vectors $s_1(x),\ldots,s_k(x)$ are linearly independent over $\mathbb{R}$ (or over $\mathbb{C}$, respectively) in V_x for all $x\in M$.*

Proof. We consider the real case; the proof in the complex case is nearly identical.
(1) Suppose $\psi\colon M\times\mathbb{R}^k \longrightarrow V$ is an isomorphism of vector bundles over M. Let $e_1,\ldots,e_k$ be the standard coordinate vectors in $\mathbb{R}^k$. Define sections $s_1,\ldots,s_k$ of V over M by

$$s_i(x) = \psi\big(x, e_i\big) \qquad \forall\, i = 1,\ldots,k,\ x\in M.$$

Since the maps $x \longrightarrow (x,e_i)$ are sections of $M\times\mathbb{R}^k$ over M and ψ is a bundle homomorphism, the maps s_i are sections of V. Since the vectors (x,e_i) are linearly independent in $x\times\mathbb{R}^k$ and ψ is an isomorphism on every fiber, the vectors $s_1(x),\ldots,s_k(x)$ are linearly independent in V_x for all $x\in M$, as needed.

(2) Suppose $s_1,\ldots,s_k$ are sections of V such that the vectors $s_1(x),\ldots,s_k(x)$ are linearly independent in V_x for all $x\in M$. Define the map

$$\psi\colon M\times\mathbb{R}^k \longrightarrow V \qquad \text{by} \qquad \psi(x,c_1,\ldots,c_k)=c_1s_1(x)+\ldots+c_ks_k(x)\in V_x.$$

Since the sections $s_1,\ldots,s_k$ and the vector space operations on V are smooth, the map h is smooth. It is immediate that $\pi(\psi(x,c))=x$ and the restriction of ψ to $x\times\mathbb{R}^k$ is linear; thus, ψ is a vector-bundle homomorphism. Since the vectors $s_1(x),\ldots,s_k(x)$ are linearly independent in V_x, the homomorphism ψ is injective and thus an isomorphism on every fiber. We conclude that ψ is an isomorphism between vector bundles over M. □

9 Transition Data

Suppose $\pi\colon V\longrightarrow M$ is a real vector bundle of rank k. By Definition 7.1, there exists a collection $\{(U_\alpha,h_\alpha)\}_{\alpha\in\mathcal{A}}$ of trivializations for V such that $\bigcup_{\alpha\in\mathcal{A}}U_\alpha=M$. Since (U_α,h_α) is a trivialization for V,

$$h_\alpha\colon V|_{U_\alpha}\longrightarrow U_\alpha\times\mathbb{R}^k$$

is a diffeomorphism such that $\pi_1\circ h_\alpha=\pi$ and the restriction $h_\alpha\colon V_x\longrightarrow x\times\mathbb{R}^k$ is linear for all $x\in U_\alpha$. Thus, for all $\alpha,\beta\in\mathcal{A}$,

$$h_{\alpha\beta}\equiv h_\alpha\circ h_\beta^{-1}\colon \big(U_\alpha\cap U_\beta\big)\times\mathbb{R}^k\longrightarrow \big(U_\alpha\cap U_\beta\big)\times\mathbb{R}^k$$

is a diffeomorphism such that $\pi_1\circ h_{\alpha\beta}=\pi_1$, i.e. $h_{\alpha\beta}$ maps $x\times\mathbb{R}^k$ to $x\times\mathbb{R}^k$, and the restriction of $h_{\alpha\beta}$ to $x\times\mathbb{R}^k$ defines an isomorphism of $x\times\mathbb{R}^k$ with itself. Such a diffeomorphism must be given by

$$(x,v)\longrightarrow \big(x,g_{\alpha\beta}(x)v\big) \qquad \forall\, v\in\mathbb{R}^k,$$

for a unique element $g_{\alpha\beta}(x)\in \mathrm{GL}_k\mathbb{R}$ (the general linear group of $\mathbb{R}^k$). The map $h_{\alpha\beta}$ is then given by

$$h_{\alpha\beta}(x,v)=\big(x,g_{\alpha\beta}(x)v\big) \qquad \forall\, x\in U_\alpha\cap U_\beta,\ v\in\mathbb{R}^k,$$

and is completely determined by the map $g_{\alpha\beta}\colon U_\alpha\cap U_\beta\longrightarrow \mathrm{GL}_k\mathbb{R}$ (and $g_{\alpha\beta}$ is determined by $h_{\alpha\beta}$). Since $h_{\alpha\beta}$ is smooth, so is $g_{\alpha\beta}$.

Example 9.1. Let $\pi\colon V\longrightarrow S^1$ be the Mobius band line bundle of Example 7.3. If $\{(U_\pm,h_\pm)\}$ is the pair of trivializations described in Example 7.3, then

$$h_-\circ h_+^{-1}\colon U_+\cap U_-\times\mathbb{R}\longrightarrow U_+\cap U_-\times\mathbb{R}, \quad (p,v)\longrightarrow \big(p,g_{-+}(p)v\big)=\begin{cases}(p,v), & \text{if } \mathrm{Im}\,p<0,\\ (p,-v), & \text{if } \mathrm{Im}\,p>0,\end{cases}$$

$$\text{where} \qquad g_{-+}\colon U_+\cap U_-=S^1-\{\pm1\}\longrightarrow \mathrm{GL}_1\mathbb{R}=\mathbb{R}^*, \quad g_{-+}(p)=\begin{cases}-1, & \text{if } \mathrm{Im}\,p>0;\\ 1, & \text{if } \mathrm{Im}\,p<0.\end{cases}$$

In this case, the transition maps $g_{\alpha\beta}$ are locally constant, which is rarely the case.

Suppose $\{(U_\alpha,h_\alpha)\}_{\alpha\in\mathcal{A}}$ is a collection of trivializations of a rank k vector bundle $\pi:V\longrightarrow M$ covering M. Any (smooth) section $s:M\longrightarrow V$ of π determines a collection of (smooth) maps $\{s_\alpha\colon U_\alpha\longrightarrow\mathbb{R}^k\}_{\alpha\in\mathcal{A}}$ such that

$$h_\alpha\circ s(x)=\big(x,s_\alpha(x)\big) \quad \forall\, x\in U_\alpha \quad \Longrightarrow \quad s_\alpha(x)=g_{\alpha\beta}(x)s_\beta(x) \quad \forall\, x\in U_\alpha\cap U_\beta\,,\alpha,\beta\in\mathcal{A}, \tag{9.1}$$

where $\{g_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{A}}$ is the transition data for the collection of trivializations $\{h_\alpha\}_{\alpha\in\mathcal{A}}$ of V. Conversely, a collection of (smooth) maps $\{s_\alpha\colon U_\alpha\longrightarrow\mathbb{R}^k\}_{\alpha\in\mathcal{A}}$ satisfying the second condition in (9.1) induces a well-defined (smooth) section of π via the first equation in (9.1). Similarly, suppose $\{(U_\alpha,h'_\alpha)\}_{\alpha\in\mathcal{A}}$ is a collection of trivializations of a rank k' vector bundle $\pi'\colon V'\longrightarrow M$ covering M. A (smooth) vector-bundle homomorphism $\tilde{f}: V\longrightarrow V'$ covering id_M as in (8.5) determines a collection of (smooth) maps

$$\{\tilde{f}_\alpha\colon U_\alpha\longrightarrow\mathrm{Mat}_{k'\times k}\mathbb{R}\}_{\alpha\in\mathcal{A}} \quad \text{s.t.} \quad h'_\alpha\circ\tilde{f}\circ h_\alpha^{-1}(x,v)=\big(x,\tilde{f}_\alpha(x)v\big) \quad \forall\,(x,v)\in U_\alpha\times\mathbb{R}^k \tag{9.2}$$

$$\Longrightarrow \qquad \tilde{f}_\alpha(x)g_{\alpha\beta}(x)=g'_{\alpha\beta}(x)\tilde{f}_\beta(x) \quad x\in U_\alpha\cap U_\beta\,,\alpha,\beta\in\mathcal{A}, \tag{9.3}$$

where $\{g'_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{A}}$ is the transition data for the collection of trivializations $\{h'_\alpha\}_{\alpha\in\mathcal{A}}$ of V'. Conversely, a collection of (smooth) maps as in (9.2) satisfying (9.3) induces a well-defined (smooth) vector-bundle homomorphism $\tilde{f}\colon V\longrightarrow V'$ covering id_M as in (8.5) via the equation in (9.2).

By the above, starting with a real rank k vector bundle $\pi\colon V\longrightarrow M$, we can obtain an open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of M and a collection of smooth transition maps

$$\big\{g_{\alpha\beta}\colon U_\alpha\cap U_\beta\longrightarrow\mathrm{GL}_k\mathbb{R}\big\}_{\alpha,\beta\in\mathcal{A}}.$$

These transition maps satisfy:

(VBT1) $g_{\alpha\alpha}\equiv\mathbb{I}_k$, since $h_{\alpha\alpha}\equiv h_\alpha\circ h_\alpha^{-1}=\mathrm{id}$;

(VBT2) $g_{\alpha\beta}g_{\beta\alpha}\equiv\mathbb{I}_k$, since $h_{\alpha\beta}\circ h_{\beta\alpha}=\mathrm{id}$;

(VBT3) $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}\equiv\mathbb{I}_k$, since $h_{\alpha\beta}\circ h_{\beta\gamma}\circ h_{\gamma\alpha}=\mathrm{id}$.

The last condition is known as the (Čech) cocycle condition (more details in Chapter 5 of Warner). It is sometimes written as

$$g_{\alpha_1\alpha_2}g_{\alpha_0\alpha_2}^{-1}g_{\alpha_0\alpha_1}\equiv\mathbb{I}_k \qquad \forall\,\alpha_0,\alpha_1,\alpha_2\in\mathcal{A}.$$

In light of (VBT2), the two versions of the cocycle condition are equivalent.

Conversely, given an open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of M and a collection of smooth maps

$$\big\{g_{\alpha\beta}\colon U_\alpha\cap U_\beta\longrightarrow\mathrm{GL}_k\mathbb{R}\big\}_{\alpha,\beta\in\mathcal{A}}$$

that satisfy (VBT1)-(VBT3), we can assemble a rank k vector bundle $\pi'\colon V'\longrightarrow M$ as follows. Let

$$V'=\Big(\bigsqcup_{\alpha\in\mathcal{A}}\alpha\times U_\alpha\times\mathbb{R}^k\Big)\Big/\sim, \qquad \text{where}$$
$$(\beta,x,v)\sim\big(\alpha,x,g_{\alpha\beta}(x)v\big) \quad \forall\;\alpha,\beta\in\mathcal{A},\;x\in U_\alpha\cap U_\beta,\;v\in\mathbb{R}^k.$$

The relation $\sim$ is reflexive by (VBT1), symmetric by (VBT2), and transitive by (VBT3) and (VBT2). Thus, $\sim$ is an equivalence relation, and V' carries the quotient topology. Let

$$q\colon\bigsqcup_{\alpha\in\mathcal{A}}\alpha\times U_\alpha\times\mathbb{R}^k\longrightarrow V' \qquad \text{and} \qquad \pi'\colon V'\longrightarrow M, \quad [\alpha,x,v]\longrightarrow x,$$

be the quotient map and the natural projection map (which is well-defined). If $\beta\in\mathcal{A}$ and W is a subset of $U_\beta\times\mathbb{R}^k$, then

$$q^{-1}\big(q(\beta\times W)\big)=\bigsqcup_{\alpha\in\mathcal{A}}\alpha\times h_{\alpha\beta}(W),\qquad\text{where}$$
$$h_{\alpha\beta}\colon\big(U_\alpha\cap U_\beta\big)\times\mathbb{R}^k\longrightarrow\big(U_\alpha\cap U_\beta\big)\times\mathbb{R}^k,\qquad h_{\alpha\beta}(x,v)=\big(x,g_{\alpha\beta}(x)v\big).$$

In particular, if $\beta\times W$ is an open subset of $\beta\times U_\beta\times\mathbb{R}^k$, then $q^{-1}\big(q(\beta\times W)\big)$ is an open subset of $\bigsqcup_{\alpha\in\mathcal{A}}\alpha\times U_\alpha\times\mathbb{R}^k$. Thus, q is an open continuous map. Since its restriction

$$q_\alpha\equiv q|_{\alpha\times U_\alpha\times\mathbb{R}^k}$$

is injective, $(q_\alpha(\alpha\times U_\alpha\times\mathbb{R}^k),q_\alpha^{-1})$ is a smooth chart on V' in the sense of Lemma 2.6. The overlap maps between these charts are the maps $h_{\alpha\beta}$ and thus smooth.[1] Thus, by Lemma 2.6, these charts induce a smooth structure on V'. The projection map $\pi'\colon V'\longrightarrow M$ is smooth with respect to this smooth structure, since it induces projection maps on the charts. Since

$$\pi_1=\pi'\circ q_\alpha\colon\alpha\times U_\alpha\times\mathbb{R}^k\longrightarrow U_\alpha\subset M,$$

the diffeomorphism q_α induces a vector-space structure in V'_x for each $x\in U_\alpha$ such that the restriction of q_α to each fiber is a vector-space isomorphism. Since the restriction of the overlap map $h_{\alpha\beta}$ to $x\times\mathbb{R}^k$, with $x\in U_\alpha\cap U_\beta$, is a vector-space isomorphism, the vector space structures defined on V'_x via the maps q_α and q_β are the same. We conclude that $\pi'\colon V'\longrightarrow M$ is a real vector bundle of rank k.

If $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ and $\big\{g_{\alpha\beta}:U_\alpha\cap U_\beta\longrightarrow \mathrm{GL}_k\mathbb{R}\big\}_{\alpha,\beta\in\mathcal{A}}$ are transition data arising from a vector bundle $\pi\colon V\longrightarrow M$, then the vector bundle V' constructed in the previous paragraph is isomorphic to V. Let $\{(U_\alpha,h_\alpha)\}$ be the trivializations as above, giving rise to the transition functions $g_{\alpha\beta}$. We define

$$\tilde f\colon V\longrightarrow V'\qquad\text{by}\qquad\tilde f(v)=\big[\alpha,h_\alpha(v)\big]\quad\text{if}\ \ \pi(v)\in U_\alpha.$$

If $\pi(v)\in U_\alpha\cap U_\beta$, then

$$\big[\beta,h_\beta(v)\big]=\big[\alpha,h_{\alpha\beta}(h_\beta(v))\big]=\big[\alpha,h_\alpha(v)\big]\in V',$$

i.e. the map $\tilde f$ is well-defined (depends only on v and not on α). It is immediate that $\pi'\circ\tilde f=\pi$. Since the map

$$q_\alpha^{-1}\circ\tilde f\circ h_\alpha^{-1}\colon U_\alpha\times\mathbb{R}^k\longrightarrow\alpha\times U_\alpha\times\mathbb{R}^k$$

is the identity (and thus smooth), $\tilde f$ is a smooth map. Since the restrictions of q_α and h_α to every fiber are vector-space isomorphisms, it follows that so is $\tilde f$. We conclude that $\tilde f$ is a vector-bundle isomorphism.

In summary, a real rank k vector bundle over M determines a set of transition data with values in $\mathrm{GL}_k\mathbb{R}$ satisfying (VBT1)-(VBT3) above (many such sets, of course) and a set of transition data satisfying (VBT1)-(VBT3) determines a real rank-k vector bundle over M. These two processes are well-defined and are inverses of each other when applied to the set of equivalence classes of vector bundles and the set of equivalence classes of transition data satisfying (VBT1)-(VBT3).

[1] Formally, the overlap map is $(\beta\longrightarrow\alpha)\times h_{\alpha\beta}$.

Two vector bundles over M are defined to be equivalent if they are isomorphic as vector bundles over M. Two sets of transition data

$$\{g_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{A}} \qquad \text{and} \qquad \{g'_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{A}},$$

with $\mathcal{A}$ consisting of *all* sufficiently small open subsets of M, are said to be equivalent if there exists a collection of smooth functions $\{f_\alpha: U_\alpha \longrightarrow \mathrm{GL}_k\mathbb{R}\}_{\alpha\in\mathcal{A}}$ such that

$$g'_{\alpha\beta} = f_\alpha g_{\alpha\beta} f_\beta^{-1}, \qquad \forall \alpha, \beta \in \mathcal{A},[2]$$

i.e. the two sets of transition data differ by the action of a Čech 0-chain (more in Chapter 5 of Warner). Along with the cocycle condition on the gluing data, this means that isomorphism classes of real rank k vector bundles over M can be identified with $\check{H}^1(M;\mathrm{GL}_k\mathbb{R})$, the quotient of the space of Čech cocycles of degree one by the subspace of Čech boundaries.

Remark 9.2. In Chapter 5 of Warner, Čech cohomology groups, $\check{H}^m$, are defined for (sheafs of) abelian groups. However, the first two groups, $\check{H}^0$ and $\check{H}^1$, generalize to non-abelian groups as well.

If $\pi: V \longrightarrow M$ is a complex rank k vector bundle over M, we can similarly obtain transition data for V consisting of an open cover $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of M and a collection of smooth maps

$$\{g_{\alpha\beta}: U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}_k\mathbb{C}\}_{\alpha,\beta\in\mathcal{A}}$$

that satisfies (VBT1)-(VBT3). Conversely, given such transition data, we can construct a complex rank k vector bundle over M. The set of isomorphism classes of complex rank k vector bundles over M can be identified with $\check{H}^1(M;\mathrm{GL}_k\mathbb{C})$.

