TOPICS IN CONFORMALLY COMPACT EINSTEIN METRICS

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1. Introduction.

Conformal compactifications of Einstein metrics were introduced by Penrose [38], as a means to study the behavior of gravitational fields at infinity, i.e. the asymptotic behavior of solutions to the vacuum Einstein equations at null infinity. This has remained a very active area of research, cf. [27], [19] for recent surveys. In the context of Riemannian metrics, the modern study of conformally compact Einstein metrics began with the work of Fefferman-Graham [26], in connection with their study of conformal invariants of Riemannian metrics. Recent mathematical work in this area has been significantly influenced by the AdS/CFT (or gravity-gauge) correspondence in string theory, introduced by Maldacena [36]. We will only comment briefly here on aspects of the AdS/CFT correspondence, and refer to [2], [42], [7] for general surveys.

In this paper, we discuss recent mathematical progress in this area, focusing mainly on global aspects of conformally compact Einstein metrics and the global existence question for the Dirichlet problem. One reason for this is that it now appears that the beginnings of a general existence theory for such metrics may be emerging, at least in dimension 4. Of course to date there is no general theory for the existence of complete Einstein metrics on manifolds, with two notable exceptions; the existence theory for Kähler-Einstein metrics due to Calabi, Yau, Aubin and others, and the existence theory in dimension 3, due to Perelman, Hamilton and Thurston. In contrast to the situation for compact 4-manifolds, an existence theory for conformally compact Einstein metrics may not be that far beyond the current horizon.

We discuss numerous open problems on this topic; some new results are also presented, cf. in particular Theorem 3.4 and the discussion and results in Sections 4 and 5.

In brief, the contents of the paper are as follows. The groundwork is laid in §2, where we discuss the moduli space of conformally compact Einstein metrics and the boundary map to the space of conformal infinities. The general situation is also illustrated by the discussion of a simple but important class of examples, the static AdS black hole metrics. Section 3 deals with the general asympototic behavior of the metrics near conformal infinity, and the control of the asymptotic behavior by the metric at infinity. It will be seen that at least in even dimensions, this issue is now quite well understood. Then in Section 4 we turn to the analysis of the behavior of the metrics on compact regions, away from infinity, mostly in dimension 4 where the possible degenerations can be described in terms of orbifold and cusp degenerations. In Section 5, we conclude with a discussion of the possibility of actually finding examples where orbifold or cusp degenerations occur.

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2. Conformally compact Einstein metrics.

Let M be the interior of a compact $(n+1)$-dimensional manifold $\bar{M}$ with boundary ∂M. A complete Riemannian metric g on M is $C^{m,\alpha}$ conformally compact if there is a defining function ρ on $\bar{M}$ such that the conformally equivalent metric

$$\widetilde{g} = \rho^2 g \tag{2.1}$$

extends to a $C^{m,\alpha}$ metric on the compactification $\bar{M}$. Here ρ is a smooth, non-negative function on $\bar{M}$ with $\rho^{-1}(0) = \partial M$ and $d\rho \neq 0$ on ∂M. The induced metric $\gamma = \widetilde{g}|_{\partial M}$ is the boundary metric associated to the compactification $\widetilde{g}$. Since there are many possible defining functions, there are many conformal compactifications of a given metric g, and so only the conformal class $[\gamma]$ of γ on ∂M, called conformal infinity, is uniquely determined by (M, g). Clearly any manifold M carries many conformally compact metrics but we are mainly concerned here with Einstein metrics g, normalized so that

$$Ric_g = -ng. \tag{2.2}$$

A simple computation for conformal changes of metric shows that if g is at least C^2 conformally compact, then the sectional curvature K_g of g satisfies

$$|K_g + 1| = O(\rho^2). \tag{2.3}$$

Thus, the local geometry of (M, g) approaches that of hyperbolic space, and conformally compact Einstein metrics are frequently called asymptotically hyperbolic (AH), or also Poincaré-Einstein. All these notions will be used here interchangeably. The natural "threshold level" for smoothness is C^2, since even if g is $C^{m,\alpha}$ conformally compact, $m > 2$, (2.3) cannot be improved to $|K_g + 1| = o(\rho^2)$ in general.

Mathematically, an obviously basic issue in this area is the Dirichlet problem for conformally compact Einstein metrics: given the topological data $(M, \partial M)$, and a conformal class $[\gamma]$ on ∂M, does there exist a conformally compact Einstein metric g on M, with conformal infinity $[\gamma]$? In one form or another, this question is the basic leitmotiv throughout this paper. As will be seen later, uniqueness of solutions with a given conformal infinity fails in general.

