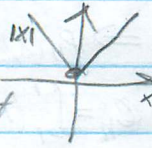


#3

Last week: Power Series $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$, $(a_n, z_0, z) \in \mathbb{C}$
 $\rightarrow R = 1/\limsup |a_n|^{1/n}$ fixed variable

$f: B_R(z_0) \rightarrow \mathbb{C}$ well-defined and smooth
 $f^{(k)}(z) = \sum_{n=k}^{\infty} a_n n(n-1)\dots(n-k+1) z^{n-k}$
 k-th derivative

(4/16/24) updated

$\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges uniformly on $B_{R'}(z_0) \forall R' < R$
 \Rightarrow good way to approximate $f(z)$ on $B_{R'}(z_0)$ with $R' > 0$
 by finite polynomials $\sum_{n=0}^N a_n(z-z_0)^n$
 Can we do this for $f(x) = |x|$ around $x=0$? 
 No, b/c $|x|$ does not exist at $x=0$, but $\sum_{n=0}^{\infty} a_n x^n$ is smooth on $B_R(0) \rightarrow$
 all derivatives exist

#1 do not erase

Weierstrass Approximation Thm: For every $f: [a,b] \rightarrow \mathbb{C}$ cont.,
 \exists polynomials $P_n(x) = \sum_{k=0}^n a_{n,k} x^k$, $a_{n,k} \in \mathbb{C}$, s.t.
 $P_n \rightarrow f$ uniformly on $[a,b]$

If $f: [a,b] \rightarrow \mathbb{R}$, then $a_{n,k}$ can be taken in \mathbb{R} .

Lemma: $\forall n \in \mathbb{Z}^+$, $c_n \equiv \int_{-1}^1 (1-x^2)^n dx \geq \frac{1}{\sqrt{n}}$

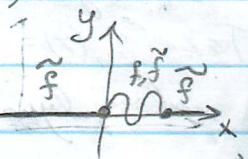
Sub-Lemma: $(1-x^2)^n \geq 1-nx^2 \forall x \in [0,1]$

Pf: enough to show $(1-x)^n \geq 1-nx \forall x \in [0,1]$
 LHS(0) = 1 = RHS(0)
 LHS'(x) = $-n(1-x)^{n-1} \geq -n \forall x \in [0,1], n \geq 1$
 RHS'(x) = $-n \leq$ LHS'(x) \leftarrow
 \rightarrow LHS(x) \geq RHS(x) \leftarrow

Pf of Lemma: $c_n = 2 \int_0^1 (1-x^2)^n dx \geq 2 \int_0^{1/\sqrt{n}} (1-x^2)^n dx$
 $\geq 2 \int_0^{1/\sqrt{n}} (1-nx^2) dx$
 Sublemma $\rightarrow \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}}$ ✓

do not erase

#4

Pf of Thm: Assume $[a,b] = [0,1], f(0), f(1) = 0$
 Extend to $\tilde{f}: \mathbb{R} \rightarrow \mathbb{C}$ by 0 
 Take $P_n(x) = \frac{1}{c_n} \int_{-\infty}^{\infty} \tilde{f}(t+x) (1-\frac{t^2}{n})^n dt$
 (1) $\forall x \in \mathbb{R}, P_n(x) \in \mathbb{C}$ is well-defined ($P_n(x) \in \mathbb{R}$ if $f: [0,1] \rightarrow \mathbb{R}$)
 (2) $P_n: \mathbb{R} \rightarrow \mathbb{C}$ is a polynomial
 (3) $P_n \rightarrow f$ uniformly on $[0,1] \Rightarrow$ done

#2

Pf of (2) $\tilde{f}(t+x) = 0$ unless $0 \leq t+x \leq 1$
 $\Rightarrow P_n(x) = \frac{1}{c_n} \int_{-x}^{1-x} \tilde{f}(t+x) (1-\frac{t^2}{n})^n dt = \frac{1}{c_n} \int_0^1 \tilde{f}(t) (1-\frac{(t-x)^2}{n})^n dt$
 \int of continuous function $g(t)$ on a bounded $[-x, 1-x]$ $\sum_{k=0}^{2n} x^k \int_0^1 \tilde{f}_{n,k}(t) dt$
 $\Rightarrow P_n(x) \in \mathbb{C}$ is well-defined
 $P_n, P_n(x)$ is a polynomial in x