10 Restrictions and Pullbacks

If N is a smooth manifold, $M \subset N$ is an embedded submanifold, and $\pi: V \longrightarrow N$ is a vector bundle of rank k (real or complex) over N, then its restriction to M,

$$\pi: V|_M \equiv \pi^{-1}(M) \longrightarrow M,$$

is a vector bundle of rank k over M. It inherits a smooth structure from V by the *Slice Lemma* (Proposition 5.3) or the *Implicit Function Theorem for Manifolds* (Theorem 6.3). If $\{(U_\alpha, h_\alpha)\}$ is a collection of trivializations for $V \longrightarrow N$, then $\{(M \cap U_\alpha, h_\alpha|_{\pi^{-1}(M\cap U_\alpha)})\}$ is a collection of trivializations for $V|_M \longrightarrow M$. Similarly, if $\{g_{\alpha\beta}\}$ is transition data for $V \longrightarrow N$, then $\{g_{\alpha\beta}|_{M\cap U_\alpha\cap U_\beta}\}$ is transition data for $V|_M \longrightarrow M$.

If $f: M \longrightarrow N$ is a smooth map and $\pi: V \longrightarrow N$ is a vector bundle of rank k, there is a pullback bundle over M:

$$f^*V \equiv M \times_N V \equiv \{(p,v) \in M \times V : f(p) = \pi(v)\} \xrightarrow{\pi_1} M. \tag{10.1}$$

[2] According to the discussion around (9.3), such a collection $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ corresponds, via trivializations, to an isomorphism between the vector bundles determined by $\{g_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{A}}$ and $\{g'_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{A}}$.

Note that f^*V is the maximal subspace of $M\times V$ so that the diagram

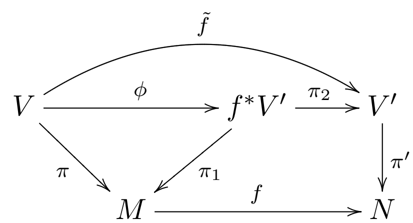
$$\begin{array}{ccc} f^*V & \xrightarrow{\pi_2} & V \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{f} & N \end{array}$$

commutes. By the *Implicit Function Theorem for Maps* (Corollary 6.7), f^*V is a smooth submanifold of $M\times V$. By construction, the fiber of π_1 over $p\in M$ is $p\times V_{f(p)}\subset M\times V$, i.e. the fiber of π over $f(p)\in N$:

$$\big(f^*V\big)_p = p\times V_{f(p)} \qquad \forall\, p\in M. \tag{10.2}$$

If $\{(U_\alpha,h_\alpha)\}$ is a collection of trivializations for $V\longrightarrow N$, then $\{(f^{-1}(U_\alpha),h_\alpha\circ f)\}$ is a collection of trivializations for $f^*V\longrightarrow M$. Similarly, if $\{g_{\alpha\beta}\}$ is transition data for $V\longrightarrow N$, then $\{g_{\alpha\beta}\circ f\}$ is transition data for $f^*V\longrightarrow M$. The case discussed in the previous paragraph corresponds to f being the inclusion map.

Lemma 10.1. *If $\tilde f\colon V\longrightarrow V'$ is a vector-bundle homomorphism covering a smooth map $f\colon M\longrightarrow N$ as in (8.4), there exists a bundle homomorphism $\phi\colon V\longrightarrow f^*V'$ so that the diagram*



commutes.

Proof. The map ϕ is defined by

$$\phi\colon V\longrightarrow M\times V', \qquad \phi(v)=\big(\pi(v),\tilde f(v)\big).$$

Since $f\circ\pi=\pi'\circ\tilde f$,

$$\phi(v)\in f^*V'\equiv M\times_N V'\equiv\big\{(p,v')\in M\times V' : f(p)=\pi'(v')\big\}.$$

Since $f^*V'\subset M\times V'$ is a smooth embedded submanifold, the map $\phi: V\longrightarrow f^*V'$ obtained by restricting the range is smooth; see Proposition 5.5. The above diagram commutes by the construction of ϕ. Since $\tilde f$ is linear on each fiber of V, so is ϕ. $\square$

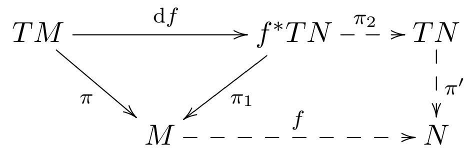
If $f: M\longrightarrow N$ is a smooth map, then $\mathrm{d}_pf\colon T_pM\longrightarrow T_{f(p)}N$ is a linear map which varies smoothly with p. It thus gives rises to a smooth map,

$$\mathrm{d}f\colon TM\longrightarrow TN, \qquad v\longrightarrow \mathrm{d}_{\pi(v)}f(v). \tag{10.3}$$

However, this description of $\mathrm{d}f$ gives no indication that $\mathrm{d}f$ maps $v\in T_pM$ to $T_{f(p)}N$ or that this map is linear on each T_pM. One way to fix this defect is to state that (10.3) is a bundle homomorphism covering the map $f\colon M\longrightarrow N$, i.e. that the diagram

$$\begin{array}{ccc} TM & \xrightarrow{\mathrm{d}f} & TN \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ M & \xrightarrow{f} & N \end{array} \tag{10.4}$$

commutes. By Lemma 10.1, $\mathrm{d}f$ then induces a vector-bundle homomorphism from TM to f^*TN so that the diagram



(10.5)

commutes. The triangular part of (10.5) is generally the preferred way of describing $\mathrm{d}f$. The description (10.4) factors through the triangular part of (10.5), as indicated by the dashed arrows. The triangular part of (10.5) also leads to a more precise statement of the *Implicit Function Theorem*, which is rather useful in topology of manifolds; see Theorem 11.11 below.

If $\pi\colon V \longrightarrow N$ is a smooth vector bundle, $f\colon M \longrightarrow N$ is a smooth map, and $s\colon N \longrightarrow V$ is a bundle section of V, then

$$f^*s\colon M \longrightarrow f^*V, \qquad \{f^*s\}(p) = \big(p, s(f(p))\big) \in f^*V \equiv M\times_N V \subset M\times V,$$

is a bundle section of $f^*V \longrightarrow M$. If s is smooth, then $f^*s\colon M \longrightarrow M\times V$ is a smooth map with the image in $M\times_N V$. Since $M\times_N V \subset M\times V$ is an embedded submanifold, $f^*s\colon M \longrightarrow f^*V$ is a smooth map by Proposition 5.5. Thus, a smooth map $f\colon M \longrightarrow N$ induces a homomorphism of vector spaces

$$f^*\colon \Gamma(N;V) \longrightarrow \Gamma(M;f^*V), \qquad s \longrightarrow f^*s, \tag{10.6}$$

which is also a homomorphism of modules with respect to the ring homomorphism

$$f^*\colon C^\infty(N) \longrightarrow C^\infty(M), \qquad g \longrightarrow g\circ f\,.$$

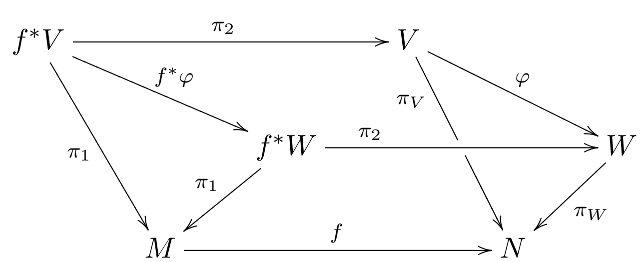
In the case of tangent bundles, the homomorphism (10.6) is compatible with the Lie algebra structures on the spaces of vector fields, as described by the following lemma.

Lemma 10.2. *Let $f: M \longrightarrow N$ be a smooth map. If $X_1, X_2 \in \mathrm{VF}(M)$ and $Y_1, Y_2 \in \mathrm{VF}(N)$ are smooth vector fields on M and N, respectively, such that $\mathrm{d}f(X_i) = f^*Y_i \in \Gamma(M; f^*TN)$ for $i=1,2$, then*

$$\mathrm{d}f\big([X_1,X_2]\big) = f^*[Y_1,Y_2].$$

This is checked directly from the relevant definitions.

The pullback operation on vector bundles also extends to homomorphisms. Let $f: M \longrightarrow N$ be a smooth map and $\pi_V: V \longrightarrow N$ and $\pi_W: W \longrightarrow N$ be vector bundles. Any vector-bundle homomorphism $\varphi\colon V \longrightarrow W$ over N induces a vector-bundle homomorphism $f^*\varphi\colon f^*V \longrightarrow f^*W$ over M so that the diagram



(10.7)

commutes. The vector-bundle homomorphism $f^*\varphi$ is given by

$$(f^*\varphi)_p = \mathrm{id}\times\varphi_{f(p)}\colon (f^*V)_p = p\times V_{f(p)} \longrightarrow (f^*W)_p = p\times W_{f(p)}, \qquad (p,v)\longrightarrow \big(p,\varphi_p(v)\big),$$

where φ_p is the restriction of φ to the fiber $V_{f(p)}=\pi_V^{-1}(f(p))$ over $f(p)\in N$.

11 Subbundles and Quotient Bundles

Definition 11.1. *Let M be a smooth manifold.*

(1) A rank k' subbundle of a vector bundle $\pi\colon V\longrightarrow M$ is a smooth submanifold V' of V such that $\pi|_{V'}\colon V'\longrightarrow M$ is a vector bundle of rank k'.

(2) A rank k distribution on M is a rank k subbundle of $TM\longrightarrow M$.

A subbundle of course cannot have a larger rank than the ambient bundle; so $\mathrm{rk}\,V'\le\mathrm{rk}\,V$ in Definition 11.1 and the equality holds if and only if $V'=V$. By Exercise 17, the requirement that $\pi|_{V'}\colon V'\longrightarrow M$ is a vector bundle of rank k' can be replaced by the condition that $V'_p\equiv V_p\cap V'$ is a k'-dimensional linear subspace of V_p for all $p\in M$.

If $f: M\longrightarrow N$ is an immersion, the bundle homomorphism $\mathrm{d}f$ as in (10.5) is injective and the image of $\mathrm{d}f$ in f^*TN is a subbundle of f^*TN. In the case $M\subset N$ is an embedded submanifold and f is the inclusion map, we identify TM with the image of $\mathrm{d}\iota$ in $f^*TN=TN|_M$. By Lemma 10.2, if $Y_1,Y_2\in\mathrm{VF}(N)$ are smooth vector fields on N, then

$$Y_1\big|_M, Y_2\big|_M\in\mathrm{VF}(M)\subset\Gamma(M;TN|_M) \quad\Longrightarrow\quad [Y_1,Y_2]\big|_M\in\mathrm{VF}(M)\subset\Gamma(M;TN|_M).$$

Definition 11.2. *Let N be a smooth manifold.*

(1) A collection $\{\iota_\alpha: M_\alpha\longrightarrow N\}_{\alpha\in\mathcal{A}}$ of injective immersions from m-manifolds is a foliation of N^n if the collection $\{\mathrm{Im}\,\iota_\alpha\}_{\alpha\in\mathcal{A}}$ covers N and for every $q\in N$ there exists a smooth chart $\psi\colon V\longrightarrow\mathbb{R}^m\times\mathbb{R}^{n-m}$ around q such that the image under ι_α of every connected subset $U\subset\iota_\alpha^{-1}(V)$ under ψ is contained in $\psi^{-1}(\mathbb{R}^m\times y)$ for some $y\in\mathbb{R}^{n-m}$ (dependent on U).

(2) A foliation $\{\iota_\alpha: M_\alpha\longrightarrow N\}_{\alpha\in\mathcal{A}}$ of N is proper if ι_α is an embedding and the images of ι_α partition N (their union covers M and any two of them are either disjoint or the same).

Thus, a foliation of N consists of regular immersions that cover N and are regular in a systematic way (all of them correspond to horizontal slices in a single coordinate chart); see Figure 2.3. Since manifolds are second-countable and the subset $\iota_\alpha^{-1}(V)\subset M_\alpha$ in Definition 11.2 is open, $\iota_\alpha(\iota_\alpha^{-1}(V))$ is contained in at most countably many of the horizontal slices $\psi^{-1}(\mathbb{R}^m\times y)$. The images of $\mathrm{d}\iota_\alpha$ in TN determine a rank m distribution $\mathcal{D}$ on N. By Lemma 10.2, if $Y_1,Y_2\in\mathrm{VF}(N)$ are vector fields on N, then

$$Y_1,Y_2\in\Gamma(N;\mathcal{D})\subset\mathrm{VF}(N) \quad\Longrightarrow\quad [Y_1,Y_2]\in\Gamma(N;\mathcal{D})\subset\mathrm{VF}(N). \tag{11.1}$$

Definition 11.3. *Let $\mathcal{D}\subset TN$ be a distribution on a smooth manifold N. An injective immersion $\iota\colon M\longrightarrow N$ is integral for $\mathcal{D}$ if*

$$\mathrm{Im}\,\mathrm{d}_p\iota=\mathcal{D}_{\iota(p)}\subset T_{\iota(p)}N \qquad \forall\,p\in M.$$

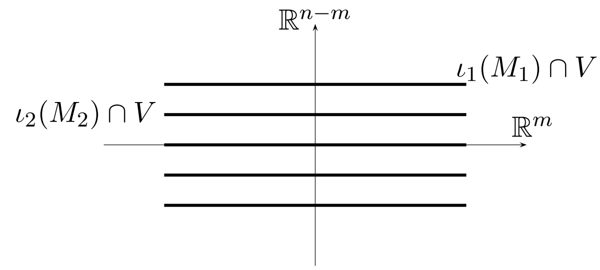


Figure 2.3: A foliation of N in a smooth chart V.

If $\iota\colon M\longrightarrow N$ is an integrable injective immersion for a distribution $\mathcal{D}$ on N, then in particular

$$\dim M = \operatorname{rk}\mathcal{D}.$$

If N admits a foliation $\{\iota_\alpha : M_\alpha \longrightarrow N\}_{\alpha\in\mathcal{A}}$ by injective immersions integral to a distribution $\mathcal{D}$ on N, then $\Gamma(N;\mathcal{D})\subset \mathrm{VF}(N)$ is a Lie subalgebra. By *Frobenius Theorem*, the converse is also true.

Example 11.4. The collection of embeddings

$$\iota_\alpha\colon \mathbb{R}^m \longrightarrow \mathbb{R}^n = \mathbb{R}^m\times\mathbb{R}^{n-m}, \quad \iota_\alpha(x)=(x,\alpha), \qquad \alpha\in\mathbb{R}^{n-m},$$

is a proper foliation of $\mathbb{R}^n$ by m-manifolds. The corresponding distribution $\mathcal{D}\subset T\mathbb{R}^n$ is described by

$$\mathcal{D}=\mathbb{R}^n\times(\mathbb{R}^m\times 0)\subset\mathbb{R}^n\times\mathbb{R}^n=T\mathbb{R}^n.$$

Example 11.5. The collection of embeddings

$$\iota_\alpha\colon S^1\longrightarrow S^{2n+1}\subset\mathbb{C}^{n+1}, \quad \iota_\alpha(\mathrm{e}^{\mathrm{i}\theta})=\mathrm{e}^{\mathrm{i}\theta}\alpha, \qquad \alpha\in S^{2n+1},$$

is a proper foliation of S^{2n+1} by circles. The corresponding distribution $\mathcal{D}\subset TS^{2n+1}$ is described by

$$\mathcal{D}=\big\{(p,\mathrm{i}rp)\colon\ p\in S^{2n+1},\ r\in\mathbb{R}\big\}\subset TS^{2n+1}\subset T\mathbb{C}^{n+1}\big|_{S^{2n+1}}=S^{2n+1}\times\mathbb{C}^{n+1}.$$

The embedded submanifolds of this foliations are the fibers of the quotient projection map

$$\pi\colon S^{2n+1}\longrightarrow S^{2n+1}/S^1=\mathbb{C}P^n$$

of Example 1.10. This is an S^1-bundle over $\mathbb{C}P^n$. In general, the fibers of the projection map $\pi\colon N\longrightarrow B$ of any smooth fiber bundle form a proper foliation of the total space N of the bundle. The corresponding distribution $\mathcal{D}\subset TN$ is then the vertical tangent bundle of π:

$$\mathcal{D}_p=\ker \mathrm{d}_p\pi\subset T_pN \qquad \forall\, p\in N.$$

Example 11.6. Let $\pi : V\longrightarrow M$ be a smooth vector bundle and $\mathcal{D}\subset TV$ the vertical tangent bundle of π as in Example 11.5. For each $p\in M$, let $\iota_p\colon V_p\longrightarrow V$ be the inclusion of the fiber over p and define

$$\tilde\iota\colon \pi^*V\equiv\big\{(v,w)\in V\times V\colon\ \pi(v)=\pi(w)\big\}\longrightarrow TV, \qquad \tilde\iota(v,w)=\mathrm{d}_v\iota_p(w)\equiv\frac{\mathrm{d}}{\mathrm{d}t}(v+tw)\Big|_{t=0}\in T_vV.$$

This map is linear on the fibers of $\pi^*V, TV \longrightarrow V$ (i.e. linear in w above) and injective (since ι_p is an immersion). If $\varphi\colon U \longrightarrow \mathbb{R}^m$ is a smooth chart on M and $(\pi, h_2)\colon V|_U \longrightarrow U \times \mathbb{R}^k$ is a trivialization of V,

$$\begin{aligned}
&\tilde{h}\colon \pi^*V|_{V|_U} \longrightarrow V|_U \times \mathbb{R}^k, && \tilde{h}(v,w) = \big(v, h_2(w)\big),\\
&H\colon TV|_{V|_U} \longrightarrow V|_U \times \mathbb{R}^m \times \mathbb{R}^k, && H(w) = \big(\pi'(w), w(\varphi\circ\pi), w(h_2)\big),
\end{aligned}$$

are trivializations of the vector bundles $\pi^*V \longrightarrow V$ and $\pi'\colon TV \longrightarrow V$. Since

$$H \circ \tilde{\iota} \circ \tilde{h}^{-1}\colon V|_U \times \mathbb{R}^k \longrightarrow V|_U \times \mathbb{R}^m \times \mathbb{R}^k, \qquad (v,w) \longrightarrow (v, 0, w),$$

is a smooth map, it follows that $\tilde{\iota}$ is a smooth injective bundle map over V. Since $\mathrm{d}_v(\tilde{\iota}(v,w)) = 0$ for all $(v,w) \in \pi^*V$, $\mathrm{Im}\,\tilde{\iota} \subset \mathcal{D}$. Since π^*V and $\mathcal{D}$ are vector bundles over π^*V of the same rank k, $\tilde{\iota}\colon \pi^*V \longrightarrow \mathcal{D}$ is an isomorphism of vector bundles over (the total space of) V. In particular, there is a short exact sequence

$$0 \longrightarrow \pi^*V \xrightarrow{\ \tilde{\iota}\ } TV \xrightarrow{\ \mathrm{d}\pi\ } \pi^*TM \longrightarrow 0 \tag{11.2}$$

of vector bundles over V.