To set the stage, we first examine the structure of the moduli space of Poincaré-Einstein metrics on a given $(n+1)$-manifold M. Let $E^{m,\alpha}$ be the space of Poincaré-Einstein metrics on M which admit a C^2 conformal compactification $\bar{g}$ as in (2.1), with $C^{m,\alpha}$ boundary metric γ on ∂M. Here $0 < \alpha < 1$, $m \geq 2$, and we allow $m = \infty$ or $m = \omega$, the latter corresponding to real-analytic. The topology on $E^{m,\alpha}$ is given by a weighted Hölder norm, cf. (2.7) below; briefly, the topology is somewhat stronger than the C^2 topology on metrics on $\bar{M}$ under a conformal compactification $\widetilde{g}$ as in (2.1). Let $\mathcal{E}^{m,\alpha} = E^{m,\alpha}/\mathrm{Diff}_1^{m+1,\alpha}(\bar{M})$, where $\mathrm{Diff}_1^{m+1,\alpha}(\bar{M})$ is the group of $C^{m+1,\alpha}$ diffeomorphisms of $\bar{M}$ inducing the identity on ∂M, acting on E in the usual way by pullback. Next, let $Met^{m,\alpha}(\partial M)$ be the space of $C^{m,\alpha}$ metrics on ∂M and $\mathcal{C}^{m,\alpha} = \mathcal{C}^{m,\alpha}(\partial M)$ the corresponding space of pointwise conformal classes.

The natural boundary map,

$$\Pi : \mathcal{E}^{m,\alpha} \to \mathcal{C}^{m,\alpha}, \quad \Pi[g] = [\gamma], \tag{2.4}$$

takes a conformally compact Einstein metric g on M to its conformal infinity on ∂M. Thus, global existence for the Dirichlet problem is equivalent to the surjectivity of Π, while uniqueness is equivalent to the injectivity of Π.

The following result describes the general structure of $\mathcal{E}$ and the map Π, building on previous work of Graham-Lee [29] and Biquard [15].

Theorem 2.1. (Manifold structure [5], [6]) *Let M be a compact, oriented $(n+1)$-manifold with boundary ∂M with $n \geq 3$. If $\mathcal{E}^{m,\alpha}$ is non-empty, then $\mathcal{E}^{m,\alpha}$ is a smooth infinite dimensional manifold. Further, the boundary map*

$$\Pi : \mathcal{E}^{m,\alpha} \to \mathcal{C}^{m,\alpha}$$

is a C^{∞} smooth Fredholm map of index 0.

When $m < \infty$, $\mathcal{E}^{m,\alpha}$ has the structure of a Banach manifold, while $\mathcal{E}^{\infty}$ has the structure of a Fréchet manifold. For $n = 3$, one expects that Theorem 2.1 also holds for $m \geq 2$, but this is an open problem.

Theorem 2.1 shows that the moduli space $\mathcal{E}$ has a very satisfactory global structure. In particular if M carries some Poincaré-Einstein metric, then it also carries a large set of them, mapping under Π to at least a variety of finite codimension in $\mathcal{C}$. Recall that a metric $g \in \mathcal{E}$ is a regular point of Π if $D_g\Pi$ is surjective. Since Π is Fredholm of index 0, $D_g\Pi$ is injective at regular points; hence, by the inverse function theorem, Π is a local diffeomorphism in a neighborhood of each regular point.

Remark 2.2. Note that Theorem 2.1 does not hold, as stated, when $n = 1$, i.e. in dimension 2. In this case, the space $\mathcal{E}$ as defined above is infinite dimensional, but it becomes finite dimensional when one divides out by the larger group of diffeomorphisms isotopic to the identity on $\bar{M}$. This space of conformally compact (geometrically finite) hyperbolic metrics on a surface Σ is a smooth, finite dimensional manifold, but the conformal infinity is unique. The boundary $\partial\Sigma$ is a collection of circles and there is only one conformal structure on S^1 up to diffeomorphism. In particular, Π is not of index 0.

When $n = 2$, Einstein metrics are again hyperbolic, and the space of such metrics, modulo diffeomorphisms isotopic to the identity, is parametrized by the Teichmüller space of conformal classes on Riemann surfaces forming ∂M. Thus, Theorem 2.1 does hold for $n = 2$. However, we point out that the map Φ in (2.7) below used in constructing $\mathcal{E}$ is not Fredholm when $n = 1, 2$. Thus, the proof of Theorem 2.1 does not extend to the case $n = 2$.

It is worthwhile to examine the local structure of the boundary map Π near singular points in more detail. To do this, we need to discuss some background material, related to the proof of Theorem 2.1. Given a boundary metric γ, one may form the standard "hyperbolic cone" metric on γ by setting, in a neighborhood of ∂M,

$$g_{\gamma} = \rho^{-2}(d\rho^2 + \gamma),$$

and extending g_{γ} to M in a fixed but arbitrary way. Given a fixed background metric $g_0 \in E^{m,\alpha}$ with boundary metric γ_0, for γ near γ_0, let $g(\gamma) = g_0 + \eta(g_{\gamma} - g_{\gamma_0})$, where η is a cutoff function supported near ∂M. Thus $g(\gamma)$ is close to g_0 and consider metrics g near g_0 of the form

$$g = g(\gamma) + h, \tag{2.5}$$

where h is a symmetric bilinear form on M which decays as $O(\rho^2)$.

