MAT 319/320: Basics of Analysis, Spring 2018
Sequences and Series (Review from Calculus C)

Extremely Important: sequences vs. series (do not mix them or their convergence/divergence tests up!!!); what it means for a sequence or series to converge or diverge;

$$\boxed{\text{“2 limits”} = \text{“no limit”} = \text{“diverge”}}$$

Very Important: convergence/divergence tests for sequences and series

Important: limit rules for sequences and series; computing limits of convergent sequences and sums of convergent series

A: Sequences

A.1 A sequence is an infinite string of numbers. It can be specified in several ways:

- list the numbers; for example, $-1, 1/2, -1/6, 1/24, -1/120, \ldots$;
- give a formula for the n-th number in the sequence; for example $a_n = (-1)^n/n!$ for $n \geq 1$;
- through recursive definition; for example, $a_1 = -1$, $a_{n+1} = -a_n/(n+1)$ for $n \geq 1$, or $f_0 = 0$, $f_1 = 1$, $f_{n+2} = f_{n+1} + f_n$ for $n \geq 0$. The first of these sequences is the same as the two sequences above; the second one is the famous Fibonacci sequence.

A sequence does not have to start with a_1; it could start with a_0 or with any other a_{n_0}, as long as a_n is specified for all $n \geq n_0$. Since it is just a string of numbers, the first number could be called a_1, or a_0, or a_{-10}; the second number in the sequence would then have to be called a_2, a_1, or a_{-9}, respectively.

A.2 Given a sequence $a_1, a_2, \ldots$, we'd like to know whether it gets closer and closer to some number a or there is no such number. In the former case, the sequence is said to converge to a and this is written as $\lim\limits_{n\longrightarrow\infty} a_n = \infty$; in the latter case, the sequence is said to diverge. In many cases, it is fairly straightforward to determine whether a sequence converges (and if so to what limit) or diverges. For example, if

$$a_n = (-1)^n \frac{\sqrt{n^4 + n^2}}{n^2 + \sqrt{n^2 + 1}}\,,$$

simply divide top and bottom by n^2 (you have to divide by the same thing!):

$$\begin{aligned} a_n &= (-1)^n \frac{\sqrt{n^4 + n^2}/n^2}{n^2/n^2 + \sqrt{n^2 + 1}/n^2} = (-1)^n \frac{\sqrt{n^4/n^4 + n^2/n^4}}{1 + \sqrt{n^2/n^4 + 1/n^4}} \\ &= (-1)^n \frac{\sqrt{1 + 1/n^2}}{1 + \sqrt{1/n^2 + 1/n^4}}\,; \end{aligned} \tag{A1}$$

so the fraction approaches $\sqrt{1}/(1 + \sqrt{0}) = 1$, but the sign alternates. So the terms a_n with n odd converge to -1, while the terms a_n with n even converge to 1. Thus, the entire sequence *diverges* (there is no *single* number to which all of the terms approach):

$$\boxed{\text{“2 limits”} = \text{“no limit”} = \text{“diverge”}}$$

The above trick of dividing top and bottom of a fraction by a power of n is the **most common approach** to dealing with sequences given by fractions. In MAT 125, a similar trick involved dividing by a power of x. In order to reduce your chances of making minor computational errors, which may even alter the qualitative answer (and thus be heavily penalized), it is best not to skip steps in a computation like (A1). In particular, be careful when taking a power of n under a square root.

A.3 Some sequences are of the form $a_n = f(b_n)$ for some fairly simple function f and some fairly simple sequence b_n. For example, if $a_n = \mathrm{e}^{1/n}$, then the sequence $b_n = 1/n$ converges to 0 and since e^x is continuous at 0,

$$\lim_{n\longrightarrow\infty} \mathrm{e}^{1/n} = \mathrm{e}^{\lim\limits_{n\longrightarrow\infty} 1/n} = \mathrm{e}^0 = 1.$$

On the other hand, the sequence $a_n = \cos(\pi n)$ is divergent, since it alternates between 1 and -1.