#3

Note: f cont. on $[0,1] \Rightarrow f$ uniformly cont. on $[0,1]$
 $\Rightarrow \tilde{f}$ is uniformly continuous on \mathbb{R}
 Pf of (3) \exists let $\epsilon > 0$. Find $N > 0$ s.t.
 $|P_n(x) - f(x)| \leq 2\epsilon \forall n \geq N, x \in [0,1]$ ✗
 Take $M = \sup_{x \in \mathbb{R}} |\tilde{f}(x)| = \sup_{x \in [0,1]} |f(x)| \in \mathbb{R}$ b/c f cont. and $[0,1]$ comp.
 \tilde{f} uniformly cont. $\Rightarrow \exists \delta > 0$ s.t.
 $|\tilde{f}(x) - \tilde{f}(x')| < \epsilon$ if $|x-x'| < \delta$ ✗✗

do not erase top 3 lines

#4

$x \in [0, 1]$, $\tilde{f}(t+x) = 0$ unless $0 \leq t+x \leq 1$

$$\Rightarrow p_n(x) = \frac{1}{C_n} \int_{-1}^1 \tilde{f}(t+x) (1-t^2)^n dt$$

$$C_n = \int_{-1}^1 (1-t^2)^n dt \Rightarrow f(x) = \tilde{f}(x) = \frac{1}{C_n} \int_{-1}^1 \tilde{f}(x) (1-t^2)^n dt$$

$$\therefore |p_n(x) - f(x)| \leq \varepsilon + \frac{4M}{C_n} (1-\delta^2)^n$$

Lemma $\rightarrow \leq \varepsilon + 4M\sqrt{n} (1-\delta^2)^n \leq 2\varepsilon$

$\forall n \geq \text{some } N(M, \varepsilon) \forall \delta \lim_{n \rightarrow \infty} \sqrt{n} (1-\delta^2)^n = 0$

$\delta < 1 - \delta^2 < 1$

\therefore Given $\varepsilon > 0$, choose $\delta = \delta(\varepsilon) > 0$ from uniform cont. of f , then choose $N = N(M, \delta(\varepsilon))$ s.t. $\forall x \in [0, 1]$

#3

do not erase top line

#4

$$C_n |p_n(x) - f(x)| = \left| \int_{-1}^1 (\tilde{f}(t+x) - \tilde{f}(x)) (1-t^2)^n dt \right|$$

$$\leq \int_{-1}^1 |\tilde{f}(t+x) - \tilde{f}(x)| (1-t^2)^n dt$$

$$\leq \int_{-\delta}^{\delta} \varepsilon (1-t^2)^n dt + \left\{ \int_{-1}^{-\delta} + \int_{\delta}^1 \right\} 2M (1-t^2)^n dt$$

$\leftarrow \sup |\tilde{f}|$

$$\leq \varepsilon \cdot C_n + 4M (1-\delta^2)^n$$

Have shown existence in this case.

Suppose $f: [a, b] \rightarrow \mathbb{C}$ any continuous map, $b > a$.

Take $g: [0, 1] \rightarrow \mathbb{C}$, $g(x) = f(a + (b-a)x) - f(a) - \frac{f(b)-f(a)}{b-a}x$

$$\Rightarrow g \text{ cont.}, g(0) = g(1) = 0$$

$$f(x) = g\left(\frac{x-a}{b-a}\right) + f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

#2

trivial case $\Rightarrow \exists$ polynomials $q_n: \mathbb{R} \rightarrow \mathbb{C}$ s.t.

$q_n \rightarrow g$ uniformly on $[0, 1]$

$$\text{Take } p_n(x) = q_n\left(\frac{x-a}{b-a}\right) + f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

also polynomials

$$|p_n(x) - f(x)| = \left| q_n\left(\frac{x-a}{b-a}\right) - g\left(\frac{x-a}{b-a}\right) \right|$$

$\in [0, 1]$

$$q_n \rightarrow g \text{ uniformly on } [0, 1] \Rightarrow p_n \rightarrow f \text{ uniformly on } [a, b]$$

Pross: explicit formula for (do) (rest) p_n 's

p218 in $[a, b] = [0, 1]$ case

from finitely many values $f\left(\frac{k}{n}\right) \in \mathbb{C}$, $k \in \mathbb{Z}$, $0 \leq k \leq n$

good for applications/computers

but more technical pf.

More general perspective: $X = \text{set}$, $(Y, d_Y) = \text{metric space}$

$\text{Maps}(X, Y) = \text{maps from } X \text{ to } Y$

$A \subset \text{Maps}(X, Y)$ some subcollection of maps from X to Y

The uniform closure of A is

$$\bar{A} = \{f \in \text{Maps}(X, Y) : \exists \text{ sequence } f_n \in A \text{ s.t.}$$

$$f_n \rightarrow f \text{ uniformly on } X\} \supset A$$

HW10: \bar{A} is "uniformly closed", i.e. $\overline{\bar{A}} = \bar{A}$

Q: For what $A \subset \text{Map}(X, Y)$, is $\bar{A} = \text{Maps}(X, Y)$?

More interesting: $(X, d_X) = \text{metric space}$

$$A \subset \mathcal{C}(X, Y) = \{\text{continuous functions } f: (X, d_X) \rightarrow (Y, d_Y)\}$$

uniform convergence then $\Rightarrow \bar{A} \subset \mathcal{C}(X, Y)$

Q: For what $A \subset \mathcal{C}(X, Y)$, is $\bar{A} = \mathcal{C}(X, Y)$?

