## Challenge Problems

(1) Define $n_1, n_2, \ldots \in \mathbb{Z}$ by

$$\sum_{r|d} n_r = \frac{(-1)^d}{2}\binom{2d}{d} \qquad \text{for all} \quad d = 1, 2, \ldots,$$

where the sum is taken over $r \in \mathbb{Z}^+$ that divide $d$ (including $r=1, d$). For example,

$$\begin{aligned} n_1 &= \frac{-1}{2}\binom{2}{1} = -1 \\ n_2 &= \frac{1}{2}\binom{4}{2} - n_1 = 4 \\ n_3 &= \frac{-1}{2}\binom{6}{3} - n_1 = -9 \\ n_4 &= \frac{1}{2}\binom{8}{4} - n_2 - n_1 = 32 \end{aligned}$$

Show that each $n_r$ is divisible by $r^2$.

(2) Let $a, b$ be relatively prime positive integers (one or both could be 1). Define $n_1, n_2, \ldots \in \mathbb{Q}$ by

$$\sum_{r|d} n_r = \frac{1}{(a+b)^2}\binom{(a+b)d}{ad}^2 \qquad \text{for all} \quad d = 1, 2, \ldots,$$

where the sum is taken over $r \in \mathbb{Z}^+$ that divide $d$ (including $r=1, d$). For example, if $a=b=1$,

$$\begin{aligned} n_1 &= \frac{1}{4}\binom{2}{1}^2 = 1 \\ n_2 &= \frac{1}{4}\binom{4}{2}^2 - n_1 = 8 \\ n_3 &= \frac{1}{4}\binom{6}{3}^2 - n_1 = 99 \\ n_4 &= \frac{1}{4}\binom{8}{4}^2 - n_2 - n_1 = 1216 \end{aligned}$$

Show that each $n_r$ is an integer divisible by $r^2$.

If you get these down, get in touch with Aleksey Zinger, azinger@ math.sunysb.edu.