A.4 If $a_n = f(n)$ for some function $f = f(x)$ defined on the positive real line, then

$$\lim_{n\longrightarrow\infty} a_n = \lim_{x\longrightarrow\infty} f(x)$$

if the second limit exists (the first limit may exist even if the second does not). This may allow using l'Hospital for limits of functions. For example, if $a_n = (\ln n)/n$, then

$$\lim_{n\longrightarrow\infty} a_n = \lim_{x\longrightarrow\infty} \frac{\ln x}{x} = \lim_{x\longrightarrow\infty} \frac{(\ln x)'}{x'} = \lim_{x\longrightarrow\infty} \frac{1/x}{1} = 0.$$

If $a_n = n\cdot\sin(1/n)$, then

$$\lim_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty} \frac{\sin(1/n)}{1/n} = \lim_{x\longrightarrow 0} \frac{\sin x}{x} = \lim_{x\longrightarrow 0} \frac{(\sin x)'}{x'} = \lim_{x\longrightarrow 0} \frac{\cos x}{1} = 1.$$

This approach is *not* suitable for many sequences, including those involving $n!$ and $(-1)^n$.

A.5 If a sequence $\{a_n\}$ is defined recursively as $a_{n+1} = f(a_n)$, for some function f and with some initial condition, and it converges to a, then $a = f(a)$; this is obtained by taking the limit of both sides of $a_{n+1} = f(a_n)$. So *if* the sequence a_n is known to have a limit, one simply needs to solve the equation $a = f(a)$; it may have several solutions, but it should be possible to rule out all but one of them as possible limits (perhaps only one solution of $a = f(a)$ is non-negative and $a_n > 0$ for all n). This trick applies in

8.1 Example 12: $\quad a_1 = 2,\ a_{n+1} = \dfrac{a_n + 6}{2}$

8.1 #48: $\quad a_1 = \sqrt{2},\ a_{n+1} = \sqrt{2a_n}, \qquad \sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$

8.1 #54: $\quad a_1 = \sqrt{2},\ a_{n+1} = \sqrt{2 + a_n}, \qquad \sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$

Note: given the right-most presentations of sequences on the second and third lines above, you should be able to convert them to the recursive definitions in the middle of the two lines.

A.6 Before applying the trick in **A.5**, one has to know that the sequence $\{a_n\}$ has a limit at all. The convergence/divergence test for sequences which is suitable for all three examples in **A.5** is the Monotonic Sequence Theorem:

if $a_n \le a_{n+1}$ and $a_n \le M$ for all n ($\ge$ some N), then $\{a_n\}$ converges and $\lim\limits_{n\longrightarrow\infty} a_n \le M$
if $a_n \ge a_{n+1}$ and $a_n \ge m$ for all n ($\ge$ some N), then $\{a_n\}$ converges and $\lim\limits_{n\longrightarrow\infty} a_n \ge m$

In the first case, the sequence is increasing with n and is climbing below some "roof" M. As it keeps climbing, but cannot escape past the roof, it must approach some level below the roof (or the roof itself). In the second case, the sequence is decreasing with n and is descending toward some "floor" m. As it keeps descending, but cannot escape past the floor, it must approach some level above the floor (or the floor itself).

A.7 The second main convergence test for sequences is the *Squeeze Theorem for Sequences:*

> if $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are sequences such that $a_n \le b_n \le c_n$ for all n ($\ge$ some N), $\{a_n\}$ and $\{c_n\}$ converge, and $\lim\limits_{n\longrightarrow\infty} a_n = \lim\limits_{n\longrightarrow\infty} c_n$, then $\{b_n\}$ also converges and
>
> $$\lim_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty} b_n = \lim_{n\longrightarrow\infty} c_n$$

You might apply this to a specified sequence $\{b_n\}$ and appropriately chosen simpler sequences $\{a_n\}$ and $\{c_n\}$ which converge to the *same* limit and squeeze $\{b_n\}$ in between. For example, if

$$b_n = \frac{n}{n+1} + \frac{\cos n}{n},$$

you might take $a_n = n/(n+1) - 1/n$ and $c_n = n/(n+1) + 1/n$ so that

$$a_n \le b_n \le c_n, \qquad \lim_{n\longrightarrow\infty} a_n = \lim_{n\longrightarrow\infty} c_n = \lim_{n\longrightarrow\infty} \frac{n/n}{n/n + 1/n} = 1.$$

This implies that $\{b_n\}$ also converges and its limit is also 1.