E.g. $A = \mathcal{P}([a, b], \mathbb{R}) \subset \mathcal{C}([a, b], \mathbb{R})$ real polynomials

$A = \mathcal{P}([a, b], \mathbb{C}) \subset \mathcal{C}([a, b], \mathbb{C})$ complex polynomials

$$\text{WAT} \Rightarrow \overline{\mathcal{P}}_{\mathbb{C}} = \mathcal{C}([a, b], \mathbb{C}) \subset \mathcal{C}([a, b], \mathbb{C})$$

⇒ Color chalk ←

Thm: Cr. of Weierstrass Approximation Thm: #4

∀ a, ε ∈ ℝ⁺; ∃ real polyn. p: ℝ → ℝ s.t.

p(0) = 0, |p(y) - y| < ε ∀ y ∈ [-a, a] (*)

Also: X = set, A ⊂ Maps(X, ℝ)

⇒ $\bar{A} \equiv \{f \in \text{Maps}(X, \mathbb{R}) : \exists f_n \in A \text{ s.t. } f_n \rightarrow f \text{ uniformly on } X\}$

A ⊂ \bar{A} , $\overline{\bar{A}} = \bar{A}$ (HW 10)

Also need: B(X, ℝ) ⊂ Maps(X, ℝ) bounded functions

Pf: Let A ⊂ Maps(X, ℝ) be an algebra (do not use)

(1) If A separates pts and does not vanish anywhere, (#2)

∃ x₁, x₂ ∈ X, x₁ ≠ x₂, c₁, c₂ ∈ ℝ, ∃ f ∈ A s.t. f(x₁) = c₁, f(x₂) = c₂

(2) If A ⊂ B(X, ℝ) = {bounded functions}, then

$\bar{A} \subset B(X, \mathbb{R})$ is also an algebra

(3) If $\bar{A} \subset B(X, \mathbb{R})$, then |f|, max(f, g), min(f, g) ∈ \bar{A} ∀ f, g ∈ \bar{A}

max(f, g) = 1/2(f + g + |f - g|), min(f, g) = 1/2(f + g - |f - g|) (#4)

\bar{A} algebra, h, g ∈ \bar{A} ⇒ max(f, g), min(f, g) ∈ \bar{A} ∀ f, g ∈ \bar{A}

Pf of (2): f_n ∈ B(X, ℝ), f_n → f uniformly ⇒ f bounded;

⇒ if A ⊂ B(X, ℝ), then $\bar{A} \subset B(X, \mathbb{R})$

f_n → f, g_n → g uniformly ⇒ f_n + g_n → f + g, f_ng_n → fg

⊂ f_n → cf uniformly ∀ c ∈ ℝ

Pf of (1): A separates x₁, x₂ ⇒ ∃ g ∈ A s.t. g(x₁) ≠ g(x₂) (#4)

A does not vanish at x₁, x₂ ⇒ ∃ f₁, f₂ ∈ A s.t. f₁(x₁) ≠ 0, f₂(x₂) ≠ 0

Let h₁ = (g - g(x₂))f₁, h₂ = (g - g(x₁))f₂: X → ℝ

h₁(x₁), h₂(x₂) ≠ 0, h₁(x₂), h₂(x₁) = 0

Take f = c₁/h₁(x₁)h₁ + c₂/h₂(x₂)h₂: X → ℝ

f(x₁) = c₁, f(x₂) = c₂

A ⊂ Maps(X, ℝ) algebra, g, f₁, f₂ ∈ A ⇒ h₁, h₂ ∈ A ⇒ f ∈ A

(#1) do not use

Def: (1) A ⊂ Maps(X, ℝ) is an algebra over ℝ if

f + g, f · g, cf ∈ A ∀ f, g ∈ A, c ∈ ℝ

(2) A ⊂ Maps(X, ℝ) separates points if

∀ x₁, x₂ ∈ X, x₁ ≠ x₂, ∃ f ∈ A s.t. f(x₁) ≠ f(x₂)

(3) A ⊂ Maps(X, ℝ) does not vanish anywhere if

∀ x ∈ X, ∃ f ∈ A s.t. f(x) ≠ 0

E.g. A = P([a, b], ℝ) = {polynomials} satisfy (1)-(3)

|f|, max(f, g), min(f, g): X → ℝ (#3)

|f|(x) = |f(x)|, max(f, g)(x) = max(f(x), g(x)), min(f, g)(x) = min(f(x), g(x))

Pf of (3): f ∈ $\bar{A} \subset B(X, \mathbb{R}) \Rightarrow a = \sup |f| \in \mathbb{R} \geq 0$

Let ε > 0 ⇒ ∃ c₁, ..., c_k ∈ ℝ s.t. |∑ c_iyⁱ - y| < ε ∀ y ∈ [a, a]

