MAT 320

3/27/18

#4

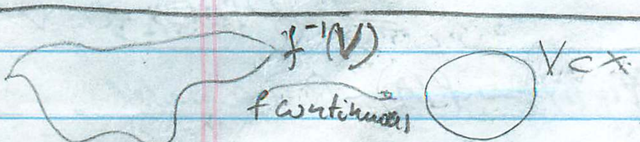
Previously: (X,d) metric space

$\rightsquigarrow$ open/closed sets, compact, connected

This week: Maps between metric spaces / continuity

Today: open set perspective

Th: metric / ε-δ perspective



$f^{-1}(V)\subset V \Longleftarrow V$ open

Examples: (1) $f: X\to Y$ constant map, i.e. $f(x)=y_0\in Y\ \forall x\in X$ is continuous

$$f^{-1}(V)=\begin{cases}\emptyset & \text{if } y_0\notin V\\ X & \text{if } y_0\in V\end{cases}\quad \text{open (even if } V \text{ is not)}$$

$f^{-1}(V)\subset X$ open $\forall V\subset X$ open $\Rightarrow f$ is cont.

(do not erase) #4

Prop: $(X,d_X),(Y,d_Y)$ metric spaces, $f: X\to Y$ continuous

(i) If $A\subset X$ is connected, then $f(A)\subset Y$ is connected

(ii) If $A\subset X$ is cmpt, then $f(A)\subset Y$ is cmpt

(iii) If (Z,d_Z) is another metric and $g: Y\to Z$ is also cont., then $g\circ f: X\to \underset{W}{Y}\to Z$ is cont.

#3

Take $U_i=f^{-1}(V_i)\subset X$ for $i=1,2$

$V_i\subset Y$ open, f continuous $\Rightarrow U_i\subset X$ open

$V_1\cap V_2=\emptyset\Rightarrow U_1\cap U_2=f^{-1}(V_1)\cap f^{-1}(V_2)=\emptyset$

$f(A)\cap V_i\neq\emptyset \Longleftrightarrow A\cap U_i=A\cap f^{-1}(V_i)\neq\emptyset$

$f(A)\subset V_1\cup V_2 \Longleftrightarrow A\subset U_1\cup U_2=f^{-1}(V_1)\cup f^{-1}(V_2)$

$\Rightarrow U_1,U_2$ separate A $\quad U_1\ \boxed{A}\quad \boxed{A}\ U_2$

#1 do not erase

$(X,d_X),(Y,d_Y)$ metric space, $f: X\to Y$ map

Dfn: (a) f is <u>continuous</u> if $\forall V\subset Y$ open, $f^{-1}(V)\subset X$ is also open

(b) f is <u>continuous at $x_0\in X$</u> if $\forall V\subset Y$ open s.t. $f(x_0)\in V$

$\exists U\subset X$ open s.t. $x_0\in U\subset f^{-1}(V) \longleftrightarrow f(U)\subset V$

(c) $\lim_{x\to x_0} f(x)=y_0\in Y$ if $\forall V\subset Y$ open s.t. $y_0\in V$

$\exists U\subset X$ open s.t. $x_0\in U$ and $U-\{x_0\}\subset f^{-1}(V)$

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(2) $f=id_X: X\to X$ is continuous

$V\subset X$ (RHS) open $\Rightarrow f^{-1}(V)=V$ open ✓

(3) $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\begin{cases}0 & \text{if } x\in\mathbb{Q}\\ 1 & \text{if } x\notin\mathbb{Q}\end{cases}$

not continuous: find $V\subset\mathbb{R}$ (RHS) open s.t. $f^{-1}(V)$ is not

E.g. $V=(-\frac{1}{2},\frac{1}{2})$, $f^{-1}(V)=\mathbb{Q}\subset\mathbb{R}$ not open

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#2

Pf of (i): $\underline{f(A)\subset Y \text{ disconnected}} \Rightarrow A\subset X$ disconnected