Essentially following [15], the Bianchi-gauged Einstein operator at g_0 is defined by

$$\Phi(g) = Ric_g + ng + \delta_g^*\beta_{g(\gamma)}(g). \tag{2.6}$$

We view Φ as a map

$$\Phi : Met^{m,\alpha}(\partial M) \times \mathbb{S}_2^{m,\alpha}(M) \to \mathbb{S}_2^{m-2,\alpha}(M), \tag{2.7}$$

$$\Phi(\gamma, h) = \Phi(g(\gamma) + h),$$

where $\mathbb{S}_2^{m,\alpha}(M)$ is the space of symmetric bilinear forms h on M, of the form $h = \rho^2 \bar{h}$, with $\bar{h}$ bounded in $C^{m,\alpha}(M)$. It turns out that if $g_0 \in E^{m,\alpha}$ then the variety $\Phi^{-1}(0)$ forms a local slice for the action of diffeomorphisms on $E^{m,\alpha}$ near g_0.

The derivative of Φ at g_0 with respect to the second factor is the linearized Einstein operator

$$L(h) = D^* D h - 2R(h), \tag{2.8}$$

$h \in \mathbb{S}_2^{m,\alpha}(M)$. By [29], this map is Fredholm, and so has finite dimensional kernel and cokernel. Let K be the kernel of L on $\mathbb{S}_2^{m,\alpha}(M)$; K is also the kernel of L on $L^2(M, g)$. To prove Theorem 2.1, it suffices to show that Φ is a submersion at any $g_0 \in E^{m,\alpha}$, and for this one needs to show that the pairing

$$\int_M \langle D\Phi^{\tilde{g}}(\dot{\gamma}_0), \kappa \rangle dV_{g_0} \tag{2.9}$$

is non-degenerate, in the sense that for any $\kappa \in K_{g_0}$, there exists a variation $\dot{\gamma}_0$ of $\gamma_0 = \Pi(g_0)$ such that (2.9) is non-zero. This is actually not so easy in general, and we refer to [6] for details.

The boundary map Π is locally, near g_0, just the projection map on the first factor of $\Phi^{-1}(0)$ in (2.7). Thus, locally, a slice for $\mathcal{E}^{m,\alpha}$ through g_0 is written as a (possibly multi-valued and singular) graph over $Met^{m,\alpha}(\partial M)$. The kernel K of $D\Pi$ at g is the subspace at which the graph is vertical, and corresponds to the kernel K of the operator L in (2.8).

To understand the singularities of Π in more detail, note that since Π is Fredholm, it is locally proper, i.e. for any $g \in \mathcal{E}^{m,\alpha}$, there exists an open set $\mathcal{U}$ with $g \in \mathcal{U}$ such that $\Pi|_{\mathcal{U}}$ is a proper map onto its image $\mathcal{V} \subset \mathcal{C}$. This means that Π has a local degree, $\deg_g \Pi \in \mathbb{Z}$, cf. [41], [13]; in fact if $\mathcal{U}$ is chosen sufficiently small, then $\deg_g \Pi = -1, 0$ or $+1$. If $\deg_g \Pi \neq 0$, then Π is locally surjective onto a neighborhood of $\gamma = \Pi(g)$; this may or not be the case if $\deg_g \Pi = 0$. Observe however that (of course) $\deg_g \Pi$ is not continuous in g.

The local degree can be calculated by examining the behavior of Π on generic, finite dimensional slices. Thus, let B be any p-dimensional local affine subspace (or submanifold) of $Met^{m,\alpha}(\partial M)$ with $\gamma = \Pi(g) \in B$ and consider the restriction of Φ to $B \times \mathbb{S}_2^{m,\alpha}(M)$, and correspondingly, the graph $E_B^{m,\alpha} = \Phi^{-1}(0) \cap \Pi^{-1}(B)$ of $E^{m,\alpha}$ over B. For a generic choice of B, $E_B^{m,\alpha}$ is a p-dimensional manifold, and thus one can examine the behavior of $\Pi|_{E_B^{m,\alpha}}$ in the context of the study of singularities of smooth mappings between equidimensional manifolds. By construction, cf. [13] for instance, one has for generic B,

$$\deg_g \Pi = \deg_g \Pi|_{E_B^{m,\alpha}}.$$

Consider for example the situation where B is 1-dimensional. Then $E_B^{m,\alpha}$ is a local curve in $Met^{m,\alpha}(\partial M) \times \mathbb{S}_2^{m,\alpha}(M)$ graphed over the interval $B = (-\varepsilon, \varepsilon)$, with 0 corresponding to γ. One sees that if $\deg_g \Pi|_{E_B^{m,\alpha}} = \pm 1$, then Π is locally surjective near γ, while if $\deg_g \Pi|_{E_B^{m,\alpha}} = 0$, then locally $\Pi|_{E_B^{m,\alpha}}$ is a fold map, equivalent to $x \to x^2$ on $(-\varepsilon, \varepsilon)$. In this case, at least in a small neighborhood $\mathcal{U}$ of g, Π is not surjective onto a neighborhood of γ; there is a local "wall" in $\mathcal{C}$, (the image of the fold locus), which $\Pi(\mathcal{U})$ does not cross.