+ |f| ≤ a ⇒ |∑ c_if(x)ⁱ - |f(x)|| < ε ∀ x ∈ X (**)

\bar{A} algebra, f ∈ $\bar{A} \Rightarrow f \in \sum_{i=1}^k c_i f^i \in \bar{A} \forall \epsilon > 0$

⇒ f_n → f uniformly, $\bar{\bar{A}} = \bar{A} \Rightarrow f \in \bar{A} \checkmark$

⇒ if A ⊂ B(X, ℝ) is algebra, then $\bar{A} \subset B(X, \mathbb{R})$ is algebra (#3)

only (2) needs f, g ∈ B(X, ℝ)

|f_n(x)g_n(x) - f(x)g(x)| ≤ |f_n(x)(g_n(x) - g(x))| + |(f_n(x) - f(x))g(x)|

≤ (sup |f_n|) |g_n(x) - g(x)| + (sup |g_n|) |f_n(x) - f(x)|

∃ n_z some N(ε), x ∈ X
↑ doesn't depend on x

Stone-Weierstrass Approximation Thm (#3)

Let (X, d_X) be a compact metric space, C(X, ℝ) = {continuous f: X → ℝ}

If A ⊂ C(X, ℝ) is an algebra that separates pts in X and does not vanish anywhere on X, then $\bar{A} = C(X, \mathbb{R})$.

Pf: uniform convergence thm ⇒ $\bar{A} \subset C(X, \mathbb{R})$.

Let f ∈ C(X, ℝ), ε > 0. Find f_z ∈ \bar{A} s.t.

|f_z(x) - f(x)| < ε ∀ x ∈ X ⇒ f_{z} ∈ \bar{A}}

#4

Claim: $\forall \epsilon > 0, \exists g_x \in \bar{A}$ s.t. $g_x(x) = f(x), g_x(x') > f(x) - \epsilon \forall x' \in X$

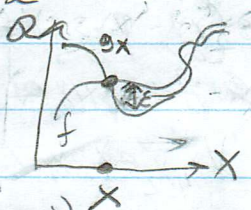
Assume claim. $\forall x \in X, \exists g_x \in \bar{A}$ s.t. $g_x(x) = f(x), g_x(x') > f(x) - \epsilon \forall x' \in X$

$\forall \epsilon > 0, \mathcal{U}_x = \{x' \in X : g_x(x') > f(x) - \epsilon\}$

in open in X (preimage of $(-\infty, \epsilon)$)

cont. $X \rightarrow \mathbb{R}, x' \rightarrow g_x(x') - f(x)$

$x \in \mathcal{U}_x$ b/c $g_x(x) = f(x) \Rightarrow \{x' \in X : g_x(x') > f(x) - \epsilon\}$ is an open cover



#1

X compact $\Rightarrow X \subset \mathcal{U}_{x_1} \cup \dots \cup \mathcal{U}_{x_m}$ for some $x_1, \dots, x_m \in X$

Take $f_\epsilon = \min(g_{x_1}, \dots, g_{x_m}) \in \bar{A}$ by Prop (3)

$\Rightarrow \forall x' \in \mathcal{U}_{x_i}, i=1, \dots, m: f(x') - \epsilon < f_\epsilon(x') \leq g_{x_i}(x') < f(x) + \epsilon$
true $\forall g_{x_i}$ by choice in claim by choice of \mathcal{U}_{x_i}

$X \subset \mathcal{U}_{x_1} \cup \dots \cup \mathcal{U}_{x_m} \Rightarrow f(x') - \epsilon < f_\epsilon(x') < f(x) + \epsilon$

#4

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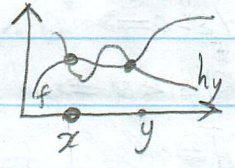
Pf of Claim: A algebra that separates pts and does not vanish on X

Prop (2) \Rightarrow so in $\bar{A} \Rightarrow \forall g \in \bar{A}, \exists h_1, h_2 \in A$ s.t. $h_1(x) = f(x), h_2(y) = f(y)$

$\forall \epsilon > 0, \mathcal{U}_y = \{x' \in X : h_2(x') > f(x) - \epsilon\}$

in open in X (preimage of $(-\epsilon, \infty)$)

cont. $h_2 - f : X \rightarrow \mathbb{R}$



$y \in \mathcal{U}_y$ b/c $h_2(y) = f(y) \Rightarrow \{x' \in X : h_2(x') > f(x) - \epsilon\}$ is open cover of X

#5

X compact $\Rightarrow X \subset \mathcal{U}_{y_1} \cup \dots \cup \mathcal{U}_{y_m}$ for some $y_1, \dots, y_m \in X$

Take $g_x = \max(h_{y_1}, \dots, h_{y_m}) \in \bar{A}$ by Prop (3)