If $b_n = 7^n/n!$, then

$$b_{7+n} = \frac{7^{7+n}}{(7+n)!} = \frac{7^7}{7!} \cdot \frac{7}{8} \cdot \frac{7}{9} \cdot \ldots \frac{7}{7+n} \le b_7 \cdot \left(\frac{7}{8}\right)^n;$$

so the sequence b_{7+n} is squeezed between the constant sequence $a_n = 0$ and the geometric sequence $c_n = b_7(7/8)^n$ which converges to 0 by **A.8** below. This implies that the sequences $\{b_n\}$ also converges to 0. The practical use of the *Squeeze Theorem for Sequences* is rather limited though. For example, in the first case above, you know that $|\cos n| \le 1$ and thus $(\cos n)/n \longrightarrow 0$; so

$$\lim_{n\longrightarrow\infty} b_n = \lim_{n\longrightarrow\infty} \frac{n}{n+1} = 1.$$

The second case is best dealt with by using the Ratio Test for Sequences; see **A.9** below.

A.8 A sequence of the form $c, cr, cr^2, cr^3, \ldots$ is called geometric. It is not difficult to determine whether it converges:

$$\boxed{\begin{array}{l}\text{the geometric } \textit{sequence } c, cr, cr^2, cr^3, \ldots \text{ with } c \neq 0 \\ \quad \bullet \text{ converges if } -1 < r \leq 1 \text{ (to 0 if } -1 < r < 1\text{; to 1 if } r = 1\text{);} \\ \quad \bullet \text{ diverges if } r \leq -1 \text{ or } r > 1. \end{array}} \tag{A2}$$

Note that the convergence statement for geometric *series*, (B3) below, is slightly different.

A.9 Another convergence test that works well for some sequences is the Ratio Test for Sequences:

- if $\displaystyle\lim_{n\longrightarrow\infty} \frac{|a_{n+1}|}{|a_n|} < 1$, then $\displaystyle\lim_{n\longrightarrow\infty} a_n = 0$;
- if $\displaystyle\lim_{n\longrightarrow\infty} \frac{|a_{n+1}|}{|a_n|} > 1$ or $a_{n+1}/a_n \longrightarrow \infty$, then the sequence a_n diverges;
- if $\displaystyle\lim_{n\longrightarrow\infty} \frac{|a_{n+1}|}{|a_n|} = 1$, then this test says nothing.

For example, for the sequence $a_n = (-1)^n 7^n/n!$,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{7^{n+1}/(n+1)!}{7^n/n!} = \frac{7^{n+1}}{7^n} \cdot \frac{n!}{(n+1)!} = \frac{7^n \cdot 7}{7^n} \cdot \frac{n!}{n! \cdot (n+1)} = \frac{7}{n+1} \longrightarrow \frac{7}{\infty + 1} = 0.$$

Since $0 < 1$, this sequence converges to 0. On the other hand, for the sequence $a_n = 2^n/n$,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{2^{n+1}/(n+1)}{2^n/n} = \frac{2^{n+1}}{2^n} \cdot \frac{n}{n+1} = \frac{2^n \cdot 2}{2^n} \cdot \frac{1}{n/n+1/n} = 2\frac{1}{1+1/n} \longrightarrow 2\frac{1}{1+1/\infty} = 2\frac{1}{1+0} = 2.$$

Since $2 > 1$, this sequence diverges (actually "converges" to ∞).

Since *RT for Sequences* can detect convergence of sequences $a_1, a_2, \ldots$ with limit 0 only (and even of only some of these), it works with few sequences. However, whenever it is applicable, *RT for Sequences* determines the limit of convergent sequences immediately. *RT for Sequences* has a good chance of working for sequences that involve factorials and powers n (e.g. $n!$, 3^n, n^n), but has *no* chance of working for sequences that involve only powers of n (e.g. n^3).

