MAT 320

$\Longrightarrow$ Color chalk $\Longleftarrow$

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Before break: (X,d) metric space #1

- $\mathcal{U}\subset X$ **open** if $\forall x\in\mathcal{U}$, $\exists \delta_x\in\mathbb{R}^+$ s.t. $B_{\delta_x}(x)\subset\mathcal{U}$
- $A\subset X$ **cmpt** if every open cover $\mathcal{C}$ of A contains a finite subcollection $\mathcal{C}'$ covering A
- $(x_n)_n$ **converges** to $x\in X$ if $\forall \mathcal{U}\subset X$ open with $x\in\mathcal{U}$, $\exists N\in\mathbb{Z}^+$ s.t. $x_n\in\mathcal{U}$ $\forall n\geq N$

$\Longrightarrow$: $x_{n_k}\to x$ and $\mathcal{U}\subset X$ open with $x\in\mathcal{U}$,

$\exists K\in\mathbb{Z}^+$ s.t. $x_{n_k}\in\mathcal{U}$ $\forall k\geq N \Longleftrightarrow n_k\in S(\mathcal{U})$ $\forall k\geq N$

$\Longrightarrow n_k<n_{k+1}<\ldots \Longrightarrow$ get ∞ many elements of $S(\mathcal{U})$

$\Longleftarrow$: $n_1=\min \underbrace{\mathbb{N}(B_1(x))}_{\text{non-empty subset of } \mathbb{N}}\in\mathbb{N}$

$n_{k+1}=\min\{n\in\mathbb{N}(B_{1/k}(x)) : n>n_k\}$ $\forall k\geq 1$

$d(x_{n_k},x)<1/N$ $\forall k\geq N \Longrightarrow (x_{n_k})_k\to x$

#2

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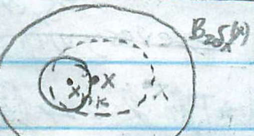
Lemma 2 (Lebesgue Number Lemma) (X,d) = metric space

If $A\subset X$ is sequentially cmpt and $\mathcal{C}$ is an open cover of A, $\exists\delta\in\mathbb{R}^+$ s.t. $\forall x\in A$, $\exists\mathcal{U}_x\in\mathcal{C}$ with $B_\delta(x)\subset\mathcal{U}_x$

: same δ for all $x\in A$!

Choose $k\geq N$ s.t. $1/n_k\leq\delta_x$



$d(x,x_{n_k})<\delta_x$

$d(x_{n_k},x')<1/n_k\leq\delta_x$ $\forall x'\in B_{1/n_k}(x_{n_k})$ #4

$\Longrightarrow d(x,x')<2\delta_x$ $\forall x'\in B_{1/n_k}(x_{n_k})$

$\Longrightarrow B_{1/n_k}(x_{n_k})\subset B_{2\delta_x}(x)\subset\mathcal{U}_x\in\mathcal{C}$

Contradiction b/c $B_{1/n}(x_n)\not\subset\mathcal{U}$ $\forall\mathcal{U}\in\mathcal{C}$

3/20/18

Thm 1: (X,d) metric space. $A\subset X$ is cmpt iff $\underbrace{\text{every sequence } (x_n)_n \text{ in } A \text{ has a subsequence convergent in } A}_{A\subset X \text{ is sequentially cmpt}}$. #2

Lemma 1: $(x_n)_n$ has a subsequence $(x_{n_k})_k$ converged to $x\in X$ iff $\forall\mathcal{U}\subset X$ open with $x\in\mathcal{U}$, $|\underbrace{\{n\in\mathbb{N}: x_n\in\mathcal{U}\}}_{\mathbb{N}(\mathcal{U})}|=\infty$

Pf of Thm $\Longrightarrow$: Suppose $(x_n)_n$ is a sequence in $A\subset X$ and contains no subsequence $(x_{n_k})_k$ convergent to any $x\in X$

$\Longrightarrow\forall x\in A$, $\exists\mathcal{U}_x\subset X$ open with $x\in\mathcal{U}_x$ and $|\mathbb{N}(\mathcal{U}_x)|<\infty$ $\quad\{n\in\mathbb{N}: x_n\in\mathcal{U}_x\}$

$\Longrightarrow\mathcal{C}=\{\mathcal{U}_x : x\in A\}$ is an open cover of A

$\mathcal{U}_x\subset X$ open $\forall\mathcal{U}_x\in\mathcal{C}$, $A\subset\bigcup_{\mathcal{U}_x\in\mathcal{C}}\mathcal{U}_x$

A cmpt $\Longrightarrow\exists\mathcal{C}'\subset\mathcal{C}$ finite s.t. $A\subset\bigcup_{\mathcal{U}_x\in\mathcal{C}'}\mathcal{U}_x$