$h_{y_i}(x) = f(x) \forall i \Rightarrow g_x(x) = f(x)$

$\forall x' \in \mathcal{U}_{y_i}, i=1, \dots, m: g_x(x') \geq h_{y_i}(x') > f(x) - \epsilon$ choice of \mathcal{U}_{y_i}

$X \subset \mathcal{U}_{y_1} \cup \dots \cup \mathcal{U}_{y_m} \Rightarrow g_x(x') > f(x) - \epsilon$

\Rightarrow Claim

#4

\mathbb{C} -case: $A \subset \text{Maps}(X, \mathbb{C})$ is an algebra over \mathbb{C}

if $f+g, f \cdot g, c f \in A \forall f, g \in A, c \in \mathbb{C}$

Such algebra is self-adjoint if $\bar{f} \in A \forall f \in A$

$\bar{f} : X \rightarrow \mathbb{C}, \bar{f}(x) = \overline{f(x)}$

$f : X \rightarrow \mathbb{C} \Rightarrow f = u + i v, u, v : X \rightarrow \mathbb{R}$

$u = \frac{1}{2}(f + \bar{f}), v = \frac{1}{2i}(f - \bar{f})$

\Rightarrow if $f \in A$ and A is self-adjoint algebra over \mathbb{C} , then $u, v \in A$

#5

Prop (1) applies to algebra A over \mathbb{C} with $a_1, a_2 \in \mathbb{C}$

that separates pts and does not vanish on X

$\Rightarrow \forall x_1, x_2 \in X, \exists f \in A \subset \text{Maps}(X, \mathbb{C})$ s.t. $f(x_1) = 1, f(x_2) = 0$

if A is self-adjoint, then $u = \text{Re } f \in A, u(x_1) = 1, u(x_2) = 0$

$\Rightarrow A_{\mathbb{R}} \in \bar{A} \cap \text{Map}(X, \mathbb{R})$ separates points and does not vanish on X .

A \mathbb{C} -algebra $\Rightarrow A_{\mathbb{R}}$ is \mathbb{R} -algebra

Thm: Let (X, d_X) be a compact metric space, $\mathcal{C}(X, \mathbb{C}) = \{f : X \rightarrow \mathbb{C}\}$

If $A \subset \mathcal{C}(X, \mathbb{C})$ is a self-adjoint algebra over \mathbb{C} that separates pts and does not vanish on X , then $\bar{A} = \mathcal{C}(X, \mathbb{C})$

Pf: last part $\Rightarrow A_{\mathbb{R}} = A \cap \text{Maps}(X, \mathbb{R}) \subset \mathcal{C}(X, \mathbb{R})$

an algebra over \mathbb{R} that separates pts and does not vanish on X

$\Rightarrow \bar{A}_{\mathbb{R}} = \mathcal{C}(X, \mathbb{R}), \bar{A}_{\mathbb{R}} = \bar{A} \cap \text{Maps}(X, \mathbb{R}) = \bar{A}_{\mathbb{R}}$

A \mathbb{C} -algebra, $\bar{A}_{\mathbb{R}} = \mathcal{C}(X, \mathbb{R}) \Rightarrow \bar{A} = \mathcal{C}(X, \mathbb{C})$

b/c $f = u + i v \leftarrow$ anything in $\bar{A} \subset \mathcal{C}(X, \mathbb{C})$
anything in $\bar{A} = \mathcal{C}(X, \mathbb{C})$

$f = u + i v : X \rightarrow \mathbb{C}$ cont. if $u, v : X \rightarrow \mathbb{R}$ cont.

Rudin's Principles of Mathematical Analysis

THE STONE-WEIERSTRASS THEOREM

7.26 Theorem *If f is a continuous complex function on $[a, b]$, there exists a sequence of polynomials P_n such that*

$$\lim_{n \rightarrow \infty} P_n(x) = f(x)$$

uniformly on $[a, b]$. If f is real, the P_n may be taken real.

This is the form in which the theorem was originally discovered by Weierstrass.

Proof We may assume, without loss of generality, that $[a, b] = [0, 1]$. We may also assume that $f(0) = f(1) = 0$. For if the theorem is proved for this case, consider

$$g(x) = f(x) - f(0) - x[f(1) - f(0)] \quad (0 \leq x \leq 1).$$

Here $g(0) = g(1) = 0$, and if g can be obtained as the limit of a uniformly convergent sequence of polynomials, it is clear that the same is true for f , since $f - g$ is a polynomial.

Furthermore, we define $f(x)$ to be zero for x outside $[0, 1]$. Then f is uniformly continuous on the whole line.