A.10 Finally, there are *Limit Rules for Convergent Sequences*, which are more or less as expected:

if $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is any number,

$$\lim_{n\longrightarrow\infty} (a_n \pm b_n) = \lim_{n\longrightarrow\infty} a_n \pm \lim_{n\longrightarrow\infty} b_n\,, \qquad \lim_{n\longrightarrow\infty} c a_n = c \cdot \lim_{n\longrightarrow\infty} a_n$$

$$\lim_{n\longrightarrow\infty} (a_n b_n) = \big(\lim_{n\longrightarrow\infty} a_n\big) \cdot \big(\lim_{n\longrightarrow\infty} b_n\big) \qquad \lim_{n\longrightarrow\infty} \frac{a_n}{b_n} = \frac{\lim_{n\longrightarrow\infty} a_n}{\lim_{n\longrightarrow\infty} b_n} \quad \text{if } \lim_{n\longrightarrow\infty} b_n \neq 0$$

The second equation on the first line is a special case of the first equation on the second line: just take $b_n = c$ for all n. Note that the sequences $\{a_n \pm b_n\}$, $\{a_n b_n\}$, and $\{a_n/b_n\}$ can converge even if the sequences $\{a_n\}$ and $\{b_n\}$ do not; in such cases, the limit rules are useless. Typically the limit rules are used to compute limits of sequences; in some cases they could also be used to test for convergence. For example, *if* the sequence $\{a_n\}$ converges, then the sequence $\{a_n \pm b_n\}$ converges if and only if the sequence $\{b_n\}$ does.

B: Series

B.1 A series (or infinite series) is the sum of all terms in a sequence:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots\,, \tag{B1}$$

where $a_1, a_2, \ldots$ is some sequence. The lower limit in the summation need not be 1; if a_0 is the first term of the corresponding sequence, then the lower limit in the sum is 0. Associated to the infinite sum (B1) is the sequence of partial sums,

$$s_1 = a_1, \quad s_2 = a_1 + a_2, \quad s_3 = a_1 + a_2 + a_3, \qquad s_n = \sum_{k=1}^{k=n} a_k.$$

The (infinite) series (B1) is said to converge to s if the sequence $\{s_n\}$ of the *partial sums* (**not** the original sequence $\{a_n\}$!!!) converges to s. If the partial sums s_n do not have a limit, the series (B1) is said to diverge. Thus, if the series (B1) converges to some number s, then the partial sums s_n approach s and so

$$a_n = s_n - s_{n-1} = \big(s_n - s\big) - \big(s_{n-1} - s\big)$$

approaches 0. This gives the **most important statement** regarding convergence of series:

$$\boxed{\begin{array}{l}\text{if the sequence } \{a_n\} \text{ diverges or it converges, but } \displaystyle\lim_{n\longrightarrow\infty} a_n \neq 0, \\ \text{then the series } \displaystyle\sum_{n=1}^{\infty} a_n \text{ diverges}\end{array}} \tag{B2}$$

For example, the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n$ and $\displaystyle\sum_{n=1}^{\infty}\cos(n\pi/2)$ diverge, because neither of the sequences $\{(-1)^n\}$ nor $\{\cos(n\pi/2)\}$ converges to 0 (in fact, neither of the two sequences converges at all). The partial sums s_n in the first case alternate between -1 and 0 and so indeed do not approach any number. In the second case, the partial sums cycle through $0, -1, -1, 0$ and so do not approach any number either.

WARNING: The most important statement about convergence of power series **can never be used** to conclude that a series converges; this is the reason that there are *lots of* other convergence tests for series. For example, the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ does not converge, according to the *p-Series Test* in (B10) below, even though $1/n \longrightarrow 0$.

