

## MAT 320: Introduction to Analysis, Spring 2019

### Homework Assignment 9

Please read Ross's Sections 23-26 thoroughly and before starting on the problem set below.

*Optional supplementary reading:* pp143-158 of Rudin's book

Problem Set 9 (**due at the start of recitation on Wednesday, 4/24**): 23.2\*, Problems P-S (below and next page); \*answers only on 23.2 (8 in total)

#### Problem P

Suppose the radius of convergence of  $\sum a_n z^n$  is  $R \in \mathbb{R}^+$ .

- (a) Suppose  $a_n \geq 0$  and the series converges for  $z = R$ . Show that the series converges for every  $z \in \mathbb{C}$  with  $|z| \leq R$ .
- (b) Give an example of a series that converges at  $z = 1$  and diverges at  $z = -1$ .

#### Problem R

Let  $X$  be a set,  $(Y, d_Y)$  be a complete metric space, and  $f_n : X \rightarrow Y$  be a uniformly Cauchy sequence of functions.

- (a) Show that the sequence  $f_n$  converges uniformly to some function  $f : X \rightarrow Y$ .
- (b) Suppose in addition that  $d_X$  is a metric on  $X$  and each  $f_n$  is continuous function from  $(X, d_X)$  to  $(Y, d_Y)$ . Show that the sequence  $f_n$  converges uniformly to some continuous function  $f$  from  $(X, d_X)$  to  $(Y, d_Y)$ .

*Hint:* A special case of (a) is HW6-13.3c. This problem is also related to Theorem 24.3, a more general version of which was proved in class, and to Theorem 25.4.

#### Problem Q

Let  $X$  be a set and  $(Y, d_Y)$  be a metric space. A function  $f : X \rightarrow Y$  is called **bounded** if  $f(X) \subset Y$  is a bounded subset (contained in some  $B_R^Y(y)$ ). Denote by  $\mathfrak{B}(X, Y)$  the set of bounded functions from  $X$  to  $Y$ . Define

$$d : \mathfrak{B}(X, Y)^2 \rightarrow \mathbb{R}, \quad d(f, g) = \sup_{x \in X} d_Y(f(x), g(x)).$$

- (a) Show that  $d$  is well-defined (takes values in  $\mathbb{R}$ ) and is a metric on  $\mathfrak{B}(X, Y)$ .
- (b) Show that  $d$  is a complete metric on  $\mathfrak{B}(X, Y)$  if  $d_Y$  is a complete metric on  $Y$ .

Suppose in addition that  $d_X$  is a metric on  $X$ . Let  $\mathfrak{CB}(X, Y) \subset \mathfrak{B}(X, Y)$  be the subset of continuous bounded functions from  $(X, d_X)$  to  $(Y, d_Y)$ .

- (c) Show that (the restriction of) the metric  $d$  on  $\mathfrak{CB}(X, Y)$  is complete if  $d_Y$  is a complete metric on  $Y$ .
- (d) Suppose  $f_n \in \mathfrak{CB}(X, Y)$  is a sequence of uniformly continuous functions which converges to a uniformly continuous function  $f \in \mathfrak{CB}(X, Y)$  with respect to the metric  $d$ . Show that the sequence of functions  $f_n$  is **equicontinuous**, i.e. for every  $\epsilon \in \mathbb{R}^+$  there exists  $\delta \in \mathbb{R}^+$  such that

$$d_Y(f_n(x), f_n(x')) < \epsilon \quad \forall n \in \mathbb{N}, x, x' \in X \text{ s.t. } d_X(x, x') < \delta;$$

thus,  $\delta$  depends only on  $\epsilon$ , not on  $n$ .

### Problem S

Power series are typically used to “break” a function into a sequence of numbers (the Taylor and Fourier coefficients of the function). However, sometimes it is useful to go in the opposite direction, assembling a sequence of numbers into a function.

Let  $f_n$  be the  $n$ -th Fibonacci number defined recursively by

$$f_0 = 0, \quad f_1 = 1, \quad f_{n+2} = f_n + f_{n+1} \quad \forall n \in \mathbb{Z}^{\geq 0}.$$

Let

$$A_n = \sum_{k=1}^{k=n} k = 1+2+\dots+n, \quad B_n = \sum_{k=1}^{k=n} k^2 = 1^2+2^2+\dots+n^2 \quad \forall n \in \mathbb{Z}^{\geq 0}.$$

- (a) Give a recursive definition of the numbers  $A_n, B_n$  with  $n \geq 0$ .
- (b) Use mathematical induction and only the recursive definitions of  $f_n, A_n, B_n$  to show that  $f_n, A_n, B_n \leq 5^n$  for all  $n \geq 0$ .
- (c) Use the *Absolute Convergence* and *Comparison* Tests and only part (b) to show that the power series

$$f(x) = \sum_{n=0}^{\infty} f_n x^n, \quad A(x) = \sum_{n=0}^{\infty} A_n x^n, \quad B(x) = \sum_{n=0}^{\infty} B_n x^n,$$

converge if  $|x| < 1/6$  (and thus determine smooth functions near  $x=0$ ).

- (d) Using only the recursive definitions of  $f_n, A_n, B_n$ , show that

$$f(x) = x + x f(x) + x^2 f(x), \quad A(x) = x A(x) + \frac{x}{(1-x)^2}, \quad B(x) = x B(x) + \frac{x}{(1-x)^2} + \frac{2x^2}{(1-x)^3}.$$

*Hint:* You'll need to use identities such as the following:

$$\frac{1}{(1-x)^3} = \frac{1}{2} \left( \frac{1}{1-x} \right)'' = \frac{1}{2} \left( \sum_{n=0}^{\infty} x^n \right)'' = \frac{1}{2} \sum_{n=0}^{\infty} n(n-1) x^{n-2}.$$

- (e) Using only part (d), express  $f_n, A_n$ , and  $B_n$  explicitly in terms of  $n$ .  
*Hint:* use (d) to solve for  $f, A$ , and  $B$  and expand them into Taylor series around  $x=0$  (partial fractions might help in the case of  $f$ ); compare the result with the definitions of  $f, A$ , and  $B$  in (c).

*Note:* For  $f_n$ , you should end up with a formula involving two square roots. There is a much simpler way of finding an explicit formula for  $A_n$ ; so you can check your answer, but please deduce this formula from (b). The answer for  $B_n$  can be confirmed using induction (or google).