

Math Club Problem of the Month (November 2009)

Let M be a square matrix consisting of 0's and 1's only. If there are precisely k 1's in each column, show that the determinant of M is divisible by k . Can the assumptions on the matrix be relaxed?

The determinant of a square matrix does not change when a row is added to another row or a column is added to another (for example, if every entry in the first row is replaced by its sum with the entry in the same column in the second row). So, add the second row to the first row, then add the third row to the first, and so on; this does not change the determinant. On the other hand, each entry in the first row of the original matrix is replaced by the sum of all entries in its column; this sum is k by assumption. Since every entry in the first row is k and the matrix has integer entries, the determinant is a multiple of k . You can compute the determinant by either expanding it by the first row or subtracting the first column from the other columns so that the resulting matrix with integer coefficients has only 1 nonzero entry in the first row and that entry is k .

The conclusion still holds a square matrix if it has integer entries and the sum of entries in every column is a multiple of k .