B.2 Computing the sum of an infinite series is usually difficult, but possible in some cases. A geometric series is the sum of a geometric sequence and so has the form $\sum_{n=0}^{\infty} cr^n$. The sequence of partial sums in this case is

$$s_0 = c, \qquad s_1 = c + cr, \qquad s_2 = c + cr + cr^2, \qquad \ldots$$

$$s_n = c + cr + \ldots + cr^n = c(1 + r + \ldots + r^n) = \begin{cases} \frac{1-r^{n+1}}{1-r}c, & \text{if } r \neq 1; \\ (n+1)c, & \text{if } r = 1. \end{cases}$$

If $c \neq 0$ and $|r| \geq 1$, by the last line the sequence s_n diverges. If $|r| < 0$, then $r^{n+1} \longrightarrow 0$ and so $s_n \longrightarrow 1/(1-r)$. Since the convergence of the series $\sum_{n=0}^{\infty} cr^n$ is the same the convergence of the sequence s_n (but **not** of a_n), we find that

$$\boxed{\begin{aligned} &\sum_{n=0}^{\infty} cr^n = \frac{c}{1-r} \quad \text{if } |r| < 1 \text{ (note the lower limit on the sum)} \\ &\sum_{n=0}^{\infty} cr^n \quad \text{diverges if } |r| \geq 1 \text{ and } c \neq 0 \end{aligned}} \tag{B3}$$

In the second case, the sequence $a_n = cr^n$ being summed does not converge to 0 by (A2). Thus, the conclusion in this case also follows from the most important statement about convergence of series in (B2) above.

As an application, we can write the number $2.1\overline{37} = 2.1373737\ldots$ as a simple fraction:

$$\begin{aligned} 2.1\overline{37} &= 2.1 + .037 + .037 \cdot \frac{1}{100} + .037 \cdot \frac{1}{100^2} + \ldots = \frac{21}{10} + \frac{37/1000}{1 - \frac{1}{100}} = \frac{21}{10} + \frac{37/10}{99} \\ &= \frac{21 \cdot 99 + 37}{990} = \frac{2116}{990} = \frac{1058}{495} \end{aligned}$$

This is another example when skipping steps might increase the chance of a computational error.

B.3 Infinite series can also be summed up in the cases of pairwise cancellation. Such series have the form

$$\sum_{n=1}^{\infty} \big(b_n - b_{n+m}\big) = (b_1 - b_{1+m}) + (b_2 - b_{2+m}) + \ldots + (b_{1+m} - b_{1+2m}) + (b_{2+m} - b_{2+2m}) + \ldots$$

for some fixed integer $m \geq 0$ or can be put into this form after some algebraic manipulations (the lower limit can be anything). Note that lots of terms above cancel in pairs. If $n \geq m$, the n-th partial sum is then

$$\begin{aligned} s_n = a_1 + a_2 + \ldots + a_n &= (b_1 - b_{1+m}) + (b_2 - b_{2+m}) + \ldots + (b_n - b_{n+m}) \\ &= \sum_{k=1}^{k=m} b_k - \sum_{k=n+1}^{k=n+m} b_k, \end{aligned} \tag{B4}$$

since the second term in the k-th pair cancels with the first term in the $(k+m)$-th, provided $k \le n-m$. This leaves the first terms in the first m pairs and the second terms in the last m pairs. As $n \longrightarrow \infty$, the first sum on the second line in (B4) does not change. So the sequence $\{s_n\}$ (and thus the series $\sum\limits_{n=1}^{\infty} \big(b_n - b_{n+m}\big)$) converges if and only if the sequence

$$s_n^- = \sum_{k=n+1}^{k=n+m} b_k = b_{n+1} + b_{n+2} + \ldots + b_{n+m}$$

does. This happens if the sequence $\{b_n\}$ converges, but may happen even if $\{b_n\}$ diverges. For example, all of the series

$$\sum_{n=1}^{\infty}\big(\sin(1/n) - \sin(1/(n+1))\big), \quad \sum_{n=1}^{\infty}\big(\cos(1/n) - \cos(1/(n+2))\big), \quad \sum_{n=1}^{\infty}(-1)^n\big(\ln(n) - \ln(n+2)\big)$$

converge, while the series

$$\sum_{n=1}^{\infty}\big(\cos(n) - \cos(n+1)\big), \quad \sum_{n=1}^{\infty}\big(\ln(n) - \ln(n+1)\big), \quad \sum_{n=1}^{\infty}\big(\mathrm{e}^n - \mathrm{e}^{n+1}\big)$$

diverge.