Some natural questions related to this discussion are the following: is the set of critical points of Π a non-degenerate critical submanifold (in sense of Bott)? Is it possible that Π maps a connected manifold or variety of dimension ≥ 1 onto a point $\gamma \in \mathcal{C}$?

At this point, it is useful to illustrate the discussion on the basis of some concrete examples.

Example 2.3. (Static AdS black hole metrics). Let N^{n-1} be any closed $(n-1)$-dimensional manifold, which carries an Einstein metric g_N satisfying

$$Ric_{g_N} = k(n-2)g_N, \tag{2.10}$$

where $k = +1, 0$ or -1. We assume $n \geq 3$. Consider the metric g_m on $\mathbb{R}^2 \times N$ defined by

$$g_m = V^{-1}dr^2 + Vd\theta^2 + r^2 g_N, \tag{2.11}$$

where

$$V(r) = k + r^2 - \frac{2m}{r^{n-2}}. \tag{2.12}$$

Here $r \in [r_+, \infty)$, where r_+ is the largest root of V, and the circular parameter $\theta \in [0, \beta]$, where

$$\beta = 4\pi r_+/(nr_+^2 + k(n-2)). \tag{2.13}$$

This choice of β is required so that the metric g_m is smooth at the locus $\{r = r_+\}$; if β is arbitrary, the metric will have cone singularities normal to the locus $\{r = r_+\}$, although the metric is otherwise smooth. Since this locus is the fixed point set of the isometric S^1 action given by rotation in θ, the set $\{r = r_+\}$ is diffeomorphic to N and is totally geodesic; it corresponds to the horizon of the black hole. A simple computation shows that the metrics g_m are Einstein, satisfying (2.2). Further, it is easy to see these metrics are smoothly conformally compact; the conformal infinity of g_m is given by the conformal class of the product metric on $S^1(\beta) \times (N, g_N)$.

We discuss the cases $k > 0$, $k = 0$, $k < 0$ in turn.

I. Suppose $k = +1$.

As a function of $m \in (0, \infty)$, observe that β has a maximum value of $\beta_0 = 2\pi(\frac{n-2}{n})^{1/2}$, and for every $m \neq m_0$, there are two values $m^{\pm}$ of m giving the same value of β. Thus two metrics have the same conformal infinity; in particular, the boundary map Π in (2.4) is not 1-1 along this curve. This behavior is the first example of non-uniqueness for the Dirichlet problem, and was discovered in [32] in the context of the AdS Schwarzschild metrics, where $N = S^2(1)$.

The map Π is a fold map, (of the form $x \to x^2$), in a neighborhood of the curve g_m near $m = m_0$. The local degree at g_{m_0} is 0 and Π is not locally surjective. In fact, Theorem 2.4 below implies that Π is globally not surjective, in that the conformal class of $S^1(L) \times (N, g_N)$, for $L > \beta_0$, is not in Im Π, cf. [5]. Observe that this result requires global smoothness of the Einstein metrics; if one allows cone singularities along the horizon $N = \{r = r_+\}$, i.e. if β is allowed to be arbitrary, then one can go past the "wall" through $S^1(\beta_0) \times (N, g_N)$. This clearly illustrates the global nature of the global existence or surjectivity problem.

II. Suppose $k = 0$.

In this case $\beta = 4\pi r_+/(nr_+^2)$ is a monotone function of r_+ or m, so that it assumes all values in $\mathbb{R}^+$ as $m \in (0, \infty)$. On the curve g_m, Π is 1-1.

However, the actual situation is somewhat more subtle than this. Suppose for instance that $N = T^{n-1}$, so that $M = \mathbb{R}^2 \times T^{n-1}$ is a solid torus. Topologically, the disc $D^2 = \mathbb{R}^2$ can be attached onto *any* simple closed curve in the boundary $\partial M = T^n$ instead of just the "trivial" S^1 factor in the product $T^n = S^1 \times T^{n-1}$. The resulting manifolds are all diffeomorphic. This can also be done metrically, preserving the Einstein condition, cf. [4], and leads to the existence of infinitely many distinct Einstein metrics on $\mathbb{R}^2 \times T^{n-1}$ with the same conformal infinity $(T^n, [g_0])$, where g_0 is any flat metric.

Each of these metrics lies in a distinct component of the moduli space $\mathcal{E}$, so that $\mathcal{E}$ has infinitely many components. This situation is closely related to the mapping class group $SL(n, \mathbb{Z})$ of T^n, i.e. the group of diffeomorphisms of T^n modulo those homotopic to the identity map, (so called "large diffeomorphisms"). Any element of $SL(n,\mathbb{Z})$ extends to a diffeomorphism of the solid torus $\mathbb{R}^2 \times T^{n-1}$, and while $SL(n,\mathbb{Z})$ acts trivially on the moduli space of flat metrics on T^n, the action on $\mathcal{E}$ is highly non-trivial, giving rise to the distinct components of $\mathcal{E}$. Similar constructions can obviously be carried out for manifolds N of the form $N = T^k \times N'$, $k \geq 1$, but it would be interesting to investigate the most general version of this phenomenon.