Example 11.7. An example of a foliation, which is not proper, is provided by the skew lines on the torus of the same irrational slope η:

$$\iota_\alpha\colon \mathbb{R} \longrightarrow S^1 \times S^1, \quad \iota_\alpha(s) = \big(\alpha \mathrm{e}^{\mathrm{i}s}, \mathrm{e}^{\mathrm{i}\eta s}\big), \qquad \alpha \in S^1 \subset \mathbb{C}.$$

If $\eta \in \mathbb{Q}$, this foliation is proper. In either case, the corresponding distribution $\mathcal{D}$ on $S^1 \times S^1$ is described by

$$\mathcal{D}_{(\mathrm{e}^{\mathrm{i}t_1}, \mathrm{e}^{\mathrm{i}t_2})} = \mathrm{d}_{(t_1,t_2)} q\big(\{(r, \eta r) \in \mathbb{R}^2 = T_{(t_1,t_2)}\mathbb{R}^2 \colon\ r \in \mathbb{R}\}\big),$$

where $q\colon \mathbb{R}^2 \longrightarrow S^1 \times S^1$ the usual covering map.

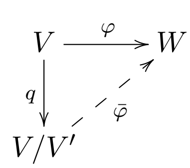
If V is a vector space (over $\mathbb{R}$ or $\mathbb{C}$) and $V' \subset V$ is a linear subspace, then we can form the quotient vector space, V/V'. If W is another vector space, $W' \subset W$ is a linear subspace, and $g\colon V \longrightarrow W$ is a linear map such that $g(V') \subset W'$, then g descends to a linear map between the quotient spaces:

$$\bar{g}\colon V/V' \longrightarrow W/W'.$$

If we choose bases for V and W such that the first few vectors in each basis form bases for V' and W', then the matrix for g with respect to these bases is of the form:

$$g = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.$$

The matrix for $\bar{g}$ is then D. If g is an isomorphism from V to W that restricts to an isomorphism from V' to W', then $\bar{g}$ is an isomorphism from V/V' to W/W'. Any vector-space homomorphism $\varphi\colon V \longrightarrow W$ such that $V' \subset \ker\varphi$ descends to a homomorphism $\bar{\varphi}$ so that the diagram



commutes.

If $V' \subset V$ is a subbundle, we can form a quotient bundle, $V/V' \longrightarrow M$, such that

$$(V/V')_p = V_p/V'_p \qquad \forall\, p \in M.$$

The topology on V/V' is the quotient topology for the natural surjective map $q: V \longrightarrow V/V'$. The vector-bundle structure on V/V' is determined from those of V and V' by requiring that q be a smooth vector-bundle homomorphism. Thus, if s is a smooth section of V, then $q \circ s$ is a smooth section of V/V'; so, there is a homomorphism

$$\Gamma(M;V) \longrightarrow \Gamma(M;V/V'), \qquad s \longrightarrow q \circ s,$$

of $C^\infty(M)$-modules. There is also a short exact sequence[3] of vector bundles over M,

$$0 \longrightarrow V' \longrightarrow V \stackrel{q}{\longrightarrow} V/V' \longrightarrow 0,$$

where the zeros denote the zero vector bundle $M \times 0 \longrightarrow M$. We can choose a system of trivializations $\{(U_\alpha, h_\alpha)\}_{\alpha \in \mathcal{A}}$ of V such that

$$h_\alpha\big(V'|_{U_\alpha}\big) = U_\alpha \times (\mathbb{R}^{k'} \times 0) \subset U_\alpha \times \mathbb{R}^k \qquad \forall\, \alpha \in \mathcal{A}. \tag{11.3}$$

Let $q_{k'}: \mathbb{R}^k \longrightarrow \mathbb{R}^{k-k'}$ be the projection onto the last $(k-k')$ coordinates. The trivializations for V/V' are then given by $\{(U_\alpha, \{\mathrm{id} \times q_{k'}\} \circ h_\alpha)\}$. Alternatively, if $\{g_{\alpha\beta}\}$ is transition data for V such that the upper-left $k' \times k'$-submatrices of $g_{\alpha\beta}$ correspond to V' (as is the case for the above trivializations h_α) and $\bar{g}_{\alpha\beta}$ is the lower-right $(k-k') \times (k-k')$ matrix of $g_{\alpha\beta}$, then $\{\bar{g}_{\alpha\beta}\}$ is transition data for V/V'. Any vector-bundle homomorphism $\varphi: V \longrightarrow W$ over M such that $\varphi(v) = 0$ for all $v \in V'$ descends to a vector-bundle homomorphism $\bar{\varphi}$ so that $\varphi = \bar{\varphi} \circ q$. We leave proofs of the following lemmas as an exercise.

Lemma 11.8. *If $f: M \longrightarrow N$ is a smooth map and $W, W' \longrightarrow N$ are smooth vector bundles,*

$$f^*\big(W/W'\big) \approx (f^*W)/(f^*W')$$

as vector bundles over M.

Lemma 11.9. *Let $V \longrightarrow M$ and $W \longrightarrow N$ be vector bundles over smooth manifolds and $f: M \longrightarrow N$ a smooth map. A vector-bundle homomorphism $\tilde{f}: V \longrightarrow W$ covering f as in (8.4) and vanishing on a subbundle $V' \subset V$ induces a vector-bundle homomorphism*

$$\bar{f}: V/V' \longrightarrow W$$

covering f; this induced homomorphism is smooth if the homomorphism $\tilde{f}$ is smooth.

If $\iota: X \longrightarrow M$ is an immersion, the image of $\mathrm{d}\iota$ in ι^*TM is a subbundle of ι^*TM. In this case, the quotient bundle,

$$\mathcal{N}_M\iota \equiv \iota^*TM \big/ \mathrm{Im}\,\mathrm{d}\iota \longrightarrow X,$$

[3] exact means that at each position the kernel of the outgoing vector-bundle homomorphism equals the image of the incoming one; short means that it consists of five terms with zeros (rank 0 vector bundles) at the ends

is called the normal bundle for the immersion ι. If X is an embedded submanifold and ι is the inclusion map, TX is a subbundle of $\iota^*TM = TM|_X$ and the quotient subbundle,

$$\mathcal{N}_M X \equiv \mathcal{N}_M \iota = \iota^*TM\big/\mathrm{Im}\,\mathrm{d}\iota = TM|_X\big/TX \longrightarrow X,$$

is called the normal bundle of X in M; its rank is the codimension of X in M.

The following lemma provides a geometric way to identify the normal bundle to a submanifold. Its converse is known as the Tubular Neighborhood Theorem; see [3, (12.11)] for the general case and Proposition 16.9 below for the compact case.

Lemma 11.10. *Suppose X is an embedded submanifold of M and $V\longrightarrow X$ is a vector bundle. If there exists a diffeomorphism between neighborhoods W and W' of X in V and in M, respectively,*

$$f\colon W \longrightarrow W' \qquad s.t. \qquad f(p)=p \quad \forall\, p\in X,$$

then V is isomorphic to the normal bundle $\mathcal{N}_M X$ of X in M. If in addition, M is a complex manifold, X is a complex submanifold, $V\longrightarrow X$ is a complex vector bundle, and the linear map

$$\mathrm{d}_p f\colon T_pV/T_pX \longrightarrow T_pM/T_pX$$

is $\mathbb{C}$-linear for all $p\in X$ (as is the case if f is a holomorphic map between complex manifolds), then V and $\mathcal{N}_M X$ are isomorphic as complex vector bundles.

Proof. The bundle map $\tilde{\iota}$ of Example 11.6 induces an isomorphism

$$V \longrightarrow \mathcal{N}_X V \equiv TV|_X\big/TX$$

of (complex) vector bundles over X; so, it is sufficient to show that $\mathcal{N}_X V, \mathcal{N}_X M \longrightarrow X$ are isomorphic vector bundles. If f is a diffeomorphism as above, the differential

$$\mathrm{d}f|_X\colon TV|_X \longrightarrow TM|_X$$

is an isomorphism that restricts to the identity on TX. Thus, $\mathrm{d}f|_X$ induces an isomorphism

$$TV|_X/TX \longrightarrow TM|_X\big/TX = \mathcal{N}_M X \tag{11.4}$$

of vector bundles over X. If V, TM, and TX are complex bundles and $\mathrm{d}f|_X$ is $\mathbb{C}$-linear, then the bundle isomorphism between the quotient bundles above is also $\mathbb{C}$-linear. Combining (11.4) with the first isomorphism, we obtain the lemma. □

If $f : M \longrightarrow N$ is a smooth map and $X \subset M$ is an embedded submanifold, the vector-bundle homomorphism $\mathrm{d}f$ in (10.5) restricts (pulls back by the inclusion map) to a vector-bundle homomorphism

$$\mathrm{d}f|_X\colon TM|_X \longrightarrow (f^*TN)\big|_X$$

over X, which can be composed with the inclusion homomorphism $TX\longrightarrow TM|_X$,

$$TX \longrightarrow TM|_X \stackrel{\mathrm{d}f|_X}{\longrightarrow} (f^*TN)\big|_X\,.$$

If in addition $Y\subset N$ is an embedded submanifold and $f(X)\subset Y$, the above sequence can be composed with the f^*-pullback of the projection homomorphism $q\colon TN|_Y\longrightarrow\mathcal{N}_NY$,

$$TX\longrightarrow TM|_X\stackrel{\mathrm{d}f|_X}{\longrightarrow}(f^*TN)\big|_X\stackrel{f^*q}{\longrightarrow}f^*\mathcal{N}_NY\,. \tag{11.5}$$

This composite vector-bundle homomorphism is 0, since $\mathrm{d}_xf(v)\in T_{f(x)}Y$ for all $x\in X$. Thus, it descends to a vector-bundle homomorphism

$$\mathrm{d}f\colon\mathcal{N}_MX\longrightarrow f^*\mathcal{N}_NY \tag{11.6}$$

over X. If $f\pitchfork_NY$ as in (6.1), then the map $TM|_X\longrightarrow f^*\mathcal{N}_NY$ in (11.5) is onto and thus the vector-bundle homomorphism (11.6) is surjective on every fiber. Finally, if $X=f^{-1}(Y)$, the ranks of the two bundles in (11.6) are the same by the last statement in Theorem 6.3, and so (11.6) is an isomorphism of vector bundles over X. Combining this observation with Theorem 6.3, we obtain a more precise statement of the latter.

Theorem 11.11. *Let $f:M\longrightarrow N$ be a smooth map and $Y\subset N$ an embedded submanifold. If $f\pitchfork_NY$ as in (6.1), then $X\equiv f^{-1}(Y)$ is an embedded submanifold of M and the differential $\mathrm{d}f$ induces a vector-bundle isomorphism*

$$\begin{array}{ccc}\mathcal{N}_MX & \xrightarrow{\quad\mathrm{d}f\quad} & f^*(\mathcal{N}_NY)\\ & {\scriptstyle\pi}\searrow\quad\swarrow{\scriptstyle\pi_1} & \\ & M & \end{array} \tag{11.7}$$

Since the ranks of $\mathcal{N}_MX$ and $f^*(\mathcal{N}_NY)$ are the codimensions of X in M and Y in N, respectively, this theorem implies Theorem 6.3. If $Y=\{q\}$ for some $q\in N$, then $\mathcal{N}_NY$ is a trivial vector bundle and thus so is $\mathcal{N}_MX\approx f^*(\mathcal{N}_NY)$. For example, the unit sphere $S^m\subset\mathbb{R}^{m+1}$ has trivial normal bundle, because

$$S^m=f^{-1}(1),\qquad\text{where}\quad f\colon\mathbb{R}^{m+1}\longrightarrow\mathbb{R},\quad f(x)=|x|^2.$$

A trivialization of the normal bundle to S^m is given by

$$T\mathbb{R}^{m+1}/TS^m\longrightarrow S^m\times\mathbb{R},\qquad(x,v)\longrightarrow(x,x\cdot v).$$

Corollary 11.12. *Let $f:X\longrightarrow M$ and $g\colon Y\longrightarrow M$ be smooth maps. If $f\pitchfork_Mg$ as in (6.5), then the space*

$$X\times_MY\equiv\big\{(x,y)\in X\times Y\colon\ f(x)=g(y)\big\}$$

is an embedded submanifold of $X\times Y$ and the differential $\mathrm{d}f$ induces a vector-bundle isomorphism

$$\begin{array}{ccc}\mathcal{N}_{X\times Y}(X\times_MY) & \xrightarrow{\ \mathrm{d}(f\circ\pi_X)+\mathrm{d}(g\circ\pi_Y)\ } & \pi_X^*f^*TM=\pi_Y^*g^*TM\\ & {\scriptstyle\pi}\searrow\qquad\swarrow & \\ & X\times_MY & \end{array} \tag{11.8}$$

Furthermore, the projection map $\pi_1=\pi_X\colon X\times_MY\longrightarrow X$ is injective (immersion) if $g\colon Y\longrightarrow M$ is injective (immersion).

This corollary is obtained by applying Theorem 11.11 to the smooth map

$$f\times g\colon X\times Y\longrightarrow M\times M.$$

All other versions of the *Implicit Function Theorem* stated in these notes are special cases of this corollary.

12 Direct Sums and Duals

If V and V' are two vector spaces, we can form a new vector space, $V\oplus V' = V\times V'$, the direct sum of V and V'. There are natural inclusions $V, V' \longrightarrow V\oplus V'$ and projections $V\oplus V' \longrightarrow V, V'$. Linear maps $f\colon V\longrightarrow W$ and $f'\colon V'\longrightarrow W'$ induce a linear map

$$f\oplus f'\colon V\oplus V' \longrightarrow W\oplus W'.$$

If we choose bases for V, V', W, and W' so that f and f' correspond to some matrices A and D, then with respect to the induced bases for $V\oplus V'$ and $W\oplus W'$,

$$f\oplus g = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}.$$

If $\pi\colon V\longrightarrow M$ and $\pi'\colon V'\longrightarrow M$ are smooth vector bundles, we can form their direct sum, $V\oplus V'$, so that

$$(V\oplus V)_p = V_p\oplus V'_p \qquad \forall\, p\in M.$$

The vector-bundle structure on $V\oplus V'$ is determined from those of V and V' by requiring that either the natural inclusion maps $V, V' \longrightarrow V\oplus V'$ or the projections $V\oplus V' \longrightarrow V, V'$ be smooth vector-bundle homomorphisms over M. Thus, if s and s' are sections of V and V', then $s\oplus s'$ is a smooth section of $V\oplus V'$ if and only if s and s' are smooth. So, the map

$$\Gamma(M;V)\oplus\Gamma(M;V') \longrightarrow \Gamma(M;V\oplus V'),$$
$$(s,s') \longrightarrow s\oplus s', \qquad \big\{s\oplus s'\big\}(p) = s(p)\oplus s'(p) \;\; \forall\, p\in M,$$

is an isomorphism of $C^\infty(M)$-modules. If $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ are transition data for V and V', transition data for $V\oplus V'$ is given by $\{g_{\alpha\beta}\oplus g'_{\alpha\beta}\}$, i.e. we put the first matrix in the top left corner and the second matrix in the bottom right corner. Alternatively,

$$\pi\times\pi'\colon V\times V' \longrightarrow M\times M$$

is a smooth vector bundle with respect to the product structures and

$$V\oplus V' = d^*(V\times V'), \tag{12.1}$$

where $d\colon M \longrightarrow M\times M, \qquad d(p) = (p,p)$ is the diagonal embedding.

The operation $\oplus$ is easily seen to be commutative and associative (the resulting vector bundles are isomorphic). If $\tau_0 = M\longrightarrow M$ is trivial rank 0 bundle,

$$\tau_0\oplus V \approx V$$

for every vector bundle $V\longrightarrow M$. If $n\in\mathbb{Z}^{\geq 0}$, let

$$nV = \underbrace{V\oplus\ldots\oplus V}_{n};$$

by convention; $0V = \tau_0$. We leave proofs of the following lemmas as an exercise.

Lemma 12.1. *If $f\colon M\longrightarrow N$ is a smooth map and $W,W'\longrightarrow N$ are smooth vector bundles,*

$$f^*\big(W\oplus W'\big)\approx (f^*W)\oplus(f^*W')$$

as vector bundles over M.

Lemma 12.2. *Let $V,V'\longrightarrow M$ and $W,W'\longrightarrow N$ be vector bundles over smooth manifolds and $f\colon M\longrightarrow N$ a smooth map. Vector-bundle homomorphisms*

$$\tilde f\colon V\longrightarrow W \qquad \text{and} \qquad \tilde f'\colon V'\longrightarrow W'$$

covering f as in (8.4) induce a vector-bundle homomorphism

$$\tilde f\oplus\tilde f'\colon V\oplus V'\longrightarrow W\oplus W'$$

covering f; this induced homomorphism is smooth if and only if $\tilde f$ and $\tilde f'$ are smooth.