$\Leftarrow \exists V_1,V_2\subset Y$ open, $V_1\cap V_2=\emptyset$ s.t. $f(A)\cap V_1, f(A)\cap V_2\neq\emptyset$

and $f(A)\subset V_1\cup V_2$

Find $U_1,U_2\subset X$ open, $U_1\cap U_2=\emptyset$ s.t. $A\cap U_1, A\cap U_2\neq\emptyset$

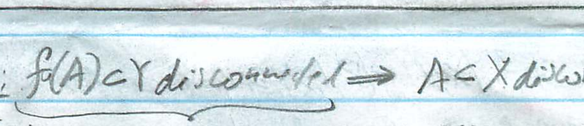
and $A\cap U_1, A\cap U_2=\emptyset$

#2

Pf of (ii): Let $\mathcal{C}=\{V_\alpha\}$ be an open cover of $f(A)\subset X$

$\therefore V_\alpha\subset Y$ open $\forall V_\alpha\in\mathcal{C}$. $\quad f(A)\subset\bigcup_{V_\alpha\in\mathcal{C}} V_\alpha$

Find finite $\mathcal{C}'\subset\mathcal{C}$ s.t. $f(A)\subset\bigcup_{V_\alpha\in\mathcal{C}'} V_\alpha$



Take $\tilde{\mathcal{C}}=\{f^{-1}(V_\alpha): V_\alpha\in\mathcal{C}\}$

$V_\alpha\subset Y$ open, f cont. $\Rightarrow f^{-1}(V_\alpha)\subset X$ open

#3

$f(A) \subset \bigcup_{V_\alpha \in \mathcal{C}} V_\alpha \iff A \subset \bigcup_{V_\alpha \in \mathcal{C}} f^{-1}(V) = \bigcup_\alpha f^{-1}(V_\alpha)$ $\quad f^{-1}(V_\alpha) \in \tilde{\mathcal{C}}$

$\Rightarrow \tilde{\mathcal{C}}$ is an open cover of A

A cmpt $\Rightarrow \exists\, \tilde{\mathcal{C}}' \subset \tilde{\mathcal{C}}$ finite s.t. $A \subset \underbrace{\bigcup f^{-1}(V_\alpha)}_{f^{-1}(V_\alpha) \in \tilde{\mathcal{C}}'}$

Pick $\underbrace{\mathcal{C}' \subset \mathcal{C}}_{\text{finite}}$ s.t. $\tilde{\mathcal{C}}' = \{f^{-1}(V_\alpha) : V_\alpha \in \mathcal{C}'\}$ $\Updownarrow$

$f(A) = \bigcup_{V_\alpha \in \mathcal{C}'} V_\alpha$

$\therefore$ get finite subcover of $f(A)$

$\Longrightarrow f(A)$ is compact

Pf of (iii): Let $W \subset Z$ be open. Show $\underbrace{\{g \circ f\}^{-1}(W)}_{} \subset X$ open

$X \xrightarrow{f} Y \xrightarrow{g} Z$

$f^{-1}(g^{-1}(W)) \subset X$, $g^{-1}(W) \subset Y$, $W \subset Z$

$\{g \circ f\}^{-1}(W)$

$W \subset Z$ open, g cont. $\Rightarrow g^{-1}(W) \subset Y$ open

$+ f$ cont. $\Rightarrow \underbrace{f^{-1}(g^{-1}(W))}_{= \{g\circ f\}^{-1}(W) \subset X \text{ open} \checkmark} \subset X$ open

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Crl 1 (Intermediate Value Thm) #3

If $f : (X, d) \to (\mathbb{R}, d_\mathbb{R})$ is cont, $A \subset X$ is connected, ($d_\mathbb{R}$ ← standard metric)

and $a, b \in X$, then $\underbrace{[f(a), f(b)] \subset f(A)}_{\text{all values b/w } f(a) \text{ and } f(b) \text{ are achieved}}$