The simplest possible case, called telescoping cancellation, occurs when $b_n \longrightarrow 0$, so that the last sum in (B4) disappears as $n \longrightarrow \infty$:

$$\boxed{\sum_{n=1}^{\infty}\big(b_n - b_{n+m}\big) = \sum_{n=1}^{n=m} b_n \quad \text{if} \lim_{n\longrightarrow\infty} b_n = 0,\ m \ge 0} \tag{B5}$$

This is frequently used in conjunction with partial fractions. For example,

$$\sum_{n=2}^{\infty}\frac{1}{n^2-1} = \sum_{n=2}^{\infty}\frac{1}{(n+1)-(n-1)}\left(\frac{1}{n-1} - \frac{1}{n+1}\right) = \frac{1}{2}\sum_{n=2}^{\infty}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$$
$$= \frac{1}{2}\left(\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \ldots\right) = \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4}$$

In this case, $b_n = 1/(n-1)$ for $n \ge 2$ and $m = 2$. Generally, re-writing LHS of (B5) as

$$\sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty} b_{n+m}$$

will constitute a serious error, since these two sums may not converge. For example,

$$\sum_{n=2}^{\infty}\frac{1}{n-1} = \sum_{n=1}^{\infty}\frac{1}{n}$$

does not converge by the *p-Series Test* in (B10) below. The condition $\lim\limits_{n\longrightarrow\infty} b_n = 0$ in (B5) is absolutely **essential**. For example, the series

$$\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^{\infty} \big(\ln(n+1) - \ln n\big)$$

does not converge at all, because the sequence of partial sums

$$s_n = \sum_{k=1}^{k=n} \big(\ln(k+1) - \ln k\big) = \ln(n+1) - \ln 1 = \ln(n+1)$$

diverges. The formula (B5) with $b_n = -\ln n$ and $m = 1$ would have produced $b_1 = 0$ for the sum of the infinite series. This is impossible in this case since all the summands are positive. The formula (B5) cannot be applied in this case because the limit condition in (B5) is not satisfied.

B.4 There are many cases when it can be determined whether a series converges, but it is hard to determine its sum (this is relatively rare for sequences). There are several convergence tests for series which, unlike the most important statement for series in (B2) above, can determine convergence. All of the tests in Section 8.3 deal with series that have only positive terms a_n (for $n \geq$ some N); the *Ratio Test for Series* does not care about the signs. Series with terms of different signs are in fact more likely to converge and will be considered in Section 8.4 in more detail after the 2nd midterm. In some cases, different tests can be used to determine whether a series converges.

B.5 The most evident and fundamental convergence test for series with positive terms is the Comparison Test:

if the sequences $\{a_n\}$ and $\{b_n\}$ have positive terms, $a_n \leq b_n$ for all n ($\geq$ some N), and the series $\sum\limits_{n=1}^{\infty} b_n$ converges, then so does the series $\sum\limits_{n=1}^{\infty} a_n$	(B6)

This test with the roles of a_n and b_n reversed leads to a divergence test for series:

if the sequences $\{a_n\}$ and $\{b_n\}$ have positive terms, $a_n \geq b_n$ for all n ($\geq$ some N), and the series $\sum\limits_{n=1}^{\infty} b_n$ diverges, then so does the series $\sum\limits_{n=1}^{\infty} a_n$

While the *Comparison Test* is the basis for most other convergence tests for series, it is often easier to apply one of the other convergence tests instead.

B.6 A close cousin to the *Comparison Test* is the Limit Comparison Test for series states that

if the sequences $\{a_n\}$ and $\{b_n\}$ have positive terms, the sequence a_n/b_n converges, and the series $\sum\limits_{n=1}^{\infty} b_n$ converges, then so does the series $\sum\limits_{n=1}^{\infty} a_n$	(B7)