III. Suppose $k = -1$.

Again β is a monotone function of m, and so takes on all values in $\mathbb{R}^+$; the boundary map Π is 1-1 on the curve g_m. Further aspects of this case are discussed later in §5.

These simple examples already show a number of subtle features of the global behavior of the boundary map Π. With regard to the global surjectivity question, the basic property that one needs to make progress is to understand whether Π is a proper map; if Π is not proper, it is important to understand exactly what possible degenerations of Poincaré-Einstein metrics can or do occur with controlled conformal infinity. Recall that Π is proper if and only if $\Pi^{-1}(K)$ is compact in $\mathcal{E}$, whenever K is compact in $\mathcal{C}$.

If Π is proper, then one has a well-defined $\mathbb{Z}_2$-valued degree, cf. [41]. In fact, since the spaces $\mathcal{E}$ and $\mathcal{C}$ can be given a well-defined orientation, one has a $\mathbb{Z}$-valued degree, given by

$$\deg \Pi = \sum_{g_i \in \Pi^{-1}[\gamma]} (-1)^{ind_{g_i}}, \tag{2.14}$$

where $[\gamma]$ is a regular value of Π and ind_{g_i} is the L^2 index of $D_{g_i}\Pi$, i.e. the number of negative eigenvalues of the operator L in (2.8) at g_i acting on L^2, cf. [5]. Of course if deg $\Pi \neq 0$, then Π is surjective; (if deg $\Pi = 0$, then Π may or may not be surjective). Note that deg Π is defined on each component $\mathcal{E}_0$ of $\mathcal{E}$ and may differ on different components.

Let $M = M^4$ be a 4-manifold, satisfying

$$H_2(\partial M, \mathbb{R}) \to H_2(M, \mathbb{R}) \to 0. \tag{2.15}$$

It is proved in [5] that Π is then proper, when restricted to the space $\mathcal{E}^0$ of Einstein metrics whose conformal infinity is of non-negative scalar curvature. More precisely,

$$\Pi^0 : \mathcal{E}^0 \to \mathcal{C}^0 \tag{2.16}$$

is proper, where $\mathcal{C}^0$ is the space of conformal classes having a *non-flat* representative of non-negative scalar curvature and $\mathcal{E}^0 = \Pi^{-1}(\mathcal{C}^0)$; in particular there are only finitely many components to $\mathcal{E}_0$, compare with Example 2.3, Case II above.

In situations where Π is proper, the degree can be calculated in a number of concrete situations by the following:

Theorem 2.4. (Isometry Extension, [5]) *Let (M^{n+1}, g) be a C^2 conformally compact Einstein metric with C^∞ boundary metric γ, $n \geq 3$. Then any connected group G of conformal isometries of $(\partial M, \gamma)$ extends to a group G of isometries of (M, g).*

This result has a number of immediate consequences. For instance, it implies that the Poincaré (or hyperbolic) metric is the unique C^2 conformally compact Einstein metric on an $(n+1)$-manifold with conformal infinity given by the round metric on S^n; see also [12], [39] for previous special cases of this result. In particular, one has on (B^4, S^3),

$$\deg \Pi^0 = 1,$$

so that Π is surjective onto $\mathcal{C}^0$. On the other hand, on $(M^4, S^3), M^4 \neq B^4$,

$$\deg \Pi^0 = 0,$$

since Π cannot be surjective in this case. Another application of Theorem 2.4 is the following:

Corollary 2.5. *Let M be any compact $(n+1)$-manifold with boundary ∂M, $n \geq 3$, and let $\hat{M} = M \cup_{\partial M} M$ be the closed manifold obtained by doubling M across its boundary. Suppose ∂M admits an effective S^1 action, but $\hat{M}$ admits no effective S^1 action. Then $\Pi = \Pi(M)$ is not surjective; in fact*

$$\mathrm{Im}\Pi \cap Met_{S^1}(\partial M) = \emptyset,$$

where $Met_{S^1}(\partial M)$ is the space of S^1 invariant metrics on ∂M. The space $Met_{S^1}(\partial M)$ is of infinite dimension and codimension in $Met(\partial M)$.

■

As a simple example, let $\hat{M} = \Sigma_g \times N$, where Σ is any surface of genus $g \geq 1$ and N is any $K(\pi, 1)$ manifold with π having no center; e.g. N has a metric of non-positive curvature. Let σ be a closed curve in Σ_g which disconnects Σ_g into two diffeomorphic components Σ^+ and Σ^- with common boundary σ, and let $M = \Sigma^+ \times N$. By [22], $\hat{M}$ does not admit an effective S^1 action, but of course $\partial M = S^1 \times N$ admits such actions. Hence, Corollary 2.5 holds for such M.