Pf: Prp (i) $\Rightarrow f(A) \subset \mathbb{R}$ is connected

Th: the connected subsets of $\mathbb{R}$ are the intervals

$+ f(a), f(b) \in f(A) \Rightarrow \underbrace{r \in f(A) \text{ if } f(a) \leq r \leq f(b)}_{[f(a), f(b)] \subset f(A)}$

#3

Crl 2: If $f : (X, d) \to (\mathbb{R}, d_\mathbb{R})$ is cont. and $A \subset X$ cmpt,

then $\inf f(A) = f(a)$ for some $a \in A$

and $\sup f(A) = f(b)$ for some $b \in A$

Pf: Prp (ii) $\Rightarrow f(A) \subset \mathbb{R}$ is cmpt

Previously $\Rightarrow \inf f(A), \sup f(A) \in f(A)$

i.e. $= f(a)$, $= f(b)$ for some $a, b \in \mathbb{R}$

#2

Prp 2: $(X, d_X), (Y, d_Y)$ metric spaces, $f : X \to Y$, $x_0 \in X$, $y_0 \in Y$

Then $\lim_{x \to x_0} f(x) = y_0$ iff

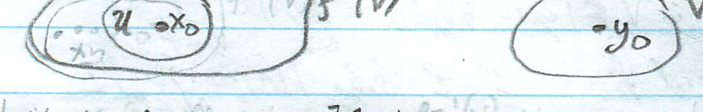
$\forall$ sequences $x_n \in X - \{x_0\}$ with $x_n \to x_0$, $f(x_n) \to y_0 \in Y$ (*)

$x_n \ldots x_0 \qquad f(x_n) \ldots y_0$

Pf $\Rightarrow$: Let $(x_n)_n$ be a sequence in $X - \{x_0\}$ with $x_n \to x_0$

Need to show $f(x_n) \to y_0$, i.e. $\forall V \subset Y$ open, $y_0 \in V$

$\exists N \in \mathbb{Z}^+$ s.t. $f(x_n) \in V \ \forall n \geq N$



$y_0 \in V \subset Y$ open, $\lim_{x \to x_0} f(x) = y_0 \Rightarrow \exists\, \mathcal{U} \subset X$ open with $x_0 \in \mathcal{U}$, $\mathcal{U} - \{x_0\} \subset f^{-1}(V)$

$x_n \to x_0 \Rightarrow \exists N \in \mathbb{Z}^+$ s.t. $\underbrace{x_n \in \mathcal{U}}$ $\forall n \geq N$

$+ x_n \neq x_0 \Longrightarrow x_n \in f^{-1}(V) \iff f(x_n) \in V \checkmark$

Pf of $\Leftarrow$: Suppose $\lim_{x \to x_0} f(x) \neq y_0 \overset{\text{Dfn(c)}}{\Longrightarrow} \exists V \subset Y$ open with $y_0 \in V$

s.t. $\nexists\, \mathcal{U} \subset X$ open w. $x_0 \in \mathcal{U}$ and $\mathcal{U} - \{x_0\} \subset f^{-1}(V)$

$\Rightarrow \forall n \in \mathbb{Z}^+$, $\underbrace{B_{1/n}(x_0) - \{x_0\} \not\subset f^{-1}(V)}$

$\exists\, x_n \in B_{1/n}(x_0)$ s.t. $\underbrace{x_n \notin f^{-1}(V)}_{f(x_n) \notin V}$, $x_n \neq x_0$

$\therefore d(x_0, x_n) < \frac{1}{n} \ \forall n \Rightarrow x_n \to x_0 \in X$

$f(x_n) \notin V \ \forall n$, $y_0 \in V \subset Y$ open $\Rightarrow f(x_n) \not\to y_0 \in Y$

$\Longrightarrow$ (*) does not hold

$x_n \ldots x_0$