$\Longrightarrow\mathbb{N}=\{n\in\mathbb{N}: x_n\in\bigcup_{\mathcal{U}_x\in\mathcal{C}'}\mathcal{U}_x\}=\left(\bigcup_{\mathcal{U}_x\in\mathcal{C}'}\mathbb{N}(\mathcal{U}_x)\right)\leftarrow$ finite $\times$

Pf: Suppose not. Then for each $n\in\mathbb{N}$,

$\exists x_n\in A$ s.t. $B_{1/n}(x_n)\not\subset\mathcal{U}$ $\forall\mathcal{U}\in\mathcal{C}$

A seq. cmpt $\Longrightarrow\exists$ subseq. $(x_{n_k})_k\to$ some $x\in A$

$\mathcal{C}$ covers $A\Longrightarrow\exists\mathcal{U}_x\in\mathcal{C}$ s.t. $x\in\mathcal{U}_x$

$\mathcal{U}_x\subset X$ open $\Longrightarrow\exists\delta_x\in\mathbb{R}^+$ s.t. $B_{2\delta_x}(x)\subset\mathcal{U}_x$

$(x_{n_k})_k\to x\Longrightarrow\exists N\in\mathbb{Z}^+$ s.t. $d(x,x_{n_k})<\delta_x$ $\forall k\geq N$

Pf of Thm $\Longleftarrow$: Let $\mathcal{C}$ be an open cover of A. #3

Suppose $\mathcal{C}$ contains no finite subcollection $\mathcal{C}'$ covering A.

Lemma 2 $\Longrightarrow\exists\delta\in\mathbb{R}^+$ s.t. $\forall x\in A$, $\exists\mathcal{U}_x\in\mathcal{C}$ with $B_\delta(x)\subset\mathcal{U}_x$

Take $x_1\in A$ any and $x_2,x_3,\ldots\in A$ inductively s.t.

$x_{n+1}\notin B_\delta(x_1)\cup\ldots\cup B_\delta(x_n)\subset\mathcal{U}_{x_1}\cup\ldots\cup\mathcal{U}_{x_n}$

exists b/c $\mathcal{C}'\equiv\{\mathcal{U}_{x_1},\ldots,\mathcal{U}_{x_n}\}\subseteq\mathcal{C}$ does not cover A

$x_n \notin B_\delta(x_i)\ \forall n > i \Rightarrow d(x_i, x_n) \geq \delta\ \forall i \neq n$ #4

$\Rightarrow$ $\underbrace{\text{no subseq. } (x_{n_k})_k \text{ of } (x_n)_n}$ is Cauchy

$\Rightarrow$ converges in A (or X)

contradicts to $A \subset X$ sequentially cmpt

HW6, Problem G-a: each d_i is a metric on $X \times Y$

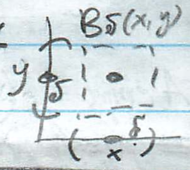
any two of these are uniformly equivalent

+ Problem F $\Rightarrow$ $W \subset X \times Y$ is d_i-open iff d_j-open

$C \subset X \times Y$ is d_i-cmpt iff d_j-cmpt

$\therefore$ enough to prove Thm 2 for $d = d_1$

$\rightarrow B_\delta(x,y) = B_\delta(x) \times B_\delta(y)$



open balls in $(X \times Y, d_1)$ are "squares"

Let $\delta_x = \min \underbrace{\{\delta_{x,y} : B_{\delta_{x,y}}(y) \in \mathcal{C}_x'\}}_{\text{finite subset of } \mathbb{R}^+} \in \mathbb{R}^+$ #3

$\Rightarrow B_{\delta_x}(x) \times B_{\delta_{x,y}}(y) \subset B_{\delta_{x,y}}(x) \times B_{\delta_{x,y}}(y) = B_{\delta_{x,y}}(x,y) \subset \mathcal{U}_{x,y}$

$\forall y \in \mathcal{C}_x',\ x \in X$ square metric

$\Rightarrow B_{\delta_x}(x) \times B \subset B_{\delta_x} \times \bigcup B_{\delta_{x,y}}(y) \subset \bigcup \mathcal{U}_{x,y}$, (***)

(**) $B_{\delta_{x,y}}(y) \times \mathcal{C}_x'$

HW6, §13.12 $B \subset A \subset X$, A cmpt, B closed $\Rightarrow$ B cmpt

Recitations yesterday and tomorrow: (X,d) = metric spaces

(i) $A \subset X$ cmpt $\Rightarrow$ A bounded, closed, complete w.r.t.

(ii) Completeness Axiom for $\mathbb{R}$ $\Rightarrow$ Bolzano-Weierstrass

MAT 319 $[a,b] \subset \mathbb{R}$ is cmpt $\Leftarrow$ Thm 1

every bounded seq. in $\mathbb{R}$ has a convergent subseq.

