MAT 320: Introduction to Analysis, Spring 2019
Homework Assignment 11

Please study Ross's Sections 32-34 before starting on the problem set below.

Optional supplementary reading: pp120-134 of Rudin's book

Problem Set 11 (**due at the start of recitation on Wednesday, 5/8**): 32.2, 32.6, 33.4, 33.14, 34.6, 34.10, Problems W and X below

Problem W

By Exercise 17.14 on HW8, the function

$$f\colon [0,1] \longrightarrow \mathbb{R}, \qquad f(x) = \begin{cases} \frac{1}{q}, & \text{if } x = p/q,\ p,q \in \mathbb{Z}^+, \gcd(p,q)=1; \\ 0, & \text{otherwise;} \end{cases}$$

is continuous at every $x \in [0,1]-\mathbb{Q}$ and discontinuous at every $x \in [0,1]\cap\mathbb{Q}$. Show that nevertheless this bounded function is Riemann integrable by computing the lower and upper sums explicitly.

Problem X

Let $f\colon [a,b] \longrightarrow \mathbb{R}^{\ge 0}$ be a bounded Riemann integrable function such that

$$\int_a^b f \, \mathrm{d}x = 0 \,.$$

For a bounded interval $I \subset \mathbb{R}$, let $\ell(I) \in \mathbb{R}^+$ denote its length.

(a) Show that for every $n \in \mathbb{Z}^+$ and $\epsilon > 0$ there exists a finite collection of intervals $I_1, \ldots, I_m \subset [a,b]$ so that

$$f^{-1}\big([1/n, \infty)\big) \subset \bigcup_{k=1}^{m} I_k \qquad \text{and} \qquad \sum_{k=1}^{m} \ell(I_k) < \epsilon \,.$$

(b) Show that for every $\epsilon > 0$ there exists a countable collection of intervals $I_1, I_2, \ldots \subset [a,b]$ so that

$$\big\{x \in \mathbb{R} : f(x) \neq 0\big\} \subset \bigcup_{k=1}^{\infty} I_k \qquad \text{and} \qquad \sum_{k=1}^{\infty} \ell(I_k) < \epsilon \,.$$