On the other hand, if $\Sigma = S^2$ is of genus 0, then $M = \mathbb{R}^2 \times N$ does admit S^1-invariant Poincaré-Einstein metrics, as discussed in Example 2.3.

A basic issue is to extend the theory described above beyond boundary metrics of non-negative scalar curvature $\mathcal{C}^0$. This will be one of the themes discussed below. We begin with the analysis of Poincaré-Einstein metrics near the boundary, i.e. conformal infinity.

3. Behavior near the Boundary.

In this section, we study the behavior of Poincaré-Einstein metrics in a neighborhood of conformal infinity $(\partial M, \gamma)$.

For many purposes, the most natural compactifications are those defined by geodesic defining functions. Thus, a compactification $\bar{g} = \rho^2 g$ as in (2.1) is called geodesic if $\rho(x) = dist_{\bar{g}}(x, \partial M)$. Each choice of boundary metric $\gamma \in [\gamma]$ determines a unique geodesic defining function ρ. For a geodesic compactification, one typically loses one derivative in the possible smoothness, but this will not be of major concern here, cf. also [11, App.B] on restoring loss of derivatives.

The Gauss Lemma gives the splitting

$$\bar{g} = d\rho^2 + g_\rho, \quad g = \rho^{-2}(d\rho^2 + g_\rho), \tag{3.1}$$

where g_ρ is a curve of metrics on ∂M. A simple and natural idea to examine the behavior of g near infinity is to expand the curve of metrics g_ρ on ∂M in a Taylor series in ρ. Surprisingly (at first), this turns out not always to be possible, as discovered in [26]. It turns out that the exact form of the expansion depends on whether n is odd or even. If n is odd, i.e M is even-dimensional, then

$$g_\rho \sim g_{(0)} + \rho^2 g_{(2)} + ... + \rho^{n-1} g_{(n-1)} + \rho^n g_{(n)} + \rho^{n+1} g_{(n+1)} + ... \tag{3.2}$$

This expansion is even in powers of ρ up to order n. The coefficients $g_{(2k)}$, $2k \leq (n-1)$ are locally determined via the Einstein equations (2.2) by the boundary metric $\gamma = g_{(0)}$. They are explicitly computable expressions in the curvature of γ and its covariant derivatives, although their complexity grows rapidly with k. The term $g_{(n)}$ is transverse-traceless, i.e.

$$tr_\gamma g_{(n)} = 0, \quad \delta_\gamma g_{(n)} = 0, \tag{3.3}$$

but is otherwise undetermined by γ and the Einstein equations; it depends on the particular structure of the AH Einstein metric (M, g) near infinity. If n is even, one has

$$g_\rho \sim g_{(0)} + \rho^2 g_{(2)} + ... + \rho^{n-2} g_{(n-2)} + \rho^n g_{(n)} + \rho^n \log\rho \, \mathcal{H} + \rho^{n+1} g_{(n+1)} + ... \tag{3.4}$$

Again (via the Einstein equations) the terms $g_{(2k)}$ up to order $n-2$ are explicitly computable from the boundary metric γ, as is the coefficient $\mathcal{H}$ of the first $\log \rho$ term. The term $\mathcal{H}$ is transverse-traceless. The term $g_{(n)}$ satisfies

$$tr_\gamma g_{(n)} = \tau, \quad \delta_\gamma g_{(n)} = \delta, \tag{3.5}$$

where again τ and δ are explicitly determined by the boundary metric γ and its derivatives; however, as before $g_{(n)}$ is otherwise undetermined by γ. There are $(\log \rho)^k$ terms that appear in the expansion at order $> n$.

Note also that these expansions (3.2) and (3.4) depend on the choice of boundary metric. Transformation properties of the coefficients $g_{(i)}$, $i \leq n$, under conformal changes have been explicitly studied in the physics literature, cf. [24]. As discovered by Fefferman-Graham [26], the term $\mathcal{H}$ is conformally invariant, or more precisely covariant: if $\widetilde{\gamma} = \phi^2\gamma$, then $\widetilde{\mathcal{H}} = \phi^{2-n}\mathcal{H}$.

Remark 3.1. Analogous to the Fefferman-Graham expansion above, there is a formal expansion of a vacuum solution to the Einstein equations near null infinity, although this has been carried out in detail only in dimension 3+1, cf. [16]. This expansion is closely related to the properties of the Penrose conformal compactification. More recently, as discussed in [20], logarithmic terms appear in the expansion in general, and these play an important role in understanding the global structure of the space-time.