#2

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Thm 2: (X, d_X), (Y, d_Y) metric space. If $A \subset X$ and $B \subset Y$ cmpt then $A \times B \subset X \times Y$ is cmpt w.r.t. metric

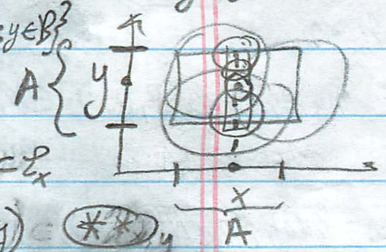
$$d_i((x,y),(x',y')) = \begin{cases} \max(d_X(x,x'), d_Y(y,y')) \\ (d_X(x,x')^2 + d_Y(y,y')^2)^{1/2} \\ d_X(x,x') + d_Y(y,y') \end{cases}$$

Pf of Thm 2: Let $\mathcal{C}$ be an open cover of $A \times B$ #4

$\Rightarrow \forall x \in A, y \in B\ \exists \delta_{x,y} \in \mathbb{R}^+$ s.t. $B_{\delta_{x,y}}(x,y) \subset$ some $\mathcal{U}_{x,y} \in \mathcal{C}$ (*)

$\Rightarrow \forall x \in X$, $\mathcal{C}_x = \{B_{\delta_{x,y}}(y) : y \in B\}$

is open cover of B



B cmpt $\Rightarrow \exists$ finite $\mathcal{C}_x' \subset \mathcal{C}_x$

s.t. $B \subset \bigcup_{B_{\delta_{x,y}}(y) \in \mathcal{C}_x'} B_{\delta_{x,y}}(y)$ (**)

#1

$\mathcal{C}_A = \{B_{\delta_x}(x) : x \in A\}$ is open cover of A

A cmpt $\Rightarrow \exists$ finite $\mathcal{C}_A' \subset \mathcal{C}_A$ s.t. $A \subset \bigcup_{B_{\delta_x}(x) \in \mathcal{C}_A'} B_{\delta_x}(x)$

$\Rightarrow A \times B \subset \bigcup_{B_{\delta_x}(x) \in \mathcal{C}_A'} B_\delta(x) \times B \overset{(***)}{\subset} \underbrace{\bigcup_{B_{\delta_x}(x) \in \mathcal{C}_A'} \bigcup_{B_{\delta_{x,y}}(y) \in \mathcal{C}_x'}} \mathcal{U}_{x,y}$

$\therefore \mathcal{C}' = \bigcup_{B_{\delta_x}(x) \in \mathcal{C}_A'} \mathcal{C}_x' \subset \mathcal{C}$ finite, covers $A \times B$

$\Rightarrow$ every open cover of $A \times B$ has finite subcover ✓

+Thm 2 $\Rightarrow$ (iii) Heine-Borel Thm for $(\mathbb{R}^n, d_{\mathbb{R}^n})$

square/round/sum metric

$A \subset \mathbb{R}^n$ is $d_{\mathbb{R}^n}$-cmpt if A is $d_{\mathbb{R}^n}$-closed and $d_{\mathbb{R}^n}$-bounded

Not true for subsets A of other metric spaces (X,d), even complete ones: e.g. HW6 §13.3ac, 13.5

J. Munkres, Topology

name. It still has not found a name on which everyone agrees. On historical grounds, some call it "Fréchet compactness"; others call it the "Bolzano-Weierstrass property." We have invented the term "limit point compactness." It seems as good a term as any; at least it describes what the property is about.

Theorem 28.1. *Compactness implies limit point compactness, but not conversely.*

Proof. Let X be a compact space. Given a subset A of X, we wish to prove that if A is infinite, then A has a limit point. We prove the contrapositive—if A has no limit point, then A must be finite.

So suppose A has no limit point. Then A contains all its limit points, so that A is closed. Furthermore, for each $a \in A$ we can choose a neighborhood U_a of a such that U_a intersects A in the point a alone. The space X is covered by the open set $X - A$ and the open sets U_a; being compact, it can be covered by finitely many of these sets. Since $X - A$ does not intersect A, and each set U_a contains only one point of A, the set A must be finite. ∎

EXAMPLE 1. Let Y consist of two points; give Y the topology consisting of Y and the empty set. Then the space $X = \mathbb{Z}_+ \times Y$ is limit point compact, for *every* nonempty subset of X has a limit point. It is not compact, for the covering of X by the open sets $U_n = \{n\} \times Y$ has no finite subcollection covering X.

