⇒ HW2 due Th ⇐
by 23:59pm

MAT 127

02/08/22

Last week: approximating solutions of 1st-order differential equations in 2 ways

Tu: Direction fields → sketch solution curves for $y'=f(x,y)$

(1) compute slopes $y'=f(x_i,y_j)$ at lots of (x_i, y_j)

(2) mark the slopes in xy-plane by short line segments

(3) fit in solution curves to roughly match the slopes

Th: Euler's method → estimate $y(x_f)$ if $y=y(x)$ solves initial value problem $y'=f(x,y)$, $y(x_0)=y_0$

(0) Do this with n steps of size $h=\Delta x=\frac{x_f-x_0}{n}$

(1) find estimates $y_1, \dots, y_n$ for $y(x_{i+1})$ with $x_1=x_0+h$, $x_2=x_0+2h, \dots, x_{i+1}=x_0+(i+1)h$ with $0\le i\le n-1$

by $\boxed{\begin{aligned} s_i &= f(x_i,y_i) \\ y_{i+1} &= y_i+s_ih \end{aligned}}$ $i=0,1,\dots,n-1$

moving along tangent line at (x_i,y_i) to (x_{i+1},y_{i+1})

Make a table:

i	x_i	y_i	$s_i=f(x_i,y_i)$	$y_{i+1}=y_i+s_ih$
0	x_0	y_0 →	compute →	compute
1	x_0+h	copy →	compute →	
⋮	⋮			
$n-1$	$x_0+(n-1)h$			→ estimate

Today: finding exact solutions to some 1st-order differential equations

Already can: $y'=f(x)$, $y=y(x)$

Just integrate both sides to get y

(Fundamental Theorem of Calculus: $\int y'dx=y+C$)

Extend to: $\boxed{y'=f(x)\cdot g(y)}$, $y=y(x)$

Step 1: Write $y'=\frac{dy}{dx}$ → $\frac{dy}{dx}=f(x)g(y)$

Step 2: Move all y's to LHS, all x's to RHS

$\frac{dy}{dx}=f(x)\cdot g(y) \rightsquigarrow \frac{dy}{g(y)}=f(x)dx$

Step 3: Integrate both sides: $\underbrace{\int\frac{dy}{g(y)}}_{G(y)+C_1} = \underbrace{\int f(x)\,dx}_{F(x)+C_2}$

Board #2

Get $\boxed{G(y)=F(x)+C}$ C = arbitrary constant

defines y implicitly as a function of x (I-§3.8):

for each fixed x, "can" solve to get $y=y(x)$

$G(y)=F(x)+C$ could have multiple solutions $y=y(x)$

E.g. $y^2=x^2+1$ → $y_1(x)=\sqrt{x^2+1}$, → $y_2(x)=-\sqrt{x^2+1}$

Caution: Step 2 involves divisions by $g(y)$

O'K as long as $g(y)\neq 0$

Roots/solutions of $g(y)=0$ give constant/equilibrium solutions $y(x)=C$ for all x of

$y'=f(x)g(y)$, $y=y(x)$

Do not forget these!

Example 1: Find the general solution to

$y'=x(y-2)$, $y=y(x)$

Step 1: $\frac{dy}{dx}=x(y-2)$

Step 2: $\frac{dy}{y-2}=x\,dx$

Step 3: $\int\frac{dy}{y-2}=\int x\,dx \Leftrightarrow \ln|y-2|=\frac{x^2}{2}+C$

Can simplify in this case

Skip

$\ln|y-2| = x^2+C \rightsquigarrow |y-2| = e^{x^2/2+C} = e^{x^2/2}\cdot \underbrace{\boxed{e^C}}_{A}$

C is any constant $\Leftrightarrow A = e^C$ is any constant >0

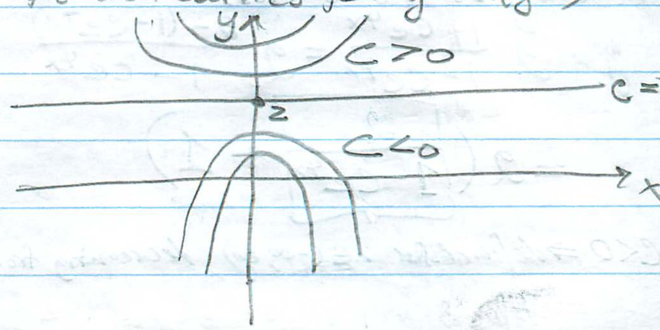
$|y-2| = Ae^{x^2/2} \Leftrightarrow y-2 = \pm Ae^{x^2/2} \Leftrightarrow y = 2 \pm Ae^{x^2/2}$

Remember the Caution: find constant solutions separately

$y' = x(y-2) = 0$ for all $x \Rightarrow y = 2 \leftrightarrow A = 0$

$\therefore$ the general solution $\boxed{y = 2 + Ce^{x^2/2}}$

Sketch of solution curves for $y' = x(y-2)$



Example 2: Find the general solution to

$y' = y^2 - 4, \quad y = y(x)$

Steps 1, 2: $\frac{dy}{dx} = y^2-4 \rightsquigarrow \frac{dy}{y^2-4} = dx$

Step 3: $\underbrace{\int \frac{dy}{y^2-4}}_{?} = \int dx = \int 1\,dx = x + C$

$\rightarrow$ partial fractions (§3.4) (Appendix G)

Partial fractions: find A, B s.t.