If $V,V'\longrightarrow M$ are vector bundles, then V and V' are vector subbundles of $V\oplus V'$. It is immediate that

$$\big(V\oplus V'\big)/V=V' \qquad \text{and} \qquad \big(V\oplus V'\big)/V'=V.$$

These equalities hold in the holomorphic category as well (i.e. when the bundles and the base manifold carry complex structures and all trivializations and transition maps are holomorphic). Conversely, if V' is a subbundle of V, by Section 14 below

$$V\approx (V/V')\oplus V'$$

as smooth vector bundles, real or complex. However, if V and V' are holomorphic bundles, V may not have the same holomorphic structure as $(V/V')\oplus V'$ (i.e. the two bundles are isomorphic as smooth vector bundles, but not as holomorphic ones).

If V is a vector space (over $\mathbb{R}$ or $\mathbb{C}$), the dual vector space is the space of the linear homomorphisms to the field ($\mathbb{R}$ or $\mathbb{C}$, respectively):

$$V^*=\mathrm{Hom}_{\mathbb{R}}\big(V,\mathbb{R}\big) \qquad \text{or} \qquad V^*=\mathrm{Hom}_{\mathbb{C}}\big(V,\mathbb{C}\big).$$

A linear map $g\colon V\longrightarrow W$ between two vector spaces induces a dual map in the "opposite" direction:

$$g^*\colon W^*\longrightarrow V^*, \qquad \big\{g^*(L)\big\}(v)=L\big(g(v)\big) \quad \forall\ L\in W^*,\ v\in V.$$

If $V=\mathbb{R}^k$ and $W=\mathbb{R}^n$, then g is given by an $n\times k$-matrix, and its dual is given by the transposed $k\times n$-matrix.

If $\pi\colon V\longrightarrow M$ is a smooth vector bundle of rank k (say, over $\mathbb{R}$), the dual bundle of V is a vector bundle $V^*\longrightarrow M$ such that

$$(V^*)_p=V_p^* \qquad \forall\, p\in M.$$

The vector-bundle structure on V^* is determined from that of V by requiring that the natural map

$$V\oplus V^*=V\times_M V^*\longrightarrow \mathbb{R}\ (\text{or } \mathbb{C}), \qquad (v,L)\longrightarrow L(v), \tag{12.2}$$

be smooth. Thus, if s and ψ are smooth sections of V and V^*,

$$\psi(s)\colon M \longrightarrow \mathbb{R}, \qquad \{\psi(s)\}(p) = \{\psi(p)\}\big(s(p)\big),$$

is a smooth function on M. So, the map

$$\Gamma(M;V) \times \Gamma(M;V^*) \longrightarrow C^\infty(M), \qquad (s,\psi) \longrightarrow \psi(s),$$

is a nondegenerate pairing of $C^\infty(M)$-modules. If $\{g_{\alpha\beta}\}$ is transition data for V, i.e. the transitions between smooth trivializations are given by

$$h_\alpha \circ h_\beta^{-1}\colon U_\alpha \cap U_\beta \times \mathbb{R}^k \longrightarrow U_\alpha \cap U_\beta \times \mathbb{R}^k, \qquad (p,v) \longrightarrow \big(p, g_{\alpha\beta}(p)v\big),$$

the dual transition maps are then given by

$$U_\alpha \cap U_\beta \times \mathbb{R}^k \longrightarrow U_\alpha \cap U_\beta \times \mathbb{R}^k, \qquad (p,v) \longrightarrow \big(p, g_{\alpha\beta}(p)^{\mathrm{tr}} v\big).$$

However, these maps reverse the direction, i.e. they go from the α-side to the β-side. To fix this problem, we simply take the inverse of $g_{\alpha\beta}(p)^{\mathrm{tr}}$:

$$U_\alpha \cap U_\beta \times \mathbb{R}^k \longrightarrow U_\alpha \cap U_\beta \times \mathbb{R}^k, \qquad (p,v) \longrightarrow \big(p, \{g_{\alpha\beta}(p)^{\mathrm{tr}}\}^{-1} v\big).$$

So, transition data for V^* is $\{(g_{\alpha\beta}^{\mathrm{tr}})^{-1}\}$. As an example, if V is a line bundle, then $g_{\alpha\beta}$ is a smooth nowhere-zero function on $U_\alpha \cap U_\beta$ and $(g^*)_{\alpha\beta}$ is the smooth function given by $1/g_{\alpha\beta}$. We leave proofs of the following lemmas as an exercise.

Lemma 12.3. *If $f\colon M \longrightarrow N$ is a smooth map and $W \longrightarrow N$ is a smooth vector bundle,*

$$f^*(W^*) \approx (f^*W)^*$$

as vector bundles over M.

Lemma 12.4. *Let $V \longrightarrow M$ and $W \longrightarrow N$ be vector bundles over smooth manifolds and $f\colon M \longrightarrow N$ a diffeomorphism. A vector-bundle homomorphism $\tilde{f}\colon V \longrightarrow W$ covering f as in (8.4) induces a vector-bundle homomorphism*

$$\tilde{f}^*\colon W^* \longrightarrow V^*$$

covering f^{-1}; this induced homomorphism is smooth if and only if the homomorphism $\tilde{f}$ is.

The cotangent bundle of a smooth manifold M, $\pi\colon T^*M \longrightarrow M$, is the dual of its tangent bundle, $TM \longrightarrow M$, i.e. $T^*M = (TM)^*$. For each $p \in M$, the fiber of the cotangent bundle over p is the cotangent space T_p^*M of M at p; see Definition 3.7. A section $\alpha\colon M \longrightarrow T^*M$ of T^*M is called a 1-form on M; it assigns to each $p \in M$ a linear map

$$\alpha_p \equiv \alpha(p)\colon T_pM \longrightarrow \mathbb{R}.$$

If in addition X is a vector field, then

$$\alpha(X)\colon M \longrightarrow \mathbb{R}, \qquad \{\alpha(X)\}(p) = \alpha_p\big(X(p)\big),$$

is a function on M. The section α is smooth if and only if $\alpha(X) \in C^\infty(M)$ for every smooth vector field X on M. If $\varphi = (x_1, \ldots, x_m)\colon U \longrightarrow \mathbb{R}^m$ is a smooth chart, the sections

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m} \in \mathrm{VF}(U)$$

form a basis for VF(U) as a $C^\infty(U)$-module. Since

$$\mathrm{d}_p x_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij} \qquad \forall\, i,j=1,2,\ldots,m,$$

$\mathrm{d}x_i(X)\in C^\infty(U)$ for all $X\in \mathrm{VF}(U)$ and $\{\mathrm{d}_p x_i\}_i$ is a basis for T_p^*M for all $p\in U$. Thus, $\mathrm{d}x_i$ is a smooth section of T^*M over U and the inverse of the map

$$U\times\mathbb{R}^m \longrightarrow T^*M|_U, \qquad (p,c_1,\ldots,c_m)\longrightarrow c_1\mathrm{d}_p x_1+\ldots+c_m\mathrm{d}_p x_m\,,$$

is a trivialization of T^*M over U; see Section 8. By (4.16), this inverse is given by

$$T^*M|_U \longrightarrow U\times\mathbb{R}^m, \qquad u\longrightarrow \left(\pi(u), u\left(\frac{\partial}{\partial x_1}\right),\ldots,u\left(\frac{\partial}{\partial x_m}\right)\right),$$

where $\pi: T^*M\longrightarrow M$ is the projection map. Thus, a 1-form α on M is smooth if and only if for every smooth chart $\varphi_\alpha=(x_1,\ldots,x_m): U_\alpha\longrightarrow\mathbb{R}^m$ the coefficient functions

$$c_1=\alpha\left(\frac{\partial}{\partial x_1}\right),\ldots,c_m=\left(\frac{\partial}{\partial x_m}\right): U\longrightarrow\mathbb{R}, \qquad \alpha_p\equiv c_1(p)\mathrm{d}_p x_1+\ldots+c_m(p)\mathrm{d}_p x_m \quad \forall\, p\in U,$$

are smooth. The $C^\infty(M)$-module of 1-forms on M is denoted by $E^1(M)$.

13 Tensor and Exterior Products

If V and V' are two vector spaces, we can form a new vector space, $V\otimes V'$, the tensor product of V and V'. If $g: V\longrightarrow W$ and $g': V'\longrightarrow W'$ are linear maps, they induce a linear map

$$g\otimes g': V\otimes V'\longrightarrow W\otimes W'.$$

If we choose bases $\{e_j\}$, $\{e'_n\}$, $\{f_i\}$, and $\{f'_m\}$ for V, V', W, and W', respectively, then $\{e_j\otimes e'_n\}_{(j,n)}$ and $\{f_i\otimes f'_m\}_{(i,m)}$ are bases for $V\otimes V'$ and $W\otimes W'$. If the matrices for g and g' with respect to the chosen bases for V, V', W, and W' are $(g_{ij})_{i,j}$ and $(g'_{mn})_{m,n}$, then the matrix for $g\otimes g'$ with respect to the induced bases for $V\otimes V'$ and $W\otimes W'$ is $(g_{ij}g'_{mn})_{(i,m),(j,n)}$. The rows of this matrix are indexed by the pairs (i,m) and the columns by the pairs (j,n). In order to actually write down the matrix, we need to order all pairs (i,m) and (j,n). If the vector spaces V and W are one-dimensional, g corresponds to a single number g_{ij}, while $g\otimes g'$ corresponds to the matrix $(g_{mn})_{m,n}$ multiplied by this number.

If $\pi: V\longrightarrow M$ and $\pi': V'\longrightarrow M$ are smooth vector bundles, we can form their tensor product, $V\otimes V'$, so that

$$(V\otimes V')_p = V_p\otimes V'_p \qquad \forall\, p\in M.$$

The topology and smooth structure on $V\otimes V'$ are determined from those of V and V' by requiring that if s and s' are smooth sections of V and V', then $s\otimes s'$ is a smooth section of $V\otimes V'$. So, the map

$$\Gamma(M;V)\otimes\Gamma(M;V')\longrightarrow\Gamma(M;V\otimes V'),$$
$$(s,s')\longrightarrow s\otimes s', \qquad \big\{s\otimes s'\big\}(p)=s(p)\otimes s'(p) \;\; \forall\, p\in M,$$

is a homomorphism of $C^\infty(M)$-modules (but not an isomorphism). If $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ are transition data for V and V', then transition data for $V\otimes V'$ is given by $\{g_{\alpha\beta}\otimes g'_{\alpha\beta}\}$, i.e. we construct a matrix-valued function $g_{\alpha\beta}\otimes g'_{\alpha\beta}$ from $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ as in the previous paragraph. If V and V' are line bundles, then $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ are smooth nowhere-zero functions on $U_\alpha\cap U_\beta$ and $(g\otimes g')_{\alpha\beta}$ is the smooth function given by $g_{\alpha\beta}g'_{\alpha\beta}$.

The operation $\otimes$ is easily seen to be commutative and associative (the resulting vector bundles are isomorphic). If $\tau_1\longrightarrow M$ is the trivial line bundle,

$$\tau_1\otimes V\approx V$$

for every vector bundle $V\longrightarrow M$ is a vector bundle. If $n\in\mathbb{Z}^+$, let

$$V^{\otimes n}=\underbrace{V\otimes\ldots\otimes V}_{n}, \qquad V^{\otimes(-n)}=(V^*)^{\otimes n}\equiv\underbrace{V^*\otimes\ldots\otimes V^*}_{n};$$

by convention, $V^{\otimes 0}=\tau_1$. We leave proofs of the following lemmas as an exercise.

Lemma 13.1. *If $f\colon M\longrightarrow N$ is a smooth map and $W,W'\longrightarrow N$ are smooth vector bundles,*

$$f^*\big(W\otimes W'\big)\approx(f^*W)\otimes(f^*W')$$

as vector bundles over M.

Lemma 13.2. *Let $V,V'\longrightarrow M$ and $W,W'\longrightarrow N$ be vector bundles over smooth manifolds and $f\colon M\longrightarrow N$ a smooth map. Vector-bundle homomorphisms*

$$\tilde f\colon V\longrightarrow W \qquad \textit{and} \qquad \tilde f\colon V'\longrightarrow W'$$

covering f as in (8.4) induce a vector-bundle homomorphism

$$\tilde f\otimes\tilde f'\colon V\otimes V'\longrightarrow W\otimes W'$$

covering f; this induced homomorphism is smooth if $\tilde f$ and $\tilde f'$ are smooth.

Lemma 13.3. *Let $V,V'\longrightarrow M$ and $W\longrightarrow N$ be vector bundles over smooth manifolds and $f\colon M\longrightarrow N$ a smooth map. A bundle map*

$$\tilde f\colon V\oplus V'=V\times_M V\longrightarrow W$$

covering f as in (8.4) such that the restriction of $\tilde f$ to each fiber $V_p\times V_p$ is linear in each component induces a vector-bundle homomorphism

$$\bar f\colon V\otimes V'\longrightarrow W$$

covering f; this induced homomorphism is smooth if the homomorphism $\tilde f$ is.

If V is a vector space and k is a nonnegative integer, we can form the k-th exterior power, Λ^kV, of V. A linear map $g\colon V\longrightarrow W$ induces a linear map

$$\Lambda^kg\colon\Lambda^kV\longrightarrow\Lambda^kW.$$

If n is a nonnegative integer, let $S_k(n)$ be the set of increasing k-tuples of integers between 1 and n:

$$S_k(n) = \big\{(i_1,\ldots,i_k)\in\mathbb{Z}^k\colon 1\le i_1<i_2<\ldots<i_k\le n\big\}.$$

If $\{e_j\}_{j=1,\ldots,n}$ and $\{f_i\}_{i=1,\ldots,m}$ are bases for V and W, then $\{e_\eta\}_{\eta\in S_k(n)}$ and $\{f_\mu\}_{\mu\in S_k(m)}$ are bases for $\Lambda^k V$ and $\Lambda^k W$, where

$$e_{(\eta_1,\ldots,\eta_k)} = e_{\eta_1}\wedge\ldots\wedge e_{\eta_k} \qquad\text{and}\qquad f_{(\mu_1,\ldots,\mu_k)} = f_{\mu_1}\wedge\ldots\wedge f_{\mu_k}.$$

If $(g_{ij})_{i=1,\ldots,m,j=1,\ldots,n}$ is the matrix for g with respect to the chosen bases for V and W, then

$$\big(\det\big((g_{\mu_r\eta_s})_{r,s=1,\ldots,k}\big)\big)_{(\mu,\eta)\in I_k(m)\times I_k(n)}$$

is the matrix for $\Lambda^k g$ with respect to the induced bases for $\Lambda^k V$ and $\Lambda^k W$. The rows and columns of this matrix are indexed by the sets $S_k(m)$ and $S_k(n)$, respectively. The (μ,η)-entry of the matrix is the determinant of the $k\times k$-submatrix of $(g_{ij})_{i,j}$ with the rows and columns indexed by the entries of μ and η, respectively. In order to actually write down the matrix, we need to order the sets $S_k(m)$ and $S_k(n)$. If $k=m=n$, then $\Lambda^k V$ and $\Lambda^k W$ are one-dimensional vector spaces, called the top exterior power of V and W, with bases

$$\big\{e_1\wedge\ldots\wedge e_k\big\} \qquad\text{and}\qquad \big\{f_1\wedge\ldots\wedge f_k\big\}.$$

With respect to these bases, the homomorphism $\Lambda^k g$ corresponds to the number $\det(g_{ij})_{i,j}$. If $k>n$ (or $k>m$), then $\Lambda^k V$ (or $\Lambda^k W$) is the zero vector space and the corresponding matrix is empty.

If $\pi\colon V\longrightarrow M$ is a smooth vector bundle, we can form its k-th exterior power, $\Lambda^k V$, so that

$$(\Lambda^k V)_p = \Lambda^k V_p \qquad \forall\, p\in M.$$

The topology and smooth structure on $\Lambda^k V$ are determined from those of $\Lambda^k V$ by requiring that if $s_1,\ldots,s_k$ are smooth sections of V, then $s_1\wedge\ldots\wedge s_k$ is a smooth section of $\Lambda^k V$. Thus, the map

$$\Lambda^k\big(\Gamma(M;V)\big)\longrightarrow\Gamma(M;\Lambda^k V),$$
$$(s_1,\ldots,s_k)\longrightarrow s_1\wedge\ldots\wedge s_k, \qquad \{s_1\wedge\ldots\wedge s_k\}(p)=s_1(p)\wedge\ldots\wedge s_k(p) \quad \forall\, p\in M,$$

is a homomorphism of $C^\infty(M)$-modules (but not an isomorphism). If $\{g_{\alpha\beta}\}$ is transition data for V, then transition data for $\Lambda^k V$ is given by $\{\Lambda^k g_{\alpha\beta}\}$, i.e. we construct a matrix-valued function $\Lambda^k g_{\alpha\beta}$ from each matrix $g_{\alpha\beta}$ as in the previous paragraph. As an example, if the rank of V is k, then the transition data for the line bundle $\Lambda^k V$, called the top exterior power of V, is $\{\det g_{\alpha\beta}\}$. By definition, $\Lambda^0 V=\tau_1^{\mathbb{R}}$ is the trivial line bundle over M.

It follows directly from the definitions that if $V\longrightarrow M$ is a vector bundle of rank k and $L\longrightarrow M$ is a line bundle (vector bundle of rank one), then

$$\Lambda^{\rm top}(V\oplus L)\equiv\Lambda^{k+1}(V\oplus L)=\Lambda^k V\otimes L\equiv\Lambda^{\rm top}V\otimes L.$$

More generally, if $V,W\longrightarrow M$ are any two vector bundles, then

$$\Lambda^{\rm top}(V\oplus W)=(\Lambda^{\rm top}V)\otimes(\Lambda^{\rm top}W) \qquad\text{and}\qquad \Lambda^k(V\oplus W)=\bigoplus_{i+j=k}(\Lambda^i V)\otimes(\Lambda^j W).$$

We leave proofs of the following lemmas as exercises.

