MAT 320: Introduction to Analysis, Spring 2018
Homework Assignment 7

Please read the posted notes and p171mid-p179 (except Example 8b) from Ross's textbook thoroughly before starting on the problem set below.

Optional supplementary reading: Chapter 2 of Rudin's book

Problem Set 7 (**due at the start of recitation on Wednesday, 3/28**): Problems H-L (below)

Problem H

Let (X,d) be a metric space and $A,B\subset X$ be disjoint closed subsets. Show that there exist open subsets $U,V\subset X$ such that $A\subset U$, $B\subset V$, and $\overline{U}\cap\overline{V}=\emptyset$.
Hint: first use open balls around points of A and B to construct disjoint U and V and show that their closures are disjoint from B and A, respectively.

Problem I

(a) Let (X,d) be a metric space, $U\subset X$ be an open dense subset, and $B\subset X$ be a nowhere dense subset. Show that the subset $U-\overline{B}\subset X$ is open and dense.

Let d be a square, round, or sum metric on $\mathbb{R}^n$.

(b) Show that $\mathbb{Q}^n\subset\mathbb{R}^n$ is not the intersection of countably many open subsets of $\mathbb{R}^n$.

(c) Show that $\mathbb{R}^n-\mathbb{Q}^n$ is not the union of countably many closed subsets of $\mathbb{R}^n$.

(d) Give an example of *disjoint* subsets $Y_1,Y_2,\ldots$ of $\mathbb{R}^n$ each of which is of the second category.

Problem J

Let (X,d) be a metric space and $Y\subset X$; thus, Y inherits a metric $d_Y\equiv d|_{Y\times Y}$ from (X,d).

(a) For a subset $A\subset Y$, let $\mathrm{Cl}_X(A)\subset X$ and $\mathrm{Cl}_Y(A)\subset Y$ denote the closures of A in (X,d) and in (Y,d_Y), respectively. Show that

$$\mathrm{Cl}_Y(A)=\mathrm{Cl}_X(A)\cap Y.$$

(b) Suppose (X,d) is a Baire space, i.e. satisfies the property of Ross's Theorem 21.7a or equivalently of 21.7b, and Y is open in (X,d). Show that (Y,d_Y) is also a Baire space.

Problem K

Let (X,d) be a connected metric space containing at least two points. Show that X is uncountable.
Hint: Look at sets of the form $B_\delta(x)$ and $X-\overline{B}_\delta(x)$.

Problem L

Suppose (X,d) is a metric space, $A\subset X$ is a connected subset, and $A\subset B\subset\overline{A}$. Show that B is also connected.