MAT 319/320: Basics of Analysis
From sup to inf

This note contains the proof of Corollary 4.5 presented in class, after David's suggestion.

4.4 Completeness Axiom for $\mathbb{R}$. If $S \subset \mathbb{R}$ is non-empty and bounded above, then there exists $\sup S \in \mathbb{R}$.

4.5 Corollary. If $S \subset \mathbb{R}$ is non-empty and bounded below, then there exists $\inf S \in \mathbb{R}$.

Proof. Let

$$S' = \big\{ s' \in \mathbb{R} \colon s' \text{ is a lower bound for } S \big\} \subset \mathbb{R}.$$

By definition,

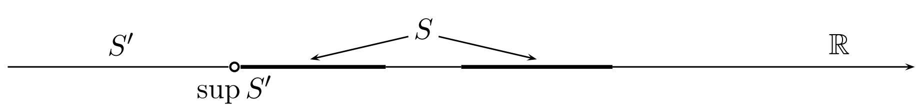
$$s' \leq s \qquad \forall\ s \in S,\ s' \leq S'. \tag{1}$$

Since S is bounded below, $S' \neq \emptyset$. Since $S \neq \emptyset$, S' is bounded above (by (1), *any* element $s \in S$ is an upper bound for S'). By the Completeness Axiom for $\mathbb{R}$, there exists $\sup S' \in \mathbb{R}$.

By definition, $s' \leq \sup S'$ for all $s' \in S'$, i.e. none of the lower bounds for S is larger than $\sup S'$. We show below that

$$\sup S' \in S', \tag{2}$$

i.e. $\sup S'$ is itself a lower bound for S. Along with the previous statement, this implies that $\sup S'$ is the *greatest* lower bound for S, i.e. $\inf S$.



Suppose (2) is not true, i.e. $\sup S'$ is not a lower bound for S. Thus, there exists $s \in S$ such that $s < \sup S'$. Along with (1), this implies that

$$s' \leq s < \sup S' \qquad \forall\ s' \in S'.$$

Thus, s is an upper bound for S' smaller than $\inf S'$. Since $\sup S'$ is the *smallest* upper bound for S', this is impossible. Therefore, (2) is true. $\square$

The derivation of 4.5 from 4.4 in the book uses

$$\inf S = -\big(\sup(-S)\big), \qquad \text{where} \quad -S \equiv \big\{ -s \colon s \in S \big\}, \tag{3}$$

after checking that this is true. The latter already requires about as much work as above, but (3) is a useful trick in other settings (so make sure to verify (3); the book does this).