For example, to determine whether the series $\sum\limits_{n=1}^{\infty} \frac{n}{4^n}$ converges, take $a_n = n/4^n$ and $b_n = 1/2^n$,

$$\lim_{n\longrightarrow\infty} \left(\frac{a_n}{b_n}\right) = \lim_{n\longrightarrow\infty} \left(\frac{n}{2^n}\right) = 0$$

and $\sum_{n=1}^{\infty} b_n$ converges by the geometric series test (B3); so $\sum_{n=1}^{\infty} a_n$ also converges. The same argument applies to $\sum_{n=1}^{\infty} \frac{n^p}{r^n}$ for any $r>1$ (but the *Ratio Test for Series* is simpler to use here). More typically, the *Limit Comparison Test* is applied to series like

$$\sum_{n=1}^{\infty} \frac{1}{2^n-n}\,, \qquad \sum_{n=2}^{\infty} \frac{1}{n^2-n}\,, \qquad \sum_{n=1} \sin^p(1/n)\,;$$

the summands in these series "asymptotically approximate" $1/2^n$, $1/n^2$, and $1/n^p$, respectively. The *Limit Comparison Test* with the roles of a_n and b_n reversed leads to a divergence test for series:

> if the sequences $\{a_n\}$ and $\{b_n\}$ have positive terms, the sequence a_n/b_n converges, $\lim_{n\longrightarrow\infty}(a_n/b_n) \neq 0$, and the series $\sum_{n=1}^{\infty} b_n$ diverges, then so does the series $\sum_{n=1}^{\infty} a_n$

In contrast to the convergence test above, there is the extra condition that a_n/b_n not approach 0; this makes sense since otherwise we could take $a_n=0$, regardless of what b_n is. The *Limit Comparison Test* follows from the *Comparison Test*, but is likely to be more useful in this course.

B.7 The Ratio Test for Series works similarly to the *Ratio Test for Sequences*:

- if $\lim_{n\longrightarrow\infty} \frac{|a_{n+1}|}{|a_n|} < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges;
- if $\lim_{n\longrightarrow\infty} \frac{|a_{n+1}|}{|a_n|} > 1$ or $a_{n+1}/a_n \longrightarrow \infty$, the series $\sum_{n=1}^{\infty} a_n$ diverges; (B8)
- if $\lim_{n\longrightarrow\infty} \frac{|a_{n+1}|}{|a_n|} = 1$, then this test says nothing.

For example, for the series $\sum(-1)^n 7^n/n!$,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{7^{n+1}/(n+1)!}{7^n/n!} = \frac{7^{n+1}}{7^n} \cdot \frac{n!}{(n+1)!} = \frac{7^n \cdot 7}{7^n} \cdot \frac{n!}{n! \cdot (n+1)} = \frac{7}{n+1} \longrightarrow \frac{7}{\infty+1} = 0.$$

Since $0<1$, the series converges to 0. On the other hand, for the series $\sum 2^n/n$,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{2^{n+1}/(n+1)}{2^n/n} = \frac{2^{n+1}}{2^n} \cdot \frac{n}{n+1} = \frac{2^n \cdot 2}{2^n} \cdot \frac{1}{n/n+1/n} = 2\frac{1}{1+1/n} \longrightarrow 2\frac{1}{1+1/\infty} = 2\frac{1}{1+0} = 2.$$

Since $2>1$, the series diverges.

Due to the last case in (B8), the *Ratio Test* does not work for many series. However, whenever it is applicable, *RT for Series* works *amazingly* well. In particular, you do not need to come up with another series to compare the given series with as you would for the *Comparison Test* and the *Limit Comparison Test*. *RT for Series* has a good chance of working for sequences that involve factorials and powers n (e.g. $n!$, 3^n, n^n), but has *no* chance of working for sequences that involve only powers of n (e.g. n^3).

B.8 Another useful convergence test for series is the Integral Test:

$$\boxed{\text{if } f \text{ is a continuous, positive, and decreasing function on } [1,\infty), \text{ then the series } \sum_{n=1}^{\infty} f(n) \text{ converges if and only if the improper integral } \int_1^{\infty} f(x)\mathrm{d}x \text{ does}} \tag{B9}$$

This test is obtained from the geometric interpretation of the integral as the area under the graph, which can be estimated by rectangles of base one and with heights determined by either left or right end points. A corollary of this test is the p-series Test:

$$\boxed{\text{the series } \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if and only if } p>1} \tag{B10}$$

B.9 The Alternating Series Test applies to a narrow, but important, set of series with terms of different signs:

$$\boxed{\text{if } \lim_{n\longrightarrow\infty} a_n = 0,\ |a_n| \geq |a_{n+1}|, \text{ and the signs of } a_n \text{ alternate } (a_n>0 \text{ for every } n \text{ odd and } a_n<0 \text{ for every } n \text{ even, or the other way around}), \text{ then the series } \sum_{n=1}^{\infty} a_n \text{ converges}}$$