Lemma 13.4. *If $f\colon M\longrightarrow N$ is a smooth map, $W\longrightarrow N$ is a smooth vector bundle, and $k\in\mathbb{Z}^{\ge0}$,*

$$f^*\big(\Lambda^kW\big)\approx\Lambda^k(f^*W)$$

as vector bundles over M.

Lemma 13.5. *Let $V\longrightarrow M$ be a vector bundle. If $k,l\in\mathbb{Z}^{\ge0}$, the map*

$$\begin{gathered}\Gamma(M;\Lambda^kV)\otimes\Gamma(M;\Lambda^lV)\longrightarrow\Gamma(M;\Lambda^{k+l}V)\\ (s_1,s_2)\longrightarrow s_1\wedge s_2,\qquad \{s_1\wedge s_2\}(p)=s_1(p)\wedge s_2(p)\quad\forall\,p\in M,\end{gathered}$$

is a well-defined homomorphism of $C^\infty(M)$-modules.

Lemma 13.6. *Let $V\longrightarrow M$ and $W\longrightarrow N$ be vector bundles over smooth manifolds and $f\colon M\longrightarrow N$ a smooth map. A vector-bundle homomorphism $\tilde f\colon V\longrightarrow W$ covering f as in (8.4) induces a vector-bundle homomorphism*

$$\Lambda^k\tilde f\colon\Lambda^kV\longrightarrow\Lambda^kW$$

covering f; this induced homomorphism is smooth if the homomorphism $\tilde f$ is.

Lemma 13.7. *Let $V\longrightarrow M$ and $W\longrightarrow N$ be vector bundles over smooth manifolds and $f\colon M\longrightarrow N$ a smooth map. A bundle homomorphism*

$$\tilde f\colon kV\equiv\underbrace{V\times_M\ldots\times_MV}_{k}\longrightarrow W$$

covering f as in (8.4) such that the restriction of $\tilde f$ to each fiber V_p^k is linear in each component and alternating induces a vector-bundle homomorphism

$$\bar f\colon\Lambda^kV\longrightarrow W$$

covering f; this induced homomorphism is smooth if the homomorphism $\tilde f$ is.

Remark 13.8. For complex vector bundles, the above constructions (exterior power, tensor product, direct sum, etc.) are always done over $\mathbb{C}$, unless specified otherwise. So if V is a complex vector bundle of rank k over M, the top exterior power of V is the complex line bundle Λ^kV over M (could also be denoted as $\Lambda^k_{\mathbb{C}}V$). In contrast, if we forget the complex structure of V (so that it becomes a real vector bundle of rank $2k$), then its top exterior power is the real line bundle $\Lambda^{2k}V$ (could also be denoted as $\Lambda^{2k}_{\mathbb{R}}V$).

If M is a smooth manifold, a section of the bundle $\Lambda^k(T^*M)\longrightarrow M$ is called a k-form on M. A smooth nowhere-vanishing section s of $\Lambda^{\mathrm{top}}(T^*M)$, i.e.

$$s(p)\in\Lambda^{\mathrm{top}}(T_p^*M)-0\qquad\forall\,p\in M,$$

is called a volume form on M; Corollary 15.2 below provides necessary and sufficient conditions for such a section to exist. The space of smooth k-forms on M is often denoted by $E^k(M)$, rather than $\Gamma(M;\Lambda^k(T^*M))$.

14 Metrics on Fibers

Definition 14.1. *A Riemannian metric in a smooth real vector bundle $\pi: V \longrightarrow M$ is a smooth map*

$$\langle , \rangle : V \times_M V \equiv \big\{ (v,w) \in V \times V :\ \pi(v) = \pi(w) \big\} \longrightarrow \mathbb{R}$$

such that the restriction

$$\langle , \rangle : V_x \times V_x \longrightarrow \mathbb{R}, \qquad (v,w) \longrightarrow \langle v,w \rangle\,,$$

is an inner-product on V_x for every $x \in M$.

Thus, a Riemannian metric in $\pi: V \longrightarrow M$ is a smoothly varying family of inner-products in the fibers $V_x \approx \mathbb{R}^k$ of V. We leave a proof of the following lemma as an exercise.

Lemma 14.2. *Let $\pi: V \longrightarrow M$ be a real vector bundle and $\langle , \rangle : V \times_M V \longrightarrow \mathbb{R}$ a map such that the restriction*

$$\langle , \rangle : V_x \times V_x \longrightarrow \mathbb{R}, \qquad (v,w) \longrightarrow \langle v,w \rangle\,,$$

is an inner-product on V_x for every $x \in M$. The following statements are equivalent:

(1) the map $\langle , \rangle$ is a Riemannian metric in V;

(2) the section $\langle , \rangle$ of the vector bundle $(V \otimes V)^* \longrightarrow M$ is smooth;

(3) if s_1, s_2 are smooth sections of the vector bundle $V \longrightarrow M$, then the map

$$\big\langle s_1, s_2 \big\rangle : M \longrightarrow \mathbb{R}, \qquad p \longrightarrow \big\langle s_1(p), s_2(p) \big\rangle,$$

is smooth;

(4) if $h: V|_U \longrightarrow U \times \mathbb{R}^k$ is a trivialization of V, then the matrix-valued function,

$$B: U \longrightarrow \mathrm{Mat}_k\mathbb{R} \quad s.t. \quad \big\langle h^{-1}(p,v), h^{-1}(p,w) \big\rangle = v^t B(p) w \quad \forall\ p \in U,\ v,w \in \mathbb{R}^k,$$

is smooth.

Every real vector bundle admits a Riemannian metric. Such a metric can be constructed by covering M by a locally finite collection of trivializations for V and patching together inner-products on each trivialization using a partition of unity; see Definition 14.3 below.

Definition 14.3. *A smooth partition of unity subordinate to the open cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of a smooth manifold M is a collection $\{\eta_\alpha\}_{\alpha \in \mathcal{A}}$ of smooth functions on M with values in $[0,1]$ such that*

(PU1) *the collection $\{\mathrm{supp}\,\eta_\alpha\}_{\alpha \in \mathcal{A}}$ is locally finite;*

(PU2) $\mathrm{supp}\,\eta_\alpha \subset U_\alpha$ *for every $\alpha \in \mathcal{A}$;*

(PU3) $\displaystyle\sum_{\alpha \in \mathcal{A}} \eta_\alpha \equiv 1$.

If $\langle,\rangle$ is a Riemannian metric on a vector bundle $\pi: V \longrightarrow M$ and $W \subset V$ is a vector subbundle, then the orthogonal complement

$$W^\perp \equiv \big\{v \in V : \langle v, w\rangle = 0 \ \forall\, w \in W_{\pi(v)}\big\}$$

of W in V is also a vector subbundle of V and

$$V = W \oplus W^\perp.$$

Furthermore, the quotient projection map $q: V \longrightarrow V/W$ induces a vector bundle isomorphism from $W^\perp$ to V/W so that

$$V \approx W \oplus (V/W).$$

Definition 14.4. *A Hermitian metric in a smooth complex vector bundle $\pi: V \longrightarrow M$ is a smooth map $\langle,\rangle: V \times_M V \longrightarrow \mathbb{C}$ such that the restriction*

$$\langle,\rangle: V_x \times V_x \longrightarrow \mathbb{C}, \qquad (v, w) \longrightarrow \langle v, w\rangle\,,$$

is a hermitian inner-product on V_x for every $x \in M$.

Thus, a Hermitian metric in $\pi: V \longrightarrow M$ is a smoothly varying family of Hermitian inner-products in the fibers $V_x \approx \mathbb{C}^k$ of V. We leave a proof of the following lemma as an exercise.

Lemma 14.5. *Let $\pi: V \longrightarrow M$ be a complex vector bundle and $\langle,\rangle: V \times_M V \longrightarrow \mathbb{C}$ a map such that the restriction*

$$\langle,\rangle: V_x \times V_x \longrightarrow \mathbb{C}, \qquad (v, w) \longrightarrow \langle v, w\rangle\,,$$

is an inner-product on V_x for every $x \in M$. The following statements are equivalent:

(1) the map $\langle,\rangle$ is a Hermitian metric in V;

(2) the section $\langle,\rangle$ of the vector bundle $(V \otimes_{\mathbb{R}} V)^* \longrightarrow M$ is smooth;

(3) if s_1, s_2 are smooth sections of the vector bundle $V \longrightarrow M$, then the map

$$\big\langle s_1, s_2\big\rangle: M \longrightarrow \mathbb{C}, \qquad p \longrightarrow \big\langle s_1(p), s_2(p)\big\rangle,$$

is smooth;

(4) if $h: V|_U \longrightarrow U \times \mathbb{C}^k$ is a trivialization of V, then the matrix-valued function,

$$B: U \longrightarrow \mathrm{Mat}_k\mathbb{C} \quad s.t. \quad \big\langle h^{-1}(p, v), h^{-1}(p, w)\big\rangle = v^t B(p) w \quad \forall\ p \in U,\ v, w \in \mathbb{C}^k,$$

is smooth.

Similarly to the real case, every complex vector bundle admits a Hermitian metric. If $\langle,\rangle$ is a Hermitian metric on a complex vector bundle $\pi: V \longrightarrow M$ and $W \subset V$ is a complex vector subbundle, then the orthogonal complement

$$W^\perp \equiv \big\{v \in V : \langle v, w\rangle = 0 \ \forall\, w \in W_{\pi(v)}\big\}$$

of W in V is also a complex vector subbundle of V and

$$V = W \oplus W^\perp.$$

Furthermore, the quotient projection map $q: V \longrightarrow V/W$ induces a vector bundle isomorphism from $W^\perp$ to V/W so that $V \approx W \oplus (V/W)$.

If $V \longrightarrow M$ is a real vector bundle of rank k with a Riemannian metric $\langle , \rangle$ or a complex vector bundle of rank k with a Hermitian metric $\langle , \rangle$, let

$$SV \equiv \big\{ v \in V : \langle v, v \rangle = 1 \big\} \longrightarrow M$$

be the sphere bundle of V. In the real case, the fiber of SV over every point of M is S^{k-1}. Furthermore, if U is a small open subset of M, then $SV|_U \approx U \times S^{k-1}$ as bundles over U, i.e. SV is an S^{k-1}-fiber bundle over M. In the complex case, SV is an S^{2k-1}-fiber bundle over M. If $V \longrightarrow M$ is a real line bundle (vector bundle of rank one) with a Riemannian metric $\langle , \rangle$, then $SV \longrightarrow M$ is an S^0-fiber bundle. In particular, if U is a small open subset of M, $SV|_U$ is diffeomorphic to $U \times \{\pm 1\}$. Thus, $SV \longrightarrow M$ is a $2:1$-covering map. If M is connected, the covering space SV is connected if and only if V is *not* orientable; see Section 15 below.

15 Orientations

If V is a real vector space of dimension k, the top exterior power of V, i.e.

$$\Lambda^{\mathrm{top}} V \equiv \Lambda^k V$$

is a one-dimensional vector space. Thus, $\Lambda^{\mathrm{top}} V - 0$ has exactly two connected components. An orientation on V is a component $\mathcal{C}$ of $\Lambda^{\mathrm{top}} V - 0$. If $\mathcal{C}$ is an orientation on V, then a basis $\{e_i\}$ for V is called oriented (with respect to $\mathcal{C}$) if

$$e_1 \wedge \ldots \wedge e_k \in \mathcal{C}.$$

If $\{f_j\}$ is another basis for V and A is the change-of-basis matrix from $\{e_i\}$ to $\{f_j\}$, i.e.

$$\big(f_1, \ldots, f_k\big) = \big(e_1, \ldots, e_k\big) A \qquad \Longleftrightarrow \qquad f_j = \sum_{i=1}^{i=k} A_{ij} e_i,$$

then

$$f_1 \wedge \ldots \wedge f_k = (\det A) e_1 \wedge \ldots \wedge e_k.$$

Thus, two different bases for V belong to the same orientation on V if and only if the determinant of the corresponding change-of-basis matrix is positive.

Suppose $V \longrightarrow M$ is a real vector bundle of rank k. An orientation for V is an orientation for each fiber $V_x \approx \mathbb{R}^k$, which varies smoothly (or continuously, or is locally constant) with $x \in M$. This means that if

$$h: V|_U \longrightarrow U \times \mathbb{R}^k$$

is a trivialization of V and U is connected, then h is either orientation-preserving or orientation-reversing (with respect to the standard orientation of $\mathbb{R}^k$) on every fiber. If V admits an orientation, V is called orientable.

Lemma 15.1. *Suppose $V \longrightarrow M$ is a smooth real vector bundle.*

(1) V is orientable if and only if V^* is orientable.

(2) V is orientable if and only if there exists a collection $\{U_\alpha, h_\alpha\}$ of trivializations that covers M such that

$$\det g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow \mathbb{R}^+,$$

where $\{g_{\alpha\beta}\}$ is the corresponding transition data.

(3) V is orientable if and only if the line bundle $\Lambda^{\mathrm{top}} V \longrightarrow M$ is orientable.

(4) If V is a line bundle, V is orientable if and only if V is (isomorphic to) the trivial line bundle $M \times \mathbb{R}$.

(5) If M is connected and V is a line bundle, V is orientable if and only if the sphere bundle SV (with respect to any Riemann metric on V) is not connected.

Proof. (1) Since $\Lambda^{\mathrm{top}}(V^*) \approx (\Lambda^{\mathrm{top}} V)^*$ and a line bundle L is trivial if and only if L^* is trivial, this claim follows from (3) and (4).

(2) If V has an orientation, we can choose a collection $\{U_\alpha, h_\alpha\}$ of trivializations that covers M such that the restriction of h_α to each fiber is orientation-preserving (if a trivialization is orientation-reversing, simply multiply its first component by -1). Then, the corresponding transition data $\{g_{\alpha\beta}\}$ is orientation-preserving, i.e.

$$\det g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow \mathbb{R}^+.$$

Conversely, suppose $\{U_\alpha, h_\alpha\}$ is a collection of trivializations that covers M such that

$$\det g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow \mathbb{R}^+.$$

Then, if $x \in U_\alpha$ for some α, define an orientation on V_x by requiring that

$$h_\alpha\colon V_x \longrightarrow x \times \mathbb{R}^k$$

is orientation-preserving. Since $\det g_{\alpha\beta}$ is $\mathbb{R}^+$-valued, the orientation on V_x is independent of α such that $x \in U_\alpha$. Each of the trivializations h_α is then orientation-preserving on each fiber.

(3) An orientation for V is the same as an orientation for $\Lambda^{\mathrm{top}} V$, since

$$\Lambda^{\mathrm{top}} V = \Lambda^{\mathrm{top}}\big(\Lambda^{\mathrm{top}} V\big).$$

Furthermore, if $\{(U_\alpha, h_\alpha)\}$ is a collection of trivializations for V such that the corresponding transition functions $g_{\alpha\beta}$ have positive determinant, then $\{(U_\alpha, \Lambda^{\mathrm{top}} h_\alpha)\}$ is a collection of trivializations for $\Lambda^{\mathrm{top}} V$ such that the corresponding transition functions $\Lambda^{\mathrm{top}} g_{\alpha\beta} = \det(g_{\alpha\beta})$ have positive determinant as well.

(4) The trivial line bundle $M \times \mathbb{R}$ is orientable, with an orientation determined by the standard orientation on $\mathbb{R}$. Thus, if V is isomorphic to the trivial line bundle, then V is orientable. Conversely, suppose V is an oriented line bundle. For each $x \in M$, let

$$\mathcal{C}_x \subset \Lambda^{\mathrm{top}} V = V$$

be the chosen orientation of the fiber. Choose a Riemannian metric on V and define a section s of V by requiring that for all $x \in M$

$$\big\langle s(x), s(x) \big\rangle = 1 \qquad \text{and} \qquad s(x) \in \mathcal{C}_x.$$

This section is well-defined and smooth (as can be seen by looking on a trivialization). Since it does not vanish, the line bundle V is trivial by Lemma 8.5.

(5) If V is orientable, then V is isomorphic to $M \times \mathbb{R}$, and thus

$$SV = S(M \times \mathbb{R}) = M \times S^0 = M \sqcup M$$

is not connected. Conversely, if M is connected and SV is not connected, let SV^+ be one of the components of SV. Since $SV \longrightarrow M$ is a covering projection, so is $SV^+ \longrightarrow M$. Since the latter is one-to-one, it is a diffeomorphism, and its inverse determines a nowhere-zero section of V. Thus, V is isomorphic to the trivial line bundle by Lemma 8.5. □

If V is a complex vector space of dimension k, V has a canonical orientation as a real vector space of dimension $2k$. If $\{e_i\}$ is a basis for V over $\mathbb{C}$, then

$$\big\{e_1, \mathrm{i}e_1, \ldots, e_k, \mathrm{i}e_k\big\}$$

is a basis for V over $\mathbb{R}$. The orientation determined by such a basis is the canonical orientation for the underlying real vector space V. If $\{f_j\}$ is another basis for V over $\mathbb{C}$, B is the complex change-of-basis matrix from $\{e_i\}$ to $\{f_j\}$, A is the real change-of-basis matrix from

$$\big\{e_1, \mathrm{i}e_1, \ldots, e_k, \mathrm{i}e_k\big\} \qquad \text{to} \qquad \big\{f_1, \mathrm{i}f_1, \ldots, f_k, \mathrm{i}f_k\big\},$$

then

$$\det A = (\det B)\overline{\det B} \in \mathbb{R}^+.$$

Thus, the two bases over $\mathbb{R}$ induced by complex bases for V determine the same orientation for V. This implies that every complex vector bundle $V \longrightarrow M$ is orientable as a real vector bundle.

A smooth manifold M is called orientable if its tangent bundle, $TM \longrightarrow M$, is orientable.

