Math Club Problem of the Month (February 2010)

Find

$$\sqrt[4]{47-\cfrac{1}{47-\frac{1}{47-\dots}}}$$

Express your answer in the form $\dfrac{a\pm\sqrt{b}}{2}$ *(specifying the sign).*

We need to compute $\sqrt[4]{x}$, where x is the limit of the sequence $x_1, x_2, \dots$ defined by

$$x_1 = 47, \qquad x_{n+1} = 47 - \frac{1}{x_n}\,.$$

By the *Monotonic Sequence Theorem*, this sequence converges to (has a limit) $x \ge 46$ because it is bounded below by 46 and is decreasing. Both properties are checked using induction:

- clearly $x_1 > 46$; if $x_n > 46$ for some $n \ge 1$, then $x_{n+1} = 47 - 1/x_n > 46$, since $1/x_n < 1$; thus, $x_n > 46$ for all n;
- clearly $x_1 > x_2$; if $x_n > x_{n+1}$ for some $n \ge 1$, then $1/x_n < 1/x_{n+1}$ and thus
$$x_{n+1} = 47 - \frac{1}{x_n} > 47 - \frac{1}{x_{n+1}} = x_{n+2}\,;$$
thus, $x_n > x_{n+1}$ for all n.

Taking the limit of the recursion equation gives

$$x = \lim_{n\longrightarrow\infty} x_n = \lim_{n\longrightarrow\infty} x_{n+1} = \lim_{n\longrightarrow\infty}\left(47 - \frac{1}{x_n}\right) = 47 - \frac{1}{\lim\limits_{n\longrightarrow\infty} x_n} = 47 - \frac{1}{x} \quad\Longrightarrow\quad x = 47 - \frac{1}{x}.$$

So our limit x is one of the roots of the quadratic equation $x^2 - 47x + 1 = 0$,

$$x_\pm = \frac{47 \pm \sqrt{47^2 - 4}}{2}\,.$$

Both of these roots are positive, and our x is the larger of the two because $x \ge 46$.

It remains to compute $\sqrt[4]{x_+}$. The roots $x_\pm$ of the quadratic equation $x^2 - 47x + 1 = 0$ satisfy

$$x_+ + x_- = 47, \quad x_+ x_- = 1$$

$$\Longrightarrow \qquad \sqrt{x_+} + \sqrt{x_-} = \sqrt{x_+ + x_- + 2\sqrt{x_+}\sqrt{x_-}} = \sqrt{47 + 2\sqrt{1}} = 7, \quad \sqrt{x_+}\sqrt{x_-} = 1$$

$$\Longrightarrow \qquad \sqrt[4]{x_+} + \sqrt[4]{x_-} = \sqrt{\sqrt{x_+} + \sqrt{x_-} + 2\sqrt[4]{x_+}\sqrt[4]{x_-}} = \sqrt{7 + 2\sqrt{1}} = 3, \quad \sqrt[4]{x_+}\sqrt[4]{x_-} = 1.$$

Thus, $y_+ = \sqrt[4]{x_+}$ and $y_- = \sqrt[4]{x_-}$ are the roots of the quadratic equation

$$y^2 - 3y + 1 = 0 \qquad \Longrightarrow \qquad y_\pm = \frac{3 \pm \sqrt{3^2 - 4}}{2} \qquad \Longrightarrow \qquad \boxed{\sqrt[4]{x} = \frac{3 + \sqrt{5}}{2}}$$

Note that we take the larger y-root because $x_+ > x_-$.

Note: The first part of this problem is a typical calculus exercise on sequences (in MAT 127/132); the second part is high-school Algebra II.