The alternating-sign condition is typically exhibited by the presence of $(-1)^n$ or $(-1)^{n-1} = -(-1)^n$. However, make sure to also check the first two conditions before concluding that the series converges. Typical examples are the series like

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}, \qquad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(\ln n)^2}{n};$$

both converge by the AST. The *Alternating Series Test* is a convergence test only: it states that a series converges if it meets 3 conditions. It can *never* be used to conclude that a series diverges. In this sense, it is the opposite of the most important divergence test, which can *never* be used to conclude that a series converges. If the first condition in the *Alternating Series Test* is not satisfied, the series does indeed diverge, but by the most important divergence test. However, there are lots of series that fail either the second or third condition (or both), but still converge. For example, there are convergent series with only positive terms, that decay to zero, but are not strictly decreasing, e.g.

$$\sum_{n=1}^{\infty} \frac{2+(-1)^n}{n^2}.$$

The *Alternating Series Test* is a consequence of the definition of convergence for series (convergence of the *sequence* of partial sums) and the Monotonic *Sequence* Theorem.

B.10 The substance of Absolute Convergence Test is that introducing some minus signs into a convergent series with positive terms does not ruin the convergence:

$$\boxed{\text{if the series } \sum_{n=1}^{\infty} |a_n| \text{ converges, then so does the series } \sum_{n=1}^{\infty} a_n}$$

This test is useful when the signs are random, as opposed to strictly alternating as required for the *Alternating Convergence Test*. For example, the series $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges by the ACT, because the series

$$\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$$

converges since $0 \le |\sin n|/n^2 \le 1/n^2$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (this argument uses 3 tests: *Absolute Convergence*, *Comparison*, and *p-Series*; *Limit Comparison Test* is less suitable in this case). The *Alternating Series Test* cannot be applied in this case, because the signs of $\sin n$ do not alternate:

$$\sin 1, \sin 2, \sin 3 > 0, \qquad \sin 4, \sin 5, \sin 6 < 0;$$

while the signs usually come in triples, occasionally there are four consecutive terms with the same sign. While the *Absolute Convergence Test* is less stringent about the alternating sign condition than the *Alternating Series Test*, the former is not a substitute of the latter. While either test can be used to conclude that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges, only the *Alternating Series Test* is applicable to the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ because the series

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

does not converge. Neither of the two tests directly implies that the series $\sum_{n=1}^{\infty} \frac{\sin n}{n}$ converges[1]. As the *Alternating Series Test*, the *Absolute Convergence Test* is a convergence test only; it can never be used to conclude that a series diverges. The *Absolute Convergence Test* is a consequence of the *Comparison Test* and the addition rule for series.

[1] this series does indeed converge because of a more general version of the *Alternating Series Test*, called *Dirichlet's Test*: if $\{b_n\}$ and $\{s_n\}$ are two sequences such that $\lim_{n\longrightarrow\infty} b_n = 0$, $b_n \ge b_{n+1}$, and there exists $C > 0$ such that $\left|\sum_{n=1}^{n=m} s_n\right| \le C$ for all m, then the series $\sum_{n=1}^{\infty} s_n b_n$ converges; in the case of the *Alternating Series Test* $s_n = \pm(-1)^n$ is just the sign, and so $C = 1$ works

B.11 There are also Rules for Convergent Series, which are more or less as expected:

if the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge and c is any number,

$$\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n\,, \qquad \sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$$

Note that these rules do not extend to multiplication and division, unlike what is the case for sequences. The series $\sum_{n=1}^{\infty}(a_n \pm b_n)$ can converge even if the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ do not; in such cases, the above rules are useless. Typically these rules are used to compute sums of series; in some cases they could also be used to test for convergence. For example, *if* the series $\sum_{n=1}^{\infty} a_n$ converges, then the series $\sum_{n=1}^{\infty}(a_n \pm b_n)$ converges if and only if $\sum_{n=1}^{\infty} b_n$ does.