Corollary 15.2. *Let M be a smooth manifold. The following statements are equivalent:*

(1) M is orientable;

(2) the bundle $T^*M \longrightarrow M$ is orientable;

(3) M admits a volume form;

(4) there exists a collection of smooth charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ that covers M such that

$$\det \mathcal{J}(\varphi_\alpha \circ \varphi_\beta^{-1})_x > 0 \qquad \forall\, x \in \varphi_\beta(U_\alpha \cap U_\beta),\ \alpha, \beta \in \mathcal{A}.$$

Proof. The equivalence of the first three conditions follows immediately from Lemma 15.1. If $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ is a collection of charts as in (4), then

$$h_\alpha = \tilde{\varphi}_\alpha \colon TM|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{R}^m, \qquad v \longrightarrow \big(\pi(v), v(\varphi_\alpha)\big),$$

is a collection of trivializations of TM as in Lemma 15.1-(2) for $V = TM$, since

$$\tilde{\varphi}_\alpha \circ \tilde{\varphi}_\beta^{-1} \colon U_\alpha \cap U_\beta \times \mathbb{R}^m \longrightarrow U_\alpha \cap U_\beta \times \mathbb{R}^m\,, \quad (p,v) \longrightarrow \big(p, \mathcal{J}(\varphi_\alpha \circ \varphi_\beta^{-1})_{\varphi_\beta(p)} v\big),$$
$$h_\alpha \circ h_\beta^{-1} \colon U_\alpha \cap U_\beta \times \mathbb{R}^m \longrightarrow U_\alpha \cap U_\beta \times \mathbb{R}^m\,, \quad (p,v) \longrightarrow \big(p, g_{\alpha\beta}(p) v\big).$$

In particular, if such a collection of charts exists, then TM is orientable. Conversely, suppose $\{(U_\alpha, h_\alpha)\}_{\alpha\in\mathcal{A}}$ is a collection of trivializations of TM as in Lemma 15.1-(2), $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ is any collection of smooth charts on M, and U_α is connected. In particular,

$$\tilde{\varphi}_\alpha \circ h_\alpha^{-1} \colon U_\alpha \times \mathbb{R}^m \longrightarrow U_\alpha \times \mathbb{R}^m\,, \qquad (p,v) \longrightarrow \big(p, \{h_\alpha^{-1}(p,v)\}(\varphi_\alpha)\big),$$

is a smooth vector-bundle isomorphism. Thus, there is a smooth map

$$A_\alpha \colon U_\alpha \longrightarrow \mathrm{GL}_m\mathbb{R} \qquad \text{s.t.} \qquad \{h_\alpha^{-1}(p,v)\}(\varphi_\alpha) = A_\alpha(p)v \quad \forall\, v \in \mathbb{R}^m.$$

Since U_α is connected, $\det A_\alpha$ does not change sign on U_α. By changing the sign of the first component of φ_α if necessary, it can be assumed that $\det A_\alpha(p) > 0$ for all $p \in U_\alpha$ and $\alpha \in \mathcal{A}$. Thus,

$$\det \mathcal{J}(\varphi_\alpha \circ \varphi_\beta^{-1})_{\varphi_\beta(p)} = \det A_\alpha(p) \cdot \det g_{\alpha\beta}(p) \cdot \det A_\beta^{-1}(p) > 0 \qquad \forall\, p \in U_\alpha \cap U_\beta,\ \alpha, \beta \in \mathcal{A}.$$

Thus, the collection $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ satisfies (4). □

An orientation for a smooth manifold M is an orientation for the vector bundle $TM \longrightarrow M$; a manifold with a choice of orientation is called oriented. A diffeomorphism $f : M \longrightarrow N$ between oriented manifolds is called orientation-preserving (orientation-reversing) if the differential

$$\mathrm{d}_p f \colon T_pM \longrightarrow T_{f(p)}N$$

is an orientation-preserving (orientation-reversing) isomorphism for every $p \in M$; if M is connected, this is the case if and only if $\mathrm{d}_p f$ is orientation-preserving (orientation-reversing) for a single point $p \in M$.

If M is a smooth manifold, the sphere bundle

$$\pi \colon S\big(\Lambda^{\mathrm{top}} T^*M\big) \longrightarrow M$$

is a two-to-one covering map. By Lemma 15.1 and Corollary 15.2, if M is connected, the domain of π is connected if and only if M is not orientable. For each $p \in M$,

$$\pi^{-1}(p) \equiv \{\Omega_p, -\Omega_p\} \subset S\big(\Lambda^{\mathrm{top}} T_p^*M\big) \subset \Lambda^{\mathrm{top}} T_p^*M$$

is a pair on nonzero top forms on T_p^*M, which define opposite orientations of T_pM. Thus, $S(\Lambda^{\mathrm{top}} T^*M)$ can be thought as the set of orientations on the fibers of M; it is called the orientation double cover of M.

Smooth maps $f, g \colon M \longrightarrow N$ are called smoothly homotopic if there exists a smooth map

$$H \colon M \times [0,1] \longrightarrow N \qquad \text{s.t.} \qquad H(p,0) = f(p), \quad H(p,1) = g(p) \quad \forall\, p \in M.$$

Diffeomorphisms $f, g \colon M \longrightarrow N$ are called isotopic if there exists a smooth map H as above such that the map

$$H_t \colon M \longrightarrow N, \qquad p \longrightarrow (p,t),$$

is a diffeomorphism for every $t \in [0,1]$. We leave proofs of the following lemmas as an exercise; both can be proved using Corollary 15.2.

Lemma 15.3. *The orientation double cover of any smooth manifold is orientable.*

Lemma 15.4. *Let $f, g\colon M \longrightarrow N$ be isotopic diffeomorphisms between oriented manifolds. If f is orientation-preserving (orientation-reversing), then so is g.*

16 Connections

Definition 16.1. *A connection in a smooth real vector bundle $V \longrightarrow M$ is an $\mathbb{R}$-linear map*

$$\nabla\colon \Gamma(M;V) \longrightarrow \Gamma(M;T^*M\otimes V) \qquad s.t.$$
$$\nabla(fs) = \mathrm{d}f\otimes s + f\nabla s \quad \forall\, f\in C^\infty(M),\; s\in\Gamma(M;V). \tag{16.1}$$

If f is a smooth function on M supported in a neighborhood U of $x\in M$ such that $f(x)=1$ and $s\in\Gamma(M;V)$, then

$$\nabla s\big|_x = \nabla\big(fs\big)\big|_x - \mathrm{d}_x f\otimes s(x) \tag{16.2}$$

by (16.1). The right-hand side of (16.2) depends only on $\xi|_U$. Thus, a connection ∇ in V is a local operator, i.e. the value of ∇s at a point $x\in M$ depends only on the restriction of s to any neighborhood U of x.

Let $h_\alpha\colon V|_{U_\alpha} \longrightarrow U_\alpha\times\mathbb{R}^k$ be a trivialization of V and

$$s_{\alpha;1},\ldots,s_{\alpha;k} \in \Gamma(U_\alpha;V), \qquad s_{\alpha;i}(x) = h_\alpha^{-1}(x,e_i), \tag{16.3}$$

be a frame for V. By definition of ∇, there exist

$$\theta^\alpha_{ij} \in \Gamma(U_\alpha;T^*M) \qquad \text{s.t.} \qquad \nabla s_{\alpha;j} = \sum_{i=1}^{i=k} s_{\alpha;i}\theta^\alpha_{ij} \equiv \sum_{i=1}^{i=k}\theta^\alpha_{ij}\otimes s_{\alpha;i} \quad \forall\, j=1,\ldots,k.$$

We will call

$$\theta^\alpha \equiv \big(\theta^\alpha_{ij}\big)_{i,j=1,\ldots,k} \in \Gamma\big(U_\alpha;T^*M\otimes_{\mathbb{R}}\mathrm{Mat}_{k\times k}\mathbb{R}\big) \tag{16.4}$$

the connection one-form of ∇ for the trivialization h_α. For an arbitrary section of $V\longrightarrow U_\alpha$, by (16.1)

$$\nabla\Big(\sum_{j=1}^{j=k} f^j s_{\alpha;j}\Big) = \sum_{i=1}^{i=k} s_{\alpha;i}\Big(\mathrm{d}f^i + \sum_{j=1}^{j=k}\theta^\alpha_{ij}f^j\Big)\,. \tag{16.5}$$

Conversely, any θ^α as in (16.4) defines a connection in $V|_{U_\alpha}\longrightarrow U_\alpha$ by (16.5). Thus, every vector bundle $V\longrightarrow M$ admits a connection, since one can be obtained by patching together connections over trivializations via partitions of unity.

If $h_\beta\colon V|_{U_\beta}\longrightarrow U_\beta\times\mathbb{R}^k$ is another trivialization of V and

$$h_\alpha\circ h_\beta^{-1}(x,w) = (x, g_{\alpha\beta}(x)w) \qquad \forall\,(x,w)\in U_\alpha\cap U_\beta\times\mathbb{R}^k,$$

then by (16.3) and (16.5)

$$s_{\beta;l}\big|_{U_\alpha\cap U_\beta} = \sum_{j=1}^{j=k}(g_{\alpha\beta})_{jl}s_{\alpha;j}\big|_{U_\alpha\cap U_\beta} \implies \nabla s_{\beta;l}\big|_{U_\alpha\cap U_\beta} = \sum_{i=1}^{i=k}s_{\alpha;i}\Big((\mathrm{d}g_{\alpha\beta})_{il} + \sum_{j=1}^{j=k}\theta^\alpha_{ij}(g_{\alpha\beta})_{jl}\Big)$$
$$\implies \qquad \theta^\beta = g_{\beta\alpha}\theta^\alpha g_{\alpha\beta} + g_{\beta\alpha}\mathrm{d}g_{\alpha\beta}\,. \tag{16.6}$$

Conversely, if $\{(U_\alpha, h_\alpha)\}_{\alpha\in\mathcal{A}}$ is a collection of trivializations covering M with transition data $\{g_{\alpha\beta}\}_{\alpha,\beta\in\mathcal{A}}$, a collection

$$\big\{\theta^\alpha \in \Gamma(U_\alpha; T^*M\otimes \mathrm{Mat}_{k\times k}\mathbb{R})\big\}_{\alpha\in\mathcal{A}}$$

satisfying (16.6) determines a connection in V by (16.5).

If ∇ is a connection in a vector bundle $\pi\colon V\longrightarrow M$, a smooth map $f\colon X\longrightarrow M$ induces a connection

$$\nabla^f\colon \Gamma(X; f^*V) \longrightarrow \Gamma(X; T^*X\otimes f^*V)$$

in the vector bundle $f^*V\longrightarrow X$ as follows. Let $\{(U_\alpha, h_\alpha)\}_{\alpha\in\mathcal{A}}$ be a collection of trivializations for V covering M with transition data $\{g_{\alpha\beta}\}_{\alpha\beta\in\mathcal{A}}$ and $\{\theta^\alpha\}_{\alpha\in\mathcal{A}}$ the corresponding collection of connection one-forms. Then, $\{(f^{-1}(U_\alpha), f^*h_\alpha)\}_{\alpha\in\mathcal{A}}$ is a collection of trivializations for the vector bundle $f^*V\longrightarrow X$ covering X with transition data $\{f^*g_{\alpha\beta}\}_{\alpha\beta\in\mathcal{A}}$, while

$$\{f^*\theta_\alpha \in \Gamma(f^{-1}(U_\alpha); T^*X\otimes \mathrm{Mat}_{k\times k}\mathbb{R})\}_{\alpha\in\mathcal{A}}$$

is a collection satisfying

$$f^*\theta^\beta = (f^*g_{\beta\alpha})(f^*\theta^\alpha)(f^*g_{\alpha\beta}) + (f^*g_{\beta\alpha})\mathrm{d}(f^*g_{\alpha\beta})\,,$$

since $f^*\mathrm{d} = \mathrm{d}f^*$. Thus, the collection $\{f^*\theta_\alpha\}_{\alpha\in\mathcal{A}}$ determines a connection ∇^f in f^*V. The connection ∇^f is independent of the choice of the collection $\{(U_\alpha, h_\alpha)\}_{\alpha\in\mathcal{A}}$, since any two such collections can be joined into one, while ∇^f is completely determined by any subcollection covering M.

Recall from Section 4 that a smooth curve on M is a smooth map $\gamma\colon (a,b)\longrightarrow M$. For $t\in(a,b)$, the tangent vector to a smooth curve γ at t is the vector

$$\gamma'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) \equiv \mathrm{d}_t\gamma\big(\partial_{e_1}|_t\big) \in T_{\gamma(t)}M,$$

where $e_1=1\in\mathbb{R}^1$ is the oriented unit vector. In particular, $\gamma'\in\Gamma((a,b);\gamma^*TM)$.

Definition 16.2. *Let M be a smooth manifold and ∇ a connection in the tangent bundle $TM\longrightarrow M$ of M. A ∇-geodesic is a smooth curve*

$$\gamma\colon (a,b)\longrightarrow M \qquad s.t. \qquad \nabla^\gamma\gamma'\big| = 0 \quad \forall\, t\in(a,b). \tag{16.7}$$

If ∇ is a connection in TM and $\varphi=(x_1,\ldots,x_m)\colon U\longrightarrow\mathbb{R}^m$ is a smooth chart on M, there exists $\Gamma_{ij}^k\in C^\infty(U)$ such that

$$\nabla\frac{\partial}{\partial x_j} = \sum_{k=1}^{k=m}\sum_{i=1}^{i=m}\Gamma_{ij}^k\mathrm{d}x_i\otimes\frac{\partial}{\partial x_k} \qquad \forall\, j=1,2,\ldots,m.$$

For any smooth map $\gamma\colon (a,b)\longrightarrow U\subset M$, let

$$(\gamma_1,\ldots,\gamma_m) = \varphi\circ\gamma\colon (a,b)\longrightarrow\mathbb{R}^m\,.$$

By the construction of ∇^γ above,

$$\nabla^\gamma\Big(\gamma^*\frac{\partial}{\partial x_j}\Big) = \sum_{k=1}^{k=m}\sum_{i=1}^{i=m}\gamma^*(\Gamma_{ij}^k\mathrm{d}x_i)\otimes\Big(\gamma^*\frac{\partial}{\partial x_k}\Big) = \sum_{k=1}^{k=m}\sum_{i=1}^{i=m}(\Gamma_{ij}^k\circ\gamma)\frac{\mathrm{d}\gamma_i}{\mathrm{d}t}\mathrm{d}t\otimes\Big(\gamma^*\frac{\partial}{\partial x_k}\Big)$$

for all $j = 1, 2, \ldots, m$. Thus, by (16.5),

$$\nabla^\gamma \gamma'(t) = \sum_{k=1}^{k=m} \left[\frac{\mathrm{d}^2\gamma_k}{\mathrm{d}t^2} + \sum_{i=1}^{i=m}\sum_{j=1}^{j=m} (\Gamma_{ij}^k \circ \gamma) \left(\frac{\mathrm{d}\gamma_i}{\mathrm{d}t} \right) \left(\frac{\mathrm{d}\gamma_j}{\mathrm{d}t} \right) \right] \mathrm{d}t \otimes \left(\gamma^* \frac{\partial}{\partial x_k} \right). \tag{16.8}$$

Thus, if $t_0 \in \mathbb{R}$ and $\gamma\colon (a,b) \longrightarrow M$ is a ∇-geodesic, then so is

$$\tilde{\gamma}\colon (a-t_0, b-t_0) \longrightarrow M, \qquad \tilde{\gamma}(t) = \gamma(t+t_0).$$

Lemma 16.3. *Let ∇ be a connection in the tangent bundle $TM \longrightarrow M$ of a smooth manifold M. For every $v \in TM$, there exists a ∇-geodesic $\gamma\colon (-\epsilon, \epsilon) \longrightarrow M$ such that $\gamma'(0) = v$. If*

$$\gamma, \tilde{\gamma} : (-\epsilon, \epsilon) \longrightarrow M$$

are two such ∇-geodesics, then $\gamma = \tilde{\gamma}$.

