Math Club Problem of the Month (April 2010)

Find the 100-th derivative of the function

$$f(x) = \frac{x^2 + 1}{x^3 - x}$$

First decompose this into partial fractions:

$$f(x) = \frac{x^2 + 1}{x(x-1)(x+1)} = \frac{x^2 + 1}{2x} \left(\frac{1}{x-1} - \frac{1}{x+1}\right) = \frac{1}{2x} \left((x+1) + \frac{2}{x-1} - (x-1) - \frac{2}{x+1}\right)$$
$$= \frac{1}{x} + \left(\frac{1}{x-1} - \frac{1}{x}\right) - \left(\frac{1}{x} - \frac{1}{x+1}\right) = \frac{1}{x-1} + \frac{1}{x+1} - \frac{1}{x}$$

Since the k-th derivative of 1/x is $(-1)^k k! / x^{k+1}$, which can be seen by induction, it follows that

$$f^{\langle 100 \rangle}(x) = 100! \left(\frac{1}{(x-1)^{101}} + \frac{1}{(x+1)^{101}} - \frac{1}{x^{101}} \right)$$