We put

$$(47) \quad Q_n(x) = c_n(1 - x^2)^n \quad (n = 1, 2, 3, \dots),$$

where c_n is chosen so that

$$(48) \quad \int_{-1}^1 Q_n(x) dx = 1 \quad (n = 1, 2, 3, \dots).$$

We need some information about the order of magnitude of c_n . Since

$$\begin{aligned} \int_{-1}^1 (1 - x^2)^n dx &= 2 \int_0^1 (1 - x^2)^n dx \geq 2 \int_0^{1/\sqrt{n}} (1 - x^2)^n dx \\ &\geq 2 \int_0^{1/\sqrt{n}} (1 - nx^2) dx \\ &= \frac{4}{3\sqrt{n}} \\ &> \frac{1}{\sqrt{n}}, \end{aligned}$$

it follows from (48) that

$$(49) \quad c_n < \sqrt{n}.$$

The inequality $(1 - x^2)^n \geq 1 - nx^2$ which we used above is easily shown to be true by considering the function

$$(1 - x^2)^n - 1 + nx^2$$

which is zero at $x = 0$ and whose derivative is positive in $(0, 1)$.

For any $\delta > 0$, (49) implies

$$(50) \quad Q_n(x) \leq \sqrt{n}(1 - \delta^2)^n \quad (\delta \leq |x| \leq 1),$$

so that $Q_n \rightarrow 0$ uniformly in $\delta \leq |x| \leq 1$.

Now set

$$(51) \quad P_n(x) = \int_{-1}^1 f(x+t)Q_n(t) dt \quad (0 \leq x \leq 1).$$

Our assumptions about f show, by a simple change of variable, that

$$P_n(x) = \int_{-x}^{1-x} f(x+t)Q_n(t) dt = \int_0^1 f(t)Q_n(t-x) dt,$$

and the last integral is clearly a polynomial in x . Thus $\{P_n\}$ is a sequence of polynomials, which are real if f is real.

Given $\varepsilon > 0$, we choose $\delta > 0$ such that $|y - x| < \delta$ implies

$$|f(y) - f(x)| < \frac{\varepsilon}{2}.$$

Let $M = \sup |f(x)|$. Using (48), (50), and the fact that $Q_n(x) \geq 0$, we see that for $0 \leq x \leq 1$,

$$\begin{aligned} |P_n(x) - f(x)| &= \left| \int_{-1}^1 [f(x+t) - f(x)]Q_n(t) dt \right| \\ &\leq \int_{-1}^1 |f(x+t) - f(x)|Q_n(t) dt \\ &\leq 2M \int_{-1}^{-\delta} Q_n(t) dt + \frac{\varepsilon}{2} \int_{-\delta}^{\delta} Q_n(t) dt + 2M \int_{\delta}^1 Q_n(t) dt \\ &\leq 4M \sqrt{n}(1 - \delta^2)^n + \frac{\varepsilon}{2} \\ &< \varepsilon \end{aligned}$$

for all large enough n , which proves the theorem.

It is instructive to sketch the graphs of Q_n for a few values of n ; also, note that we needed uniform continuity of f to deduce uniform convergence of $\{P_n\}$.

In the proof of Theorem 7.32 we shall not need the full strength of Theorem 7.26, but only the following special case, which we state as a corollary.

7.27 Corollary For every interval $[-a, a]$ there is a sequence of real polynomials P_n such that $P_n(0) = 0$ and such that

$$\lim_{n \rightarrow \infty} P_n(x) = |x|$$

uniformly on $[-a, a]$.

Proof By Theorem 7.26, there exists a sequence $\{P_n^*\}$ of real polynomials which converges to $|x|$ uniformly on $[-a, a]$. In particular, $P_n^*(0) \rightarrow 0$ as $n \rightarrow \infty$. The polynomials

$$P_n(x) = P_n^*(x) - P_n^*(0) \quad (n = 1, 2, 3, \dots)$$

have desired properties.

We shall now isolate those properties of the polynomials which make the Weierstrass theorem possible.

7.28 Definition A family \mathcal{A} of complex functions defined on a set E is said to be an *algebra* if (i) $f + g \in \mathcal{A}$, (ii) $fg \in \mathcal{A}$, and (iii) $cf \in \mathcal{A}$ for all $f \in \mathcal{A}$, $g \in \mathcal{A}$ and for all complex constants c , that is, if \mathcal{A} is closed under addition, multiplication, and scalar multiplication. We shall also have to consider algebras of real functions; in this case, (iii) is of course only required to hold for all real c .

If \mathcal{A} has the property that $f \in \mathcal{A}$ whenever $f_n \in \mathcal{A}$ ($n = 1, 2, 3, \dots$) and $f_n \rightarrow f$ uniformly on E , then \mathcal{A} is said to be *uniformly closed*.

Let \mathcal{B} be the set of all functions which are limits of uniformly convergent sequences of members of \mathcal{A} . Then \mathcal{B} is called the *uniform closure* of \mathcal{A} . (See Definition 7.14.)

For example, the set of all polynomials is an algebra, and the Weierstrass theorem may be stated by saying that the set of continuous functions on $[a, b]$ is the uniform closure of the set of polynomials on $[a, b]$.