Mathematically, it is of some importance to keep in mind that the expansions (3.2), (3.4) are only formal, obtained by conformally compactifiying the Einstein equations and taking iterated Lie derivatives of $\bar{g}$ at $\rho = 0$;

$$g_{(k)} = \frac{1}{k!}\mathcal{L}_T^{(k)}\bar{g}, \tag{3.6}$$

where $T = \nabla\rho$. If $\bar{g} \in C^{m,\alpha}(\bar{M})$, then the expansions hold up to order $m+\alpha$. However, boundary regularity results are needed to ensure that if an AH Einstein metric g with boundary metric γ satisfies $\gamma \in C^{m,\alpha}(\partial M)$, then the compactification $\bar{g} \in C^{m,\alpha}(\bar{M})$ or $C^{m',\alpha'}(\bar{M})$.

In both cases n odd or even, the Einstein equations determine all higher order coefficients $g_{(k)}$ (and coefficients of the log terms), in terms of $g_{(0)}$ and $g_{(n)}$, so that an AH Einstein metric is formally determined by $g_{(0)}$ and $g_{(n)}$ near ∂M. The term $g_{(0)}$ corresponds to Dirichlet boundary data on ∂M, while $g_{(n)}$ corresponds to Neumann boundary data, (in analogy with the scalar Laplace operator). Thus, on AH Einstein metrics, the formal correspondence

$$g_{(0)} \to g_{(n)} \tag{3.7}$$

is analogous to the Dirichlet-to-Neumann map for harmonic functions. However, the map (3.7) is only well-defined if there is a unique AH Einstein metric with boundary data $\gamma = g_{(0)}$; as seen above on the curve of AdS Schwarzschild metrics for example, this is not always the case. Understanding the correspondence (3.7) is a basic issue, both mathematically and in certain aspects of the AdS/CFT correspondence. Again in a formal sense, knowing $g_{(0)}$ and $g_{(n)}$ allows one to locally construct the bulk gravitational field, i.e. the Poincaré-Einstein metric, at least near ∂M via the expansion (3.2) or (3.4).

To begin to make some of the discussion above more rigorous, we next discuss the boundary regularity issue; many aspects of this have been resolved over the past few years. Suppose first $n = 3$, so dim $M = 4$. If $g \in E^{m,\alpha}$, $m \geq 2$, then by definition g has a C^2 conformal compactification to a $C^{m,\alpha}$ boundary metric γ. In [4], it is proved that there is a $C^{m,\alpha}$ conformal compactification $\widetilde{g} \in C^{m,\alpha}(\bar{M})$ of g, cf. also [6]. This result also holds if

$m = \infty$ or $m = \omega$. It is proved using the fact that 4-dimensional Einstein metrics satisfy the Bach equations, cf. [14], which are conformally invariant. In suitable gauges, the Bach equation can be recast as a non-degenerate elliptic system of equations for a conformal compactification $\widetilde{g}$, and the result follows from elliptic boundary regularity.

In dimension 4, the Bach tensor is the Fefferman-Graham obstruction tensor $\mathcal{H}$ above. In any even dimension, the system of equations

$$\mathcal{H} = 0 \tag{3.8}$$

is conformally invariant, and is satisfied by metrics conformal to Einstein metrics. Thus, one might expect that the method using the Bach equation in [4], [6] when $n = 3$ can be extended to all n odd. This is in fact the case, and has been worked out in detail by Helliwell [31]. Thus, essentially the same regularity results hold for n odd.

When n is even, so that dim M is odd, this type of boundary regularity cannot hold of course, due to the presence of the logarithmic terms in the FG expansion. A result of Lee [35] shows that if $g \in E^{m,\alpha}$ and $m < n$, then g is $C^{m,\alpha}$ conformally compact. This is optimal, but does not reach the important threshold level $m = n$, where logarithmic terms and the important $g_{(n)}$ term first appear. Recently, Chruściel et al. [21] have proved that when $g \in E^{\infty}$, i.e. g has a C^{∞} boundary metric γ, then g has a C^{∞} polyhomogeneous conformal compactification, so that the expansion (3.4) exists as an asymptotic series. Moreover, if $\gamma \in C^{m,\alpha}(\partial M)$, then the expansion exists up to order k, where k can be made large by choosing m sufficiently large; (in general m must be much larger than k). Finally, it has recently been proved by Kichenassamy [34] that when $g \in E^{\omega}$ and $g_{(n)}$ is real-analytic, the formal series (3.4) exists, i.e. it is summable, and it converges to g_{ρ}.

These results have the following immediate consequence. Suppose n is odd. Given any real-analytic symmetric bilinear forms $g_{(0)}$ and $g_{(n)}$ on ∂M, satisfying (3.3), there exists a unique C^{ω} conformally compact Einstein metric g defined in a thickening $\partial M \times [0, \varepsilon)$ of ∂M. If instead n is even, given any analytic symmetric bilinear forms $g_{(0)}$ and $g_{(n)}$ on ∂M, satisfying (3.5), there exists a unique C^{∞} polyhomogeneous conformally compact Einstein metric g defined in a thickening $\partial M \times [0, \varepsilon)$ of ∂M. In both cases, the expansions (3.2) or (3.4) converge to the metric g_{ρ}. These results follow from the work in [4], [6], [31] when n is odd, and [34] when n is even. Since analytic data $g_{(0)}$ and $g_{(n)}$ may be specified arbitrarily and independently of each other, subject only to the constraint (3.3) or (3.5), to give "local" AH Einstein metrics, defined in a neighborhood of ∂M, this shows that the correspondence (3.7) must depend highly on global properties of Poincaré-Einstein metrics.