Proof. Let $\varphi = (x_1, \ldots, x_m)\colon (U, p) \longrightarrow (\mathbb{R}^m, \mathbf{0})$ be a smooth chart on M. By (16.8), $\gamma\colon (-\epsilon, \epsilon) \longrightarrow U$ is a ∇-geodesic such that $\gamma(0) = p$ and $\gamma'(0) = v$ if and only if

$$\begin{cases} \frac{\mathrm{d}^2\gamma_k}{\mathrm{d}t^2} = -\sum\limits_{i=1}^{i=m}\sum\limits_{j=1}^{j=m} (\Gamma_{ij}^k \circ \gamma) \left(\frac{\mathrm{d}\gamma_i}{\mathrm{d}t} \right) \left(\frac{\mathrm{d}\gamma_j}{\mathrm{d}t} \right) \\ \gamma_k(0) = 0, \quad \left. \frac{\mathrm{d}\gamma_k}{\mathrm{d}t} \right|_{t=0} = v(x_k) \end{cases} \qquad \forall\, k = 1, 2, \ldots, m. \tag{16.9}$$

This system of m second-order ODEs is equivalent to a system of $2m$ first-order ODEs. By the *Existence Theorem for First-Order Differential Equations* [1, A.2], this system has a solution

$$(\gamma_1, \ldots, \gamma_m)\colon (-\epsilon, \epsilon) \longrightarrow \mathbb{R}^m$$

for some $\epsilon > 0$. By the *Uniqueness Theorem for First-Order Differential Equations* [1, A.1], any two solutions of this initial-value problem must agree on the intersection of the domains of their definition. □

Corollary 16.4. *Let ∇ be a connection in the tangent bundle $TM \longrightarrow M$ of a smooth manifold M. If $a, \tilde{a} \in \mathbb{R}^-$, $b, \tilde{b} \in \mathbb{R}^+$, and $\gamma : (a,b) \longrightarrow M$ and $\tilde{\gamma} : (\tilde{a}, \tilde{b}) \longrightarrow M$ are ∇-geodesics such that $\gamma'(0) = \tilde{\gamma}'(0)$, then*

$$\gamma|_{(a,b)\cap(\tilde{a},\tilde{b})} = \tilde{\gamma}|_{(a,b)\cap(\tilde{a},\tilde{b})}.$$

Proof. The subset

$$A \equiv \big\{ t \in (a,b) \cap (\tilde{a}, \tilde{b})\colon\ \gamma(t) = \tilde{\gamma}(t) \big\} \subset (a,b) \cap (\tilde{a}, \tilde{b})$$

is nonempty (as it contains 0) and closed (as γ and $\tilde{\gamma}$ are continuous). Since $(a,b) \cap (\tilde{a}, \tilde{b})$ is connected, it is sufficient to show that S is open. If

$$t_0 \in S \qquad \text{and} \qquad (t_0-\epsilon, t_0+\epsilon) \subset (a,b) \cap (\tilde{a}, \tilde{b}),$$

define smooth curves

$$\alpha, \beta\colon (-\epsilon, \epsilon) \longrightarrow M \qquad \text{by} \qquad \alpha(t) = \gamma(t+t_0), \quad \beta(t) = \tilde{\gamma}(t+t_0).$$

Since γ and $\tilde{\gamma}$ are ∇-geodesics, so are α and β; see the sentence preceding Lemma 16.3. Since

$$\alpha'(0) = \gamma'(t_0) = \tilde{\gamma}'(t_0) = \beta'(0),$$

$\alpha = \beta$ by Lemma 16.3 and thus $(t_0-\epsilon, t_0+\epsilon) \subset (a,b) \cap (\tilde{a}, \tilde{b})$. □

Corollary 16.5. *Let ∇ be a connection in the tangent bundle $TM\longrightarrow M$ of a smooth manifold M. For every $v\in TM$, there exists a unique maximal ∇-geodesic $\gamma_v\colon(a_v,b_v)\longrightarrow M$ such that $\gamma_v'(0)=v$, where $a_v\in[-\infty,0)$ and $b_v\in(0,\infty]$. If $t\in(a_v,b_v)$, then*

$$\big(a_{\gamma_v'(t)},b_{\gamma_v'(t)}\big)=(a_v-t,b_v-t),\qquad \gamma'_{\gamma_v'(t)}(-t)=v. \tag{16.10}$$

Proof. (1) Let $\{\gamma_\alpha\colon(a_\alpha,b_\alpha)\longrightarrow M\}_{\alpha\in\mathcal{A}}$ be the collection of all ∇-geodesics such that $\gamma_\alpha'(0)=v$. Define

$$(a_v,b_v)=\bigcup_{\alpha\in\mathcal{A}}(a_\alpha,b_\alpha),\qquad \gamma_v\colon(a_v,b_v)\longrightarrow M,\quad \gamma_v(t)=\gamma_\alpha(t)\quad\forall\,t\in(a_\alpha,b_\alpha),\ \alpha\in\mathcal{A}.$$

By Corollary 16.4, $\gamma_\alpha(t)$ is independent of the choice of $\alpha\in\mathcal{A}$ such that $t\in(a_\alpha,b_\alpha)$. Thus, γ_v is well-defined. It is smooth, since its restriction to each open subset (a_α,b_α) is smooth and these subsets cover (a_v,b_v). It is a ∇-geodesic, since this is the case on the open subsets (a_α,b_α). It is immediate that $\gamma_v'(0)=v$. By construction, γ_v is a maximal ∇-geodesic.

(2) If $t\in(a_v,b_v)$, define

$$\gamma\colon(a_v-t,b_v-t)\longrightarrow M\qquad\text{by}\qquad \gamma(\tau)=\gamma_v(\tau+t).$$

By the sentence preceding Lemma 16.3, γ is a ∇-geodesic. Furthermore,

$$\gamma'(0)=\gamma_v'(t),\qquad \gamma'(-t)=\gamma_v'(0)=v.$$

Thus, by the first statement of Corollary 16.5,

$$\begin{aligned}\big(a_{\gamma_v'(t)},b_{\gamma_v'(t)}\big)\supset(a_v-t,b_v-t),\ \gamma_{\gamma_v'(t)}\big|_{(a_v-t,b_v-t)}=\gamma \quad&\Longrightarrow\quad -t\in\big(a_{\gamma_v'(t)},b_{\gamma_v'(t)}\big),\ \gamma'_{\gamma_v'(t)}(-t)=v\\ &\Longrightarrow\quad (a_v,b_v)=\big(a_{\gamma'_{\gamma_v'(t)}(-t)},b_{\gamma'_{\gamma_v'(t)}(-t)}\big)\supset\big(a_{\gamma_v'(t)}+t,b_{\gamma_v'(t)}+t\big).\end{aligned}$$

This confirms (16.10). ☐

If ∇ is a connection in the tangent bundle $TM\longrightarrow M$ of a smooth manifold M and $t\in\mathbb{R}$, let

$$\mathrm{Dom}_t(\nabla)=\big\{v\in TM\colon t\in(a_v,b_v)\big\},\qquad \Psi_t\colon\mathrm{Dom}_t(\nabla)\longrightarrow TM,\quad \Psi_t(v)=\gamma_v'(t).$$

Proposition 16.6. *If ∇ is a connection in the tangent bundle $\pi\colon TM\longrightarrow M$ of a smooth manifold M, then*

(1) $\mathrm{Dom}_0(\nabla)=TM$, $\exp_0=\mathrm{id}_{TM}$, $M\subset\mathrm{Dom}_t(\nabla)$ *for all* $t\in\mathbb{R}$, *and*

$$TM=\bigcup_{t>0}\mathrm{Dom}_t(\nabla)=\bigcup_{t<0}\mathrm{Dom}_t(\nabla);$$

(2) for all $s,t\in\mathbb{R}$, $\Psi_{s+t}=\Psi_s\circ\Psi_t\colon\mathrm{Dom}(\Psi_s\circ\Psi_t)=\Psi_t^{-1}\big(\mathrm{Dom}_s(\nabla)\big)\longrightarrow TM$;

(3) for all $v\in TM$, there exist an open neighborhood U of v in TM and $\epsilon\in\mathbb{R}^+$ such that the map

$$\Psi\colon(-\epsilon,\epsilon)\times U\longrightarrow TM,\qquad (t,v')\longrightarrow\Psi_t(v')\equiv\gamma_{v'}'(t), \tag{16.11}$$

is defined and smooth;

(4) for all $t\in\mathbb{R}$, $\mathrm{Dom}_t(\nabla)\subset TM$ *is an open subset;*

(5) for all $t\in\mathbb{R}$, $\Psi_t\colon \mathrm{Dom}_t(\nabla)\longrightarrow \mathrm{Dom}_{-t}(\nabla)$ *is a diffeomorphism with inverse* Ψ_{-t}*.*

Proof. (1) By Lemma 16.3, for each $v\in TM$ there exists a ∇-geodesic $\gamma\colon(-\epsilon,\epsilon)\longrightarrow M$ such that $\gamma'(0)=v$. Thus, $v\in\mathrm{Dom}_{\pm\epsilon/2}(\nabla)\subset\mathrm{Dom}_0(\nabla)$; this implies the first and last claims in (1). For the third claim, note that any constant map $\mathbb{R}\longrightarrow M$ is a ∇-geodesic. The second claim follows from the requirement that $\Psi_0(v)\equiv\gamma_v'(0)=v$ for all $v\in TM$.

(2) Since $\mathrm{Dom}(\Psi_s)=\mathrm{Dom}_s(\nabla)$, $\mathrm{Dom}(\Psi_s\circ\Psi_t)=\Psi_t^{-1}(\mathrm{Dom}_s(\nabla))$. If $v\in\Psi_t^{-1}(\mathrm{Dom}_s(\nabla))$,

$$s\in\big(a_{\Psi_t(v)},b_{\Psi_t(v)}\big)=\big(a_{\gamma_v'(t)},b_{\gamma_v'(t)}\big).$$

Thus, $s+t\in(a_v,b_v)$ by (16.10) and $\Psi_t^{-1}(\mathrm{Dom}_s(\nabla))\subset\mathrm{Dom}(\Psi_{s+t})$. Define

$$\gamma\colon\big(a_{\Psi_t(v)},b_{\Psi_t(v)}\big)\longrightarrow TM \qquad \text{by} \qquad \gamma(\tau)=\gamma_v(\tau+t);$$

by (16.10), $\gamma_v(\tau+t)$ is defined for all $\tau\in(a_{\Psi_t(v)},b_{\Psi_t(v)})$. By the sentence preceding Lemma 16.3, γ is a ∇-geodesic. Furthermore, $\gamma'(0)=\gamma_v'(t)=\Psi_t(v)$. Thus, by Corollary 16.5, $\gamma=\gamma_{\Psi_t(v)}$ and so

$$\Psi_{s+t}(v)\equiv\gamma_v(s+t)=\gamma(s)=\gamma_{\Psi(v)}(s)\equiv\Psi_s\big(\Psi_t(v)\big)$$

for all $s\in(a_{\Psi_t(v)},b_{\Psi_t(v)})$.

(3) As in the proof of Lemma 16.3, the requirement for a smooth map $\gamma:(a,b)\longrightarrow M$ to be a ∇-geodesic with $\gamma'(0)=v$ corresponds to an initial-value problem (16.9) in a smooth chart around $\pi(v)$. Thus, the claim follows from the smooth dependence of solutions of (16.9) on the parameters [1, A.4].

(4) Since $\mathrm{Dom}_0(\nabla)=TM$, it is sufficient to prove this statement for $t\in\mathbb{R}^*$. We consider the case $t\in\mathbb{R}^+$; the case $t\in\mathbb{R}^-$ is proved similarly. Let $v\in\mathrm{Dom}_t(\nabla)$ and $W\subset TM$ be an open neighborhood of $\Psi_t(v)=\gamma_v'(t)$ in TM. Since the interval $[0,t]$ is compact, by (3) and Lebesgue Number Lemma (Lemma B.1.2), there exist $\epsilon>0$ and a neighborhood U of $\gamma_v'([0,t])$ such that the map (16.11) is defined and smooth. Let $n\in\mathbb{Z}^+$ be such that $t/n<\epsilon$. We inductively define subsets $W_i\subset TM$ by

$$W_n=W,\qquad W_i=\Psi_{t/n}^{-1}(W_{i+1})\cap U=\big\{\Psi_{t/n}|_U\big\}^{-1}(W_{i+1})\quad\forall\,i=0,1,\ldots,n-1.$$

By induction, $W_i\subset U$ is an open neighborhood of $\gamma_v'(it/n)$, $W_i\subset\Psi_{t/n}^{-1}(\mathrm{Dom}(\Psi_{(n-1-i)t/n}))$, and thus

$$\Psi_{(n-i)t/n}=\Psi_{t/n}\circ\Psi_{(n-1-i)t/n}\colon W_i\longrightarrow U\subset TM$$

by (2). It follows that $W_0\subset TM$ is an open neighborhood of v in TM such that $W_0\subset\mathrm{Dom}_t(\nabla)$.

(5) By (16.10) and (2), $\mathrm{Im}\,\Psi_t=\mathrm{Dom}_{-t}(\nabla)$ and Ψ_{-t} is the inverse of Ψ_t. If $v\in\mathrm{Dom}_t(\nabla)$ and W_0 is a neighborhood of v in TM as in the proof of (4), $\Psi_t|_{W_0}$ is a smooth map. Thus, Ψ_t is smooth on the open subset $\mathrm{Dom}_t(\nabla)\subset TM$. □

Definition 16.7. *Let* ∇ *be a connection in the tangent bundle* $\pi\colon TM\longrightarrow M$ *of* M *of a smooth manifold* M*. The* exponential map for ∇ *is the map*

$$\exp^\nabla\colon\mathrm{Dom}_1(\nabla)\longrightarrow M,\qquad v\longrightarrow\pi(\Psi_1(v))=\gamma_v(1).$$

Remark 16.8. A connection ∇ in a vector bundle $\pi: V \longrightarrow M$ provides a splitting of the short exact sequence (11.2), i.e. a vector-bundle homomorphism

$$j_\nabla: \pi^*TM \longrightarrow TV \qquad \text{s.t.} \qquad \mathrm{d}\pi \circ j_\nabla = \mathrm{id}_{\pi^*TM}$$

over (the total space of) V, as follows. If $s \in \Gamma(M;V)$, $p \in M$, and $w \in T_pM$, let

$$j_\nabla\big(s(p), w\big) \equiv \mathrm{d}_p s(w) - \tilde{\iota}(\nabla_v s).$$

By a direct check in a trivialization, $j_\nabla(fs(p), w) = j_\nabla(s(p), w)$ for any $f \in C^\infty(M)$ with $f(p)=1$. Thus, the bundle homomorphism j is well-defined. A connection ∇ in $TM \longrightarrow M$ also determines a smooth vector field X_∇ on TM by

$$X_\nabla(v) = j_\nabla(v,v) \in T_v(TM).$$

A smooth curve $\gamma: (a,b) \longrightarrow M$ is a ∇-geodesic if and only if $\gamma': (a,b) \longrightarrow TM$ is an integral flow for vector field X_∇ on TM; see Definition 17.1. Thus, Lemma 16.3, Corollaries 16.4 and 16.5, and Proposition 16.6 are special cases of Lemma 17.2, Corollaries 17.3 and 17.4, and Proposition 17.7, respectively. We include their proofs for the same of completeness, since the primary purpose of Section 17 is completely independent of the primary purpose of this section.

By Proposition 16.6, $\exp^\nabla$ is a smooth map from an open neighborhood of M in TM to M restricts to the identity on M. By the construction of $\exp^\nabla$,

$$\mathrm{d}_p \exp^\nabla = \big(\mathrm{id}_{T_pM}, \mathrm{id}_{T_pM}\big): T_p(TM) \approx T_pM \oplus T_pM \longrightarrow T_pM \qquad \forall\, p \in M \tag{16.12}$$

under the canonical isomorphism $T_p(TM) \approx T_pM \oplus T_pM$ induced by the map $\tilde{\iota}$ of Example 11.6.

Proposition 16.9. *If X is a compact submanifold of a smooth manifold M, there exists a diffeomorphism between neighborhoods W and W' of X in $\mathcal{N}_X M$ and in M, respectively,*

$$f: W \longrightarrow W' \qquad s.t. \qquad f(p)=p \quad \forall\, p \in X.$$

Proof. (1) Let ∇ be a connection in the tangent bundle $\pi: TM \longrightarrow M$ and $\exp^\nabla: U \longrightarrow M$ its exponential map, where U is a neighborhood of M in TM. Let

$$TX^\perp \equiv \{v \in TM|_X: \langle v, w\rangle = 0 \ \forall\, w \in T_{\pi(v)}X\}$$

be the orthogonal complement of the subbundle $TX \subset TM|_X$ with respect to a Riemannian metric $\langle , \rangle$ in $TM|_X$. Since $TX^\perp \cap U \subset U$ is a smooth submanifold, the restriction

$$\exp: TX^\perp \cap U \longrightarrow M$$

is a smooth map which restricts to the identity on X. By (16.12),

$$\mathrm{d}_p \exp: T_p(TX^\perp) = T_pX \oplus T_pX^\perp \longrightarrow T_pM$$

is the inclusion map on each component and thus an isomorphism. By the *Inverse Function Theorem for Manifolds* (Corollary 4.9), for each $p \in X$ there are neighborhoods U_p and U_p' of p in $TX^\perp$ and M, respectively, such that the restriction $\exp^\nabla: U_p \longrightarrow U_p'$ is a diffeomorphism. Let

$$U_0 = \bigcup_{p \in X} U_p, \qquad U_k = \big\{v \in U_0: \langle v, v\rangle < 1/k\big\} \ \ \forall\, k = 1, 2, \ldots;$$

these are neighborhoods of X in $TX^\perp$. Since exp is a local diffeomorphism on U_0, $\exp(U_k)\subset M$ is an open subset. We show below that exp is injective on U_k if k is sufficiently large and thus a diffeomorphism from the neighborhood U_k of X in $TX^\perp$ to the neighborhood $\exp(U_k)$ of X in M. Since $\mathcal{N}_X M, TX^\perp \longrightarrow X$ are isomorphic vector bundles, this implies the claim.

(2) Let $v_k, w_k \in U_k$ be two sequences such that $v_k \neq w_k$, but $\exp(v_k)=\exp(w_k)$. Since X is compact, after passing to subsequences if necessary, we can assume that $v_k \longrightarrow p$ and $w_k \longrightarrow q$ for some $p,q\in X$. Since exp is injective on U_p and $v_k\in U_p$ for all k sufficiently large, $w_k\notin U_p$ for all k sufficiently large and thus $p\neq q$. Let U_p' and U_q' be disjoint neighborhoods of p and q in M. Since $v_k\in\exp^{-1}(U_p')$ and $w_k\in\exp^{-1}(U_q')$ for all k sufficiently large, $\exp(v_k)\neq\exp(w_k)$ for such values of k, contrary to the assumption. □

Exercises

1. Let $\pi\colon V\longrightarrow M$ be a vector bundle. Show that

(a) the scalar-multiplication map (7.1) is smooth;

(b) the space $V\times_M V$ is a smooth submanifold of $V\times V$ and the addition map (7.2) is smooth.

2. Let $\pi: V \longrightarrow M$ be a smooth vector bundle of rank k and $\{(U_\alpha, h_\alpha)\}_{\alpha\in\mathcal{A}}$ a collection of trivializations covering M. Show that a section s of π is continuous (smooth) if and only if the map

$$s_\alpha \equiv \pi_2\circ h_\alpha\circ s\colon U_\alpha \longrightarrow \mathbb{R}^k,$$

where $\pi_2\colon U_\alpha\times\mathbb{R}^k\longrightarrow\mathbb{R}^k$ is the projection on the second component, is continuous (smooth) for every $\alpha\in\mathcal{A}$.