7.29 Theorem Let \mathcal{B} be the uniform closure of an algebra \mathcal{A} of bounded functions. Then \mathcal{B} is a uniformly closed algebra.

Proof If $f \in \mathcal{B}$ and $g \in \mathcal{B}$, there exist uniformly convergent sequences $\{f_n\}, \{g_n\}$ such that $f_n \rightarrow f, g_n \rightarrow g$ and $f_n \in \mathcal{A}, g_n \in \mathcal{A}$. Since we are dealing with bounded functions, it is easy to show that

$$f_n + g_n \rightarrow f + g, \quad f_n g_n \rightarrow fg, \quad cf_n \rightarrow cf,$$

where c is any constant, the convergence being uniform in each case.

Hence $f + g \in \mathcal{B}, fg \in \mathcal{B}$, and $cf \in \mathcal{B}$, so that \mathcal{B} is an algebra.

By Theorem 2.27, \mathcal{B} is (uniformly) closed.

7.30 Definition Let \mathcal{A} be a family of functions on a set E . Then \mathcal{A} is said to *separate points* on E if to every pair of distinct points $x_1, x_2 \in E$ there corresponds a function $f \in \mathcal{A}$ such that $f(x_1) \neq f(x_2)$.

If to each $x \in E$ there corresponds a function $g \in \mathcal{A}$ such that $g(x) \neq 0$, we say that \mathcal{A} *vanishes at no point of E* .

The algebra of all polynomials in one variable clearly has these properties on R^1 . An example of an algebra which does not separate points is the set of all even polynomials, say on $[-1, 1]$, since $f(-x) = f(x)$ for every even function f .

The following theorem will illustrate these concepts further.

7.31 Theorem Suppose \mathcal{A} is an algebra of functions on a set E , \mathcal{A} separates points on E , and \mathcal{A} vanishes at no point of E . Suppose x_1, x_2 are distinct points of E , and c_1, c_2 are constants (real if \mathcal{A} is a real algebra). Then \mathcal{A} contains a function f such that

$$f(x_1) = c_1, \quad f(x_2) = c_2.$$

Proof The assumptions show that \mathcal{A} contains functions g, h , and k such that

$$g(x_1) \neq g(x_2), \quad h(x_1) \neq 0, \quad k(x_2) \neq 0.$$

Put

$$u = gk - g(x_1)k, \quad v = gh - g(x_2)h.$$

Then $u \in \mathcal{A}, v \in \mathcal{A}, u(x_1) = v(x_2) = 0, u(x_2) \neq 0$, and $v(x_1) \neq 0$. Therefore

$$f = \frac{c_1 v}{v(x_1)} + \frac{c_2 u}{u(x_2)}$$

has the desired properties.

We now have all the material needed for Stone's generalization of the Weierstrass theorem.

7.32 Theorem Let \mathcal{A} be an algebra of real continuous functions on a compact set K . If \mathcal{A} separates points on K and if \mathcal{A} vanishes at no point of K , then the uniform closure \mathcal{B} of \mathcal{A} consists of all real continuous functions on K .

We shall divide the proof into four steps.

STEP 1 If $f \in \mathcal{B}$, then $|f| \in \mathcal{B}$.

Proof Let

$$(52) \quad a = \sup |f(x)| \quad (x \in K)$$

and let $\varepsilon > 0$ be given. By Corollary 7.27 there exist real numbers c_1, \dots, c_n such that

$$(53) \quad \left| \sum_{t=1}^n c_t y^t - |y| \right| < \varepsilon \quad (-a \leq y \leq a).$$

Since \mathcal{B} is an algebra, the function

$$g = \sum_{t=1}^n c_t f^t$$

is a member of \mathcal{B} . By (52) and (53), we have

$$|g(x) - |f(x)|| < \varepsilon \quad (x \in K).$$

Since \mathcal{B} is uniformly closed, this shows that $|f| \in \mathcal{B}$.

STEP 2 If $f \in \mathcal{B}$ and $g \in \mathcal{B}$, then $\max(f, g) \in \mathcal{B}$ and $\min(f, g) \in \mathcal{B}$.

By $\max(f, g)$ we mean the function h defined by

$$h(x) = \begin{cases} f(x) & \text{if } f(x) \geq g(x), \\ g(x) & \text{if } f(x) < g(x), \end{cases}$$

and $\min(f, g)$ is defined likewise.