EXAMPLE 2. Here is a less trivial example. Consider the minimal uncountable well-ordered set S_Ω, in the order topology. The space S_Ω is not compact, since it has no largest element. However, it is limit point compact: Let A be an infinite subset of S_Ω. Choose a subset B of A that is countably infinite. Being countable, the set B has an upper bound b in S_Ω; then B is a subset of the interval $[a_0, b]$ of S_Ω, where a_0 is the smallest element of S_Ω. Since S_Ω has the least upper bound property, the interval $[a_0, b]$ is compact. By the preceding theorem, B has a limit point x in $[a_0, b]$. The point x is also a limit point of A. Thus S_Ω is limit point compact.

We now show these two versions of compactness coincide for metrizable spaces; for this purpose, we introduce yet another version of compactness called *sequential compactness*. This result will be used in Chapter 7.

Definition. Let X be a topological space. If (x_n) is a sequence of points of X, and if

$$n_1 < n_2 < \cdots < n_i < \cdots$$

is an increasing sequence of positive integers, then the sequence (y_i) defined by setting $y_i = x_{n_i}$ is called a ***subsequence*** of the sequence (x_n). The space X is said to be ***sequentially compact*** if every sequence of points of X has a convergent subsequence.

***Theorem 28.2.** *Let X be a metrizable space. Then the following are equivalent:*

(1) *X is compact.*

(2) *X is limit point compact.*

(3) *X is sequentially compact.*

Proof. We have already proved that (1) $\Rightarrow$ (2). To show that (2) $\Rightarrow$ (3), assume that X is limit point compact. Given a sequence (x_n) of points of X, consider the set $A = \{x_n \mid n \in \mathbb{Z}_+\}$. If the set A is finite, then there is a point x such that $x = x_n$ for infinitely many values of n. In this case, the sequence (x_n) has a subsequence that is constant, and therefore converges trivially. On the other hand, if A is infinite, then A has a limit point x. We define a subsequence of (x_n) converging to x as follows: First choose n_1 so that

$$x_{n_1} \in B(x, 1).$$

Then suppose that the positive integer n_{i-1} is given. Because the ball $B(x, 1/i)$ intersects A in infinitely many points, we can choose an index $n_i > n_{i-1}$ such that

$$x_{n_i} \in B(x, 1/i).$$

Then the subsequence $x_{n_1}, x_{n_2}, \ldots$ converges to x.

Finally, we show that (3) $\Rightarrow$ (1). This is the hardest part of the proof.

First, we show that if X is sequentially compact, then the Lebesgue number lemma holds for X. (This would follow from compactness, but compactness is what we are trying to prove!) Let $\mathcal{A}$ be an open covering of X. We assume that there is no $\delta > 0$ such that each set of diameter less than δ has an element of $\mathcal{A}$ containing it, and derive a contradiction.

Our assumption implies in particular that for each positive integer n, there exists a set of diameter less than $1/n$ that is not contained in any element of $\mathcal{A}$; let C_n be such a set. Choose a point $x_n \in C_n$, for each n. By hypothesis, some subsequence (x_{n_i}) of the sequence (x_n) converges, say to the point a. Now a belongs to some element A of the collection $\mathcal{A}$; because A is open, we may choose an $\epsilon > 0$ such that $B(a, \epsilon) \subset A$. If i is large enough that $1/n_i < \epsilon/2$, then the set C_{n_i} lies in the $\epsilon/2$-neighborhood of x_{n_i}; if i is also chosen large enough that $d(x_{n_i}, a) < \epsilon/2$, then C_{n_i} lies in the ϵ-neighborhood of a. But this means that $C_{n_i} \subset A$, contrary to hypothesis.

Second, we show that if X is sequentially compact, then given $\epsilon > 0$, there exists a finite covering of X by open ϵ-balls. Once again, we proceed by contradiction. Assume that there exists an $\epsilon > 0$ such that X cannot be covered by finitely many ϵ-balls. Construct a sequence of points x_n of X as follows: First, choose x_1 to be any point of X. Noting that the ball $B(x_1, \epsilon)$ is not all of X (otherwise X could be covered by a single ϵ-ball), choose x_2 to be a point of X not in $B(x_1, \epsilon)$. In general, given $x_1, \ldots, x_n$, choose x_{n+1} to be a point not in the union

$$B(x_1, \epsilon) \cup \cdots \cup B(x_n, \epsilon),$$

using the fact that these balls do not cover X. Note that by construction $d(x_{n+1}, x_i) \geq \epsilon$ for $i = 1, \ldots, n$. Therefore, the sequence (x_n) can have no convergent subsequence; in fact, any ball of radius $\epsilon/2$ can contain x_n for at most *one* value of n.

Finally, we show that if X is sequentially compact, then X is compact. Let $\mathcal{A}$ be an open covering of X. Because X is sequentially compact, the open covering $\mathcal{A}$ has a Lebesgue number δ. Let $\epsilon = \delta/3$; use sequential compactness of X to find a finite