$\frac{1}{y^2-4} = \frac{1}{(y-2)(y+2)} = \frac{A}{y-2} + \frac{B}{y-2}$

Quick PFs: $\frac{1}{(y-2)(y+2)} = \frac{1}{+2-(-2)}\left(\frac{1}{y-2} - \frac{1}{y+2}\right)$

do not cross

Works only if coefficients of y are the same

E.g. $(3y^2-2)(3y^2+2)$ O'K, $(y-2)(3y+2)$ not O'K

$\therefore \int \frac{dy}{y^2-4} = \frac14\left(\int\frac{dy}{y-2} - \int\frac{dy}{y+2}\right)$

$= \frac14\left(\ln|y-2| - \ln|y+2|\right) + C' = \frac14 \ln\left|\frac{y-2}{y+2}\right| + C'$

$\Rightarrow \frac14 \ln\left|\frac{y-2}{y+2}\right| = x + C \Rightarrow \ln\left|\frac{y-2}{y+2}\right| = 4x + 4C$

$\left|\frac{y-2}{y+2}\right| = e^{4x+4C} = e^{4x}\cdot e^{4C} = Ae^{4x}$

C = any constant $\Leftrightarrow A = e^{4C}$ any constant >0

$\therefore \quad \frac{y-2}{y+2} = \pm Ae^{4x}$ Simplify more!

$\frac{y+2-4}{y+2} = \pm Ae^{4x} \rightsquigarrow 1 - \frac{4}{y+2} = \pm Ae^{4x}$

$\rightarrow 1 \mp Ae^{4x} = \frac{4}{y+2} \rightsquigarrow \frac{1}{1\mp Ae^{4x}} = \frac{y+2}{4}$

$\therefore y = \frac{4}{1\mp Ae^{4x}} - 2 = \frac{2 \pm 2Ae^{4x}}{1 \mp Ae^{4x}}$

$= 2\,\frac{1+Ce^{4x}}{1-Ce^{4x}} \quad C \neq 0$

To check: Compute y', y^2-4 and compare

do not erase

Remember the caution: check for constant solutions

$y' = y^2 - 4 = (y-2)(y+2) = 0$ for all x

$\Rightarrow y = -2, 2$ are the constant solutions

$C = 0$

$\therefore$ the general solution of $y' = y(y-2)$, $y = y(x)$ is

$\boxed{y = -2, \quad y = 2\,\frac{1+Ce^{4x}}{1-Ce^{4x}} \quad C = \text{any constant}}$

"the general solution" = the set of all solutions

over

Sketching solution curves:

$$y = 2\,\frac{1+ce^{4x}}{1-ce^{4x}} = 2\,\frac{2-(1-ce^{4x})}{1-ce^{4x}}$$

$$= 2\left(\boxed{\frac{2}{1-ce^{4x}}} - 1\right)$$

$c<0 \Rightarrow$ defined for $x \in (-\infty, \infty)$, decreasing from 2 to 0

$\underline{c>0}$: $\frac{2}{1-ce^{4x}}$ is defined for

$$1 \neq ce^{4x} \Leftrightarrow \underbrace{\ln 1}_{0} \neq \ln(ce^{4x}) = \ln c + \underbrace{\ln e^{4x}}_{4x}$$

$$x \neq -\tfrac{1}{4}\ln c$$

$\therefore \boxed{\frac{2}{1-ce^{4x}}}$ is defined for $x \in (-\infty, -\frac{1}{4}\ln c) \cup \underbrace{(-\frac{1}{4}\ln c, \infty)}$

increasing from 2 to ∞, from $-\infty$ to 0

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do not erase

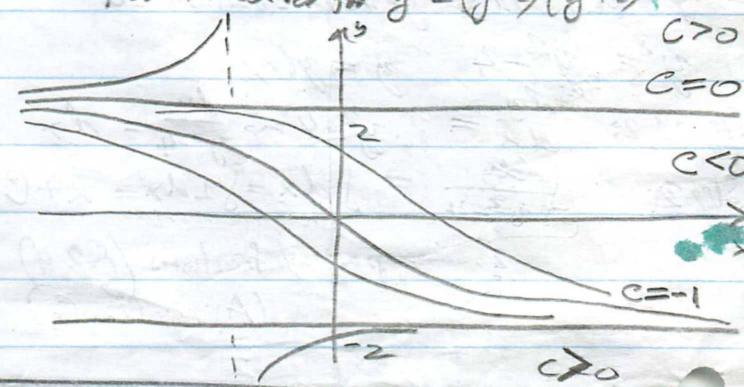
$\therefore y(x) = 2\left(\frac{2}{1-ce^{4x}} - 1\right)$

$\underline{c<0}$: defined for $x \in (-\infty, \infty)$, decreasing from 2 to -2

$\underline{c>0}$: defined for $x \in (-\infty, -\frac{1}{4}\ln c)$, increasing from 2 to ∞

and $x \in (-\frac{1}{4}\ln c, \infty)$, increasing from $-\infty$ to -2

Sketch of solution curves for $y' = (y-2)(y+2)$



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invariant under shifts

can get directly by looking at the equation

Find solution with $y(1) = 4 \to$ Find c

$$y = 2\,\frac{1+ce^{4x}}{1-ce^{4x}}$$

$$2\,\cancel{4} = \cancel{2}\,\frac{1+ce^{4}}{1-ce^{4}}$$

$$2-2ce^{4} = 1+ce^{4} \quad 1 = 3ce^{4}$$

$$c = \frac{1}{3e^{4}} = \frac{1}{3}e^{-4}$$

$$y = 2\,\frac{1+\frac{1}{3}e^{-4}\cdot e^{4x}}{1-\frac{1}{3}e^{-4}\cdot e^{4x}} = \boxed{2\,\frac{3+e^{4x-4}}{3-e^{4x-4}} = y}$$

$\to e^{4x-4}$

check: $y(1) = 2\,\frac{3+e^{4-4}}{3-e^{4-4}} = 2\cdot\frac{3+1}{3-1} = 4$