Proof Step 2 follows from step 1 and the identities

$$\max(f, g) = \frac{f+g}{2} + \frac{|f-g|}{2},$$

$$\min(f, g) = \frac{f+g}{2} - \frac{|f-g|}{2}.$$

By iteration, the result can of course be extended to any finite set of functions: If $f_1, \dots, f_n \in \mathcal{B}$, then $\max(f_1, \dots, f_n) \in \mathcal{B}$, and

$$\min(f_1, \dots, f_n) \in \mathcal{B}.$$

STEP 3 Given a real function f , continuous on K , a point $x \in K$, and $\varepsilon > 0$, there exists a function $g_x \in \mathcal{B}$ such that $g_x(x) = f(x)$ and

$$(54) \quad g_x(t) > f(t) - \varepsilon \quad (t \in K).$$

Proof Since $\mathcal{A} \subset \mathcal{B}$ and \mathcal{A} satisfies the hypotheses of Theorem 7.31 so does \mathcal{B} . Hence, for every $y \in K$, we can find a function $h_y \in \mathcal{B}$ such that

$$(55) \quad h_y(x) = f(x), \quad h_y(y) = f(y).$$

By the continuity of h , there exists an open set J_y , containing y , such that

$$(56) \quad h_y(t) > f(t) - \varepsilon \quad (t \in J_y).$$

Since K is compact, there is a finite set of points y_1, \dots, y_n such that

$$(57) \quad K \subset J_{y_1} \cup \dots \cup J_{y_n}.$$

Put

$$g_x = \max(h_{y_1}, \dots, h_{y_n}).$$

By step 2, $g_x \in \mathcal{B}$, and the relations (55) to (57) show that g_x has the other required properties.

STEP 4 Given a real function f , continuous on K , and $\varepsilon > 0$, there exists a function $h \in \mathcal{B}$ such that

$$(58) \quad |h(x) - f(x)| < \varepsilon \quad (x \in K).$$

Since \mathcal{B} is uniformly closed, this statement is equivalent to the conclusion of the theorem.

Proof Let us consider the functions g_x , for each $x \in K$, constructed in step 3. By the continuity of g_x , there exist open sets V_x containing x , such that

$$(59) \quad g_x(t) < f(t) + \varepsilon \quad (t \in V_x).$$

Since K is compact, there exists a finite set of points x_1, \dots, x_m such that

$$(60) \quad K \subset V_{x_1} \cup \dots \cup V_{x_m}.$$

Put

$$h = \min(g_{x_1}, \dots, g_{x_m}).$$

By step 2, $h \in \mathcal{B}$, and (54) implies

$$(61) \quad h(t) > f(t) - \varepsilon \quad (t \in K),$$

whereas (59) and (60) imply

$$(62) \quad h(t) < f(t) + \varepsilon \quad (t \in K).$$

Finally, (58) follows from (61) and (62).

Theorem 7.32 does not hold for complex algebras. A counterexample is given in Exercise 21. However, the conclusion of the theorem does hold, even for complex algebras, if an extra condition is imposed on \mathcal{A} , namely, that \mathcal{A} be *self-adjoint*. This means that for every $f \in \mathcal{A}$ its complex conjugate \bar{f} must also belong to \mathcal{A} ; \bar{f} is defined by $\bar{f}(x) = \overline{f(x)}$.

7.33 Theorem *Suppose \mathcal{A} is a self-adjoint algebra of complex continuous functions on a compact set K , \mathcal{A} separates points on K , and \mathcal{A} vanishes at no point of K . Then the uniform closure \mathcal{B} of \mathcal{A} consists of all complex continuous functions on K . In other words, \mathcal{A} is dense $\mathcal{C}(K)$.*

Proof Let \mathcal{A}_R be the set of all real functions on K which belong to \mathcal{A} .

If $f \in \mathcal{A}$ and $f = u + iv$, with u, v real, then $2u = f + \bar{f}$, and since \mathcal{A} is self-adjoint, we see that $u \in \mathcal{A}_R$. If $x_1 \neq x_2$, there exists $f \in \mathcal{A}$ such that $f(x_1) = 1, f(x_2) = 0$; hence $0 = u(x_2) \neq u(x_1) = 1$, which shows that \mathcal{A}_R separates points on K . If $x \in K$, then $g(x) \neq 0$ for some $g \in \mathcal{A}$, and there is a complex number λ such that $\lambda g(x) > 0$; if $f = \lambda g, f = u + iv$, it follows that $u(x) > 0$; hence \mathcal{A}_R vanishes at no point of K .

Thus \mathcal{A}_R satisfies the hypotheses of Theorem 7.32. It follows that every real continuous function on K lies in the uniform closure of \mathcal{A}_R , hence lies in \mathcal{B} . If f is a complex continuous function on $K, f = u + iv$, then $u \in \mathcal{B}, v \in \mathcal{B}$, hence $f \in \mathcal{B}$. This completes the proof.

EXERCISES

1. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.
2. If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E , prove that $\{f_n + g_n\}$ converges uniformly on E . If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E .
3. Construct sequences $\{f_n\}, \{g_n\}$ which converge uniformly on some set E , but such that $\{f_n g_n\}$ does not converge uniformly on E (of course, $\{f_n g_n\}$ must converge on E).
4. Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x}.$$

For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?