3. Let $\pi\colon V\longrightarrow M$ be a submersion satisfying ($\mathbb{R}$VB1)-($\mathbb{R}$VB3) in Definition 7.1. Show that

(a) if $s_1,\ldots,s_k\colon U\longrightarrow V|_U$ are smooth sections over an open subset $U\subset M$ such that $\{s_i(x)\}_i$ is a basis for V_x for all $x\in U$, then the map (8.2) is a diffeomorphism;

(b) $\pi: V\longrightarrow M$ is a vector bundle of rank k if and only if for every $p\in M$ there exist a neighborhood U of p in M and smooth sections $s_1,\ldots,s_k\colon U\longrightarrow V|_U$ such that $\{s_i(p)\}_i$ is a basis for V_p.

4. Show that the two versions of the last condition on $\tilde f$ in (2) in Definition 8.2 are indeed equivalent.

5. Let M be a smooth manifold and $X,Y,Z\in\mathrm{VF}(M)$. Show that

(a) $[X,Y]$ is indeed a smooth vector field on M and

$$[fX,gY]=fg[X,Y]+f(Xg)Y-g(Yf)X \qquad \forall\, f,g\in C^\infty(M);$$

(b) $[\cdot,\cdot]$ is bilinear, anti-symmetric, and

$$\big[X,[Y,Z]\big]+\big[Y,[Z,X]\big]+\big[Z,[X,Y]\big]=0.$$

6. Verify all claims made in Example 7.5, thus establishing that the tangent bundle TM of a smooth manifold is indeed a vector bundle. What is its transition data?

7. Show that the tangent bundle TS^1 of S^1 is isomorphic to the trivial real line bundle over S^1.

8. Show that the tautological line bundle $\gamma_n \longrightarrow \mathbb{R}P^n$ is non-trivial for $n \geq 1$.

9. Show that the complex tautological line bundle $\gamma_n \longrightarrow \mathbb{C}P^n$ is indeed a complex line bundle as claimed in Example 7.8. What is its transition data? Why is it non-trivial for $n \geq 1$?

10. Let $q\colon \tilde{M} \longrightarrow M$ be a smooth covering projection. Show that

(a) the map $\mathrm{d}q\colon \tilde{M} \longrightarrow M$ is a covering projection and a bundle homomorphism covering q as in (8.4);

(b) there is a natural isomorphism

$$\mathrm{VF}(M) \approx VF(\tilde{M})^{\mathrm{d}q} \equiv \big\{X \in \mathrm{VF}\colon\ \mathrm{d}_{p_1}q(X(p_1)) = \mathrm{d}_{p_2}q(X(p_2))\ \forall\, p_1, p_2 \in M \text{ s.t. } q(p_1) = q(p_2)\big\}.$$

11. Let M be a smooth m-manifold. Show that

(TM1) the topology on TM constructed in Example 7.5 is the unique one so that $\pi\colon TM \longrightarrow M$ is a topological vector bundle with the canonical vector-space structure on the fibers and so that for every vector field X on TM and smooth function $f\colon U \longrightarrow \mathbb{R}$, where U is an open subset of $\mathbb{R}$, the function $X(f)\colon U \longrightarrow \mathbb{R}$ is continuous if and only if X is continuous;

(TM2) the smooth structure on TM constructed in Example 7.5 is the unique one so that $\pi\colon TM \longrightarrow M$ is a smooth vector bundle with the canonical vector-space structure on the fibers and so that for every vector field X on TM and smooth function $f\colon U \longrightarrow \mathbb{R}$, where U is an open subset of $\mathbb{R}$, the function $X(f)\colon U \longrightarrow \mathbb{R}$ is smooth if and only if X is smooth.

12. Suppose that $f\colon M \longrightarrow N$ is a smooth map and $\pi\colon V \longrightarrow N$ is a smooth vector bundle of rank k with transition data $\{g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}_n\mathbb{R}\}_{\alpha,\beta\in\mathcal{A}}$. Show that

(a) the space f^*V defined by (10.1) is a smooth submanifold of $M \times V$ and the projection $\pi_1\colon f^*V \longrightarrow M$ is a vector bundle of rank k with transition data

$$\big\{f^*g_{\alpha\beta} = g_{\alpha\beta} \circ f\colon f^{-1}(U_\alpha) \cap f^{-1}(U_\beta) \longrightarrow \mathrm{GL}_n\mathbb{R}\}_{\alpha,\beta\in\mathcal{A}}\,;$$

(b) if M is an embedded submanifold of N and f is the inclusion map, then the projection $\pi_2\colon f^*V \longrightarrow V$ induces an isomorphism $f^*V \longrightarrow V|_M$ of vector bundles over M.

13. Let $f\colon M \longrightarrow V$ be a smooth map and $V \longrightarrow N$ a vector bundle. Show that

(a) if $V \longrightarrow N$ is a trivial vector bundle, then so is $f^*V \longrightarrow M$;

(b) $f^*V \longrightarrow M$ may be trivial even if $V \longrightarrow N$ is not.

14. Let $f\colon M \longrightarrow N$ be a smooth map. Show that the bundle homomorphisms in diagrams (10.4) and (10.5) are indeed smooth.

15. Verify Lemma 10.2.

16. Let $f\colon M \longrightarrow N$ be a smooth map and $\varphi\colon V \longrightarrow W$ a smooth vector-bundle homomorphism over N. Show that the pullback vector-bundle homomorphism $f^*\varphi\colon f^*V \longrightarrow f^*W$ is also smooth.

17. Let $\pi\colon V \longrightarrow M$ be a smooth vector bundle of rank k and $V' \subset V$ a smooth submanifold so that $V'_p \equiv V_p \cap V'$ is a k'-dimensional linear subspace of V_p for every $p \in M$. Show that

(a) for every $p \in M = s_0(M)$ there exist an open neighborhood U of p in V' and smooth charts

$$\varphi\colon U \longrightarrow \mathbb{R}^m \times \mathbb{R}^{k'} \quad \text{and} \quad \psi\colon U \cap M \longrightarrow \mathbb{R}^m \qquad \text{s.t.} \qquad \psi \circ \pi = \pi_1 \circ \varphi\,,$$

where $\pi_1\colon \mathbb{R}^m \times \mathbb{R}^{k'} \longrightarrow \mathbb{R}^m$ is the projection on the first component;

(b) $V' \subset V$ is a vector subbundle of rank k'.

18. Let $\varphi : V \longrightarrow W$ be a smooth surjective vector-bundle homomorphism over a smooth manifold M. Show that

$$\ker \varphi \equiv \big\{ v \in V :\ \varphi(v) = 0 \big\} \longrightarrow M$$

is a subbundle of V.

19. Let $\mathcal{D} \subset TM$ a rank 1 distribution on a smooth manifold M. Show that $\Gamma(M;\mathcal{D}) \subset \mathrm{VF}(M)$ is a Lie subalgebra. *Hint:* use Exercise 5.

20. Let $\{\iota_\alpha\colon M_\alpha \longrightarrow N\}_{\alpha\in\mathcal{A}}$ be a foliation of N^n by immersions from m-manifolds. Show that

$$\mathcal{D} \equiv \bigcup_{\alpha\in\mathcal{A}} \bigcup_{p\in M_\alpha} \mathrm{Im}\, \mathrm{d}_p\iota_\alpha \subset TN$$

is a subbundle of rank m.

21. Verify all claims made in Examples 11.4 and 11.5.

22. Verify all claims made in Example 11.7.

23. Let $V \longrightarrow M$ be a vector bundle of rank k and $V' \subset V$ a smooth subbundle of rank k'. Show that

(a) there exists a collection $\{(U_\alpha, h_\alpha)\}_{\alpha\in\mathcal{A}}$ of trivializations for V covering M so that (11.3) holds and thus the corresponding transition data has the form

$$g_{\alpha\beta} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} : U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}_k\mathbb{R},$$

where the top left block is $k' \times k'$;

(b) the vector-bundle structure on V/V' described in Section 11 is the unique one so that the natural projection map $V \longrightarrow V/V'$ is a smooth vector-bundle homomorphism;

(c) if $\varphi\colon V \longrightarrow W$ is a vector-bundle homomorphism over M such that $\varphi(v) = 0$ for all $v \in V'$, then the induced vector-bundle homomorphism $\bar{\varphi}\colon V/V' \longrightarrow W$ is smooth.

24. Verify Lemmas 11.8 and 11.9.

25. Obtain Corollary 11.12 from Theorem 11.11.

26. Let $f = (f_1, \ldots, f_k) : \mathbb{R}^m \longrightarrow \mathbb{R}^k$ be a smooth map, $q \in \mathbb{R}^k$ a regular value of f, and $X = f^{-1}(q)$. Denote by ∇f_i the gradient of f_i. Show that

$$TX = \big\{ (p,v) \in X \times \mathbb{R}^m :\ \nabla f_i|_p \cdot v = 0\ \forall\, i = 1, 2, \ldots, k \big\}$$

under the canonical identifications $TX \subset T\mathbb{R}^m|_X$ and $T\mathbb{R}^m = \mathbb{R}^m \times \mathbb{R}^m$. Use this description of TX to give a trivialization of $\mathcal{N}_{\mathbb{R}^m}X$.

27. Let $V, V' \longrightarrow M$ be smooth vector bundles. Show that the two constructions of $V \oplus V'$ in Section 12 produce the same vector bundle and that this is the unique vector-bundle structure on the total space of

$$V \oplus V' = \bigsqcup_{p \in M} V_p \oplus V'_p$$

so that

(VB$\bigoplus$1) the projection maps $V \oplus V' \longrightarrow V, V'$ are smooth bundle homomorphisms over M;

(VB$\bigoplus$2) the inclusion maps $V, V' \longrightarrow V \oplus V'$ are smooth bundle homomorphisms over M.

28. Let $\pi_V : V \longrightarrow M$ and $\pi_W : W \longrightarrow N$ be smooth vector bundles and $\pi_M, \pi_N : M \times N \longrightarrow M, N$ the component projection maps. Show that the total of the vector bundle

$$\pi : \pi_M^* V \oplus \pi_N^* W \longrightarrow M \times N$$

is $V \times W$ (with the product smooth structure) and $\pi = \pi_V \times \pi_W$.

29. Verify Lemmas 12.1 and 12.2.

30. Let M and N be smooth manifolds and $\pi_M, \pi_N : M \times N \longrightarrow M, N$ the projection maps. Show that $\mathrm{d}\pi_M$ and $\mathrm{d}\pi_N$ viewed as maps from $T(M \times N)$ to

(a) TM and TN, respectively, induce a diffeomorphism $T(M \times N) \longrightarrow TM \times TN$ that commutes with the projections from the tangent bundles to the manifolds and is linear on the fibers of these projections;

(b) $\pi_M^* TM$ and $\pi_N^* TN$, respectively, induce a vector-bundle isomorphism

$$T(M \times N) \longrightarrow \pi_M^* TM \oplus \pi_N^* TN.$$

Why are the above two statements the same?

31. Verify Lemmas 12.3 and 12.4.

32. Show that the vector-bundle structure on the total space of V^* constructed in Section 12 is the unique one so that the map (12.2) is smooth.

33. Verify Lemmas 13.1-13.3.

34. Show that the sets of isomorphism classes of real and complex line bundles form abelian group under the tensor product.

35. Let $V \longrightarrow M$ be a smooth vector bundle of rank k and $W \subset V$ a smooth subbundle of V of rank k'. Show that

$$\mathrm{Ann}(W) \equiv \big\{ \alpha \in V_p^* : \ \alpha(w) = 0 \ \forall \, w \in W, \, p \in M \big\}$$

is a smooth subbundle of V^* of rank $k - k'$.

36. Verify Lemmas 13.4-13.7.

37. Let $\pi : V \longrightarrow M$ be a vector bundle. Show that there is an isomorphism

$$\Lambda^k(V^*) \longrightarrow \big(\Lambda^k V\big)^*$$

of vector bundles over M.

38. Let Ω be a volume form an m-manifold M. Show that for every $p\in M$ there exists a chart $(x_1,\ldots,x_m)\colon U\longrightarrow\mathbb{R}^m$ around p such that

$$\Omega|_U = \mathrm{d}x_1\wedge\ldots\wedge\mathrm{d}x_m.$$

39. Verify Lemmas 14.2 and 14.5.

40. Show that every real vector bundle over a smooth manifold admits a Riemannian metric and every complex vector bundle over a smooth manifold admits a Hermitian metric.

41. Let $\pi\colon L\longrightarrow M$ be a real line bundle over a smooth manifold. Show that $L^{\otimes 2}\approx\tau_1^{\mathbb{R}}$ as real line bundles over M.

42. Let $V,W\longrightarrow M$ be vector bundles. Show that

(a) if V is orientable, then W is orientable if and only if $V\oplus W$ is;

(b) if V and W are non-orientable, then $V\oplus W$ may be orientable or non-orientable.

43. Let M be a connected manifold. Show that every real line bundle $L\longrightarrow M$ is orientable if and only if $\pi_1(M)$ contains no subgroup of index 2.

44. Let M and N be nonempty smooth manifolds. Show that $M\times N$ is orientable if and only if M and N are.

45. (a) Let $\varphi\colon M\longrightarrow\mathbb{R}^N$ be an immersion. Show that M is orientable if and only if the normal bundle to the immersion φ is orientable.

(b) Show that the unit sphere S^n with its natural smooth structure is orientable.

46. Verify Lemmas 15.3 and 15.4.

47. (a) Show that the antipodal map on $S^n\subset\mathbb{R}^{n+1}$ (i.e. $x\longrightarrow -x$) is orientation-preserving if n is odd and orientation-reversing if n is even.

(b) Show that $\mathbb{R}P^n$ is orientable if and only if n is odd.

(c) Describe the orientable double cover of $\mathbb{R}P^n\times\mathbb{R}P^n$ with n even.

48. Let $\gamma_n\longrightarrow\mathbb{C}P^n$ be the tautological line bundle as in Example 7.8. If $P:\mathbb{C}^{n+1}\longrightarrow\mathbb{C}$ is a homogeneous polynomial of degree $d\geq 0$, let

$$s_P\colon\mathbb{C}P^n\longrightarrow\gamma_n^*,\qquad \big\{s_P(\ell)\big\}(\ell,v^{\otimes d})=P(v)\;\;\forall\,(\ell,v)\in\gamma_n\subset\mathbb{C}P^n\times\mathbb{C}^{n+1}\,.$$

Show that

(a) s_P is a well-defined holomorphic section of $\gamma_n^{*\otimes d}$;

(b) if s is a holomorphic section of $\gamma_n^{*\otimes d}$ with $d\geq 0$, then $s=s_P$ for some homogeneous polynomial $P\colon\mathbb{C}^{n+1}\longrightarrow\mathbb{C}$ of degree d;

(c) the line bundle $\gamma_n^{\otimes}\longrightarrow\mathbb{C}P^n$ admits no nonzero holomorphic section for any $d\in\mathbb{Z}^+$.

49. Let $\gamma_n\longrightarrow\mathbb{C}P^n$ be the tautological line bundle as in Example 7.8. Show that there is a short exact sequence

$$0\longrightarrow\mathbb{C}P^n\times\mathbb{C}\longrightarrow(n+1)\gamma_n^*\longrightarrow T\mathbb{C}P^n\longrightarrow 0$$

of complex (even holomorphic) vector bundles over $\mathbb{C}P^n$.

50. Suppose $k<n$ and let $\gamma_k \longrightarrow \mathbb{C}P^k$ be the tautological line bundle as in Example 7.8. Show that the map

$$\iota\colon \mathbb{C}P^k \longrightarrow \mathbb{C}P^n, \qquad [X_0,\ldots,X_k] \longrightarrow [X_0,\ldots,X_k,\underbrace{0,\ldots,0}_{n-k}],$$

is a complex embedding (i.e. a smooth embedding that induces holomorphic maps between the charts that determine the complex structures on $\mathbb{C}P^k$ and $\mathbb{C}P^n$) and that the normal bundle to this immersion, $\mathcal{N}_\iota$, is isomorphic to

$$(n-k)\gamma_k^* \equiv \underbrace{\gamma_k^* \oplus \ldots \oplus \gamma_k^*}_{n-k}$$

as a complex (even holomorphic) vector bundle over $\mathbb{C}P^k$. *Hint:* there are a number of ways of doing this, including:

(i) use Exercise 49;

(ii) construct an isomorphism between the two vector bundles;

(iii) determine transition data for $\mathcal{N}_\iota$ and $(n-k)\gamma_k^*$;

(iv) show that there exists a holomorphic diffeomorphism between $(n-k)\gamma_k^*$ and a neighborhood of $\iota(\mathbb{C}P^k)$ in $\mathbb{C}P^n$, fixing $\iota(\mathbb{C}P^k)$, and use Lemma 11.10.

51. Let $\gamma_n \longrightarrow \mathbb{C}P^n$ and $\Lambda^n_{\mathbb{C}} T\mathbb{C}P^n \longrightarrow \mathbb{C}P^n$ be the tautological line bundle as in Example 7.8 and the top exterior power of the vector bundle $T\mathbb{C}P^n$ taken over $\mathbb{C}$, respectively. Show that there is an isomorphism

$$\Lambda^n_{\mathbb{C}} T\mathbb{C}P^n \approx \gamma_n^{*\,\otimes(n+1)} \equiv \underbrace{\gamma_n^* \otimes \ldots \otimes \gamma_n^*}_{n+1}$$

of complex (even holomorphic) line bundles over $\mathbb{C}P^n$. *Hint:* see suggestions for Exercise 50.