MAT 320: Introduction to Analysis, Spring 2019
Homework Assignment 10

Please study Ross's Section 27 and Rudin's pp159-165 before starting on the problem set below.

Problem Set 10 (**due at the start of recitation on Wednesday, 5/1**): 26.8*, 27.2, Problems T-V (below and next page); *in (a), $(1)^n$ should be $(-1)^n$

Problem T

Let X be a set and (Y,d_Y) be a metric space. Denote by $\mathcal{M}(X,Y)$ the set of all functions from X to Y. Define

$$d\colon \mathcal{M}(X,Y)^2 \longrightarrow \mathbb{R}, \qquad d(f,g) = \sup_{x\in X} \min\big(d_Y\big(f(x),g(x)\big),1\big)\,.$$

Show that

(a) d is well-defined (takes values in $\mathbb{R}$) and is a metric on $\mathcal{M}(X,Y)$;

(b) a sequence of functions $f_n \in \mathcal{M}(X,Y)$ converges uniformly to $f \in \mathcal{M}(X,Y)$ if and only if f_n converges to f with respect to the metric d;

(c) the uniform closure $\overline{\mathcal{A}}$ of any subcollection $\mathcal{A}\subset\mathcal{M}(X,Y)$, as defined in class (and on p161 in Rudin), is uniformly closed, i.e. $\overline{\overline{\mathcal{A}}}=\overline{\mathcal{A}}$.

Hint: (c) can be obtained from basic properties of metric spaces via (b) or by unwinding the relevant definitions; the former would be about a line.

Problem U

Let $S^1\subset\mathbb{C}$ be the unit circle centered at $0\in\mathbb{C}$ with a metric obtained by restricting a standard metric on $\mathbb{C}$ and

$$q\colon \mathbb{R}\longrightarrow S^1, \qquad q(t)=\mathrm{e}^{2\pi \mathrm{i}t}\,.$$

Denote by $\mathcal{C}(S^1;\mathbb{C})$ the vector space of continuous $\mathbb{C}$-valued functions and by $\mathcal{P}_1(\mathbb{R};\mathbb{C})$ the vector space of continuous 1-periodic functions $\widetilde{f}$ on $\mathbb{R}$, i.e. $\widetilde{f}(t+1)=\widetilde{f}(t)$ for all $t\in\mathbb{R}$. Show that

(a) the map q is continuous, $f\circ q\in\mathcal{P}_1(\mathbb{R};\mathbb{C})$ for every $f\in\mathcal{C}(S^1;\mathbb{C})$, and for every $\widetilde{f}\in\mathcal{P}_1(\mathbb{R};\mathbb{C})$ there exists a unique $f\in\mathcal{C}(S^1;\mathbb{C})$ such that $\widetilde{f}=f\circ q$;

(b) for every $\widetilde{f}\in\mathcal{P}_1(\mathbb{R};\mathbb{C})$ and every $\epsilon>0$ there exist $N\in\mathbb{Z}^+$ and $a_n\in\mathbb{C}$ with $n\in\mathbb{Z}$ such that

$$\bigg|\widetilde{f}(t)-\sum_{n=-N}^{n=N}a_n\mathrm{e}^{2\pi\mathrm{i}\,nt}\bigg|\le\epsilon \qquad \forall\ t\in\mathbb{R}.$$

Hint: apply Rudin's Theorem 7.33 via part (a).

Problem V

Let $\mathcal{P}_1(\mathbb{R};\mathbb{C})$ be as in Problem U. For $f,g\in\mathcal{P}_1(\mathbb{R};\mathbb{C})$ and $n\in\mathbb{Z}$, define

$$\langle\!\langle f,g\rangle\!\rangle_2=\int_0^1 f\overline{g}\,\mathrm{d}t\in\mathbb{C},\qquad c_n(f)=\langle\!\langle f,\mathrm{e}^{2\pi\mathrm{i}\,nt}\rangle\!\rangle_2\,.$$

(a) Show that the collection $\{\mathrm{e}^{2\pi\mathrm{i}\,nt}\colon n\in\mathbb{Z}^+\}$ consists of orthonormal elements of the Hermitian vector space $(\mathcal{P}_1(\mathbb{R};\mathbb{C}),\langle\!\langle\cdot,\cdot\rangle\!\rangle_2)$, i.e.

$$\langle\!\langle \mathrm{e}^{2\pi\mathrm{i}\,mt},\mathrm{e}^{2\pi\mathrm{i}\,nt}\rangle\!\rangle_2=\begin{cases}1, & \text{if } m=n;\\ 0, & \text{if } m\neq n.\end{cases}$$

(b) Suppose $f\in\mathcal{P}_1(\mathbb{R};\mathbb{C})$ is twice continuously differentiable. Show that

$$c_n(f)=\frac{1}{2\pi\mathrm{i}\,n}c_n(f')=-\frac{1}{4\pi^2\,n^2}c_n(f'')\qquad\forall\;n\neq0$$

and that the sum

$$\sum_{n\in\mathbb{Z}}c_n(f)\mathrm{e}^{2\pi\mathrm{i}\,nt}\equiv\lim_{k,m\longrightarrow\infty}\sum_{n=-k}^{n=m}c_n(f)\mathrm{e}^{2\pi\mathrm{i}\,nt}$$

converges uniformly to a continuous function $h_f\colon\mathbb{R}\longrightarrow\mathbb{C}$ with $c_n(h_f)=c_n(f)$ for all $n\in\mathbb{Z}$.

(c) Suppose $f\in\mathcal{P}_1(\mathbb{R};\mathbb{C})$ is twice continuously differentiable. Show that

$$f(t)=\sum_{n\in\mathbb{Z}}c_n(f)\mathrm{e}^{2\pi\mathrm{i}\,nt}\qquad\forall\;t\in\mathbb{R}$$

with the sum converging uniformly on $\mathbb{R}$.

Hint: for the first statement in (b), use integration by parts; for (c), show that $\|f-h_f\|_2^2=0$ by applying the conclusion of Problem U(b) to $\widetilde{f}=f-h_f$ and using the Cauchy-Schwartz inequality

$$\langle\!\langle\widetilde{f},g\rangle\!\rangle_2\le\|f\|_2\|g\|_2$$

which holds for all Hermitian vector spaces.

Note: This says that the Fourier series of a function f as in (c) converges to f uniformly on $\mathbb{R}$. If f is smooth, then a similar argument shows that

$$\begin{aligned}\frac{\mathrm{d}^\ell}{\mathrm{d}t^\ell}f&=\sum_{n\in\mathbb{Z}}\big(2\pi\mathrm{i}n\big)^\ell c_n(f)\mathrm{e}^{2\pi\mathrm{i}\,nt}\equiv\lim_{k,m\longrightarrow\infty}\sum_{n=-k}^{n=m}\big(2\pi\mathrm{i}n\big)^\ell c_n(f)\mathrm{e}^{2\pi\mathrm{i}\,nt}\\&=\lim_{k,m\longrightarrow\infty}\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}\sum_{n=-k}^{n=m}c_n(f)\mathrm{e}^{2\pi\mathrm{i}\,nt}\end{aligned}$$

i.e. the Fourier series of f converges to f uniformly with all derivatives.