On the other hand, it is well-known that the use of analytic data to solve elliptic-type problems is misleading. While the Dirichlet or Neumann problem is formally well-posed, the Cauchy problem is not. Standard examples involving Laplace operator and harmonic functions show that even if Cauchy data on a boundary converge smoothly to limit Cauchy data on the boundary, the corresponding solutions do not converge to a limit in any neighborhood of the boundary.

To pass from analytic to smooth boundary data, one needs apriori estimates or equivalently a stability result. In this respect, one has the following:

Theorem 3.2. (Local Stability, [4], [6]) *Let g be a C^2 conformally compact Einstein metric, defined in a region $\Omega = [0, \rho_0] \times \partial M$ containing ∂M, where ρ is a geodesic compactification. Suppose there exists a compactification $\widetilde{g}$ with $C^{m,\alpha}$ boundary metric γ, such that*

$$||\widetilde{g}||_{C^{1,\alpha}(\Omega)} \leq K. \tag{3.9}$$

If $n = 3$, then there is a (possibly different) compactification, also called $\widetilde{g}$, such that, in $\Omega' = [0, \frac{\rho_0}{2}] \times \partial M$, one has the estimate

$$||\widetilde{g}||_{C^{m,\alpha}(\Omega')} \leq C, \tag{3.10}$$

where the constant C depends only on K, m, α, n and ρ_0.

This result is proved simultaneously with the boundary regularity result itself, i.e. using the fact that $\widetilde{g}$ is a solution of the Bach equations together with standard estimates for solutions of elliptic systems of PDE's.

Using similar ideas as discussed above in connection with (3.8), Theorem 3.2 also holds for all n odd, at least if C^2 is replaced by $C^{n,\alpha}$ and $C^{1,\alpha}$ is replaced by $C^{n,\alpha}$ in (3.9), with $m > n$ in (3.10), cf. [31].

Theorem 3.2 shows that if two solutions are close in a weak norm, ($C^{1,\alpha}$ or $C^{n,\alpha}$), then they are close in a strong norm, $C^{m,\alpha}$, $m > n$, provided the boundary metrics are close in a strong norm. It would be very interesting if a similar result can be proved when n is even, (i.e. in odd dimensions). A direct generalization is of course not possible, due again to the logarithmic terms. Redefining the Hölder norms to take such logarithmic terms into account, it would be very surprising if such a stability result did not hold; however, a proof remains to be established.

In even dimensions, the local stability theorem allows one to pass to limits in the analytic data problem above. Thus, suppose $\gamma = g_{(0)}$ and $g_{(n)}$ are arbitrary $C^{m,\alpha}$ data on ∂M, subject to the constraint (3.3). Let γ_i and $(g_{(n)})_i$ be a sequence of analytic data satisfying (3.3) converging to γ and $g_{(n)}$ in the $C^{m,\alpha}$ topology, (such sequences always exist), and let $\widetilde{g}_i$ be the corresponding sequence of conformal compactifications of Poincaré-Einstein metrics defined in regions Ω_i. If the hypothesis (3.9), (with $C^{1,\alpha}$ replaced by $C^{n,\alpha}$ for $n > 3$), held on the sequence $\{\widetilde{g}_i\}$, i.e. $\Omega_i = \Omega$ is uniform, then it follows that $\{\widetilde{g}_i\}$ converges in the $C^{m,\alpha}$ topology on Ω to a limit $\widetilde{g} \in C^{m,\alpha}(\Omega)$. The metric $\widetilde{g}$ is a conformal compactification of a Poincaré-Einstein metric g, defined at least on Ω. In other words, it would then follow that arbitrary smooth γ and $g_{(n)}$ can be realized as local boundary data.

However, the following result shows this cannot be the case:

Theorem 3.3. (Unique Continuation, [8]) *Let data $(g_{(0)}, g_{(n)})$ be arbitrarily given, satisfying the constraints (3.3) or (3.5), in some open set $U \subset \partial M$, with $(g_{(0)}, g_{(n)}) \in C^{m,\alpha}(U)$, for $m > n$ and any $n \geq 3$. If g is a $C^{m,\alpha}$ conformally compact Einstein metric realizing the data $(g_{(0)}, g_{(n)})$, defined in a neighborhood Ω with $\Omega \cap \partial M = U$, then g is the unique such metric, up to local isometry.*

This result implies in particular that local Cauchy data in an open set $U \subset \partial M$ determine the global behavior of the metric, and the topology of the manifold, up to covering spaces; here we use the fact that Einstein metrics are real-analytic in the interior, and so trivially satisfy a unique continuation property. It follows that $(g_{(0)}, g_{(n)})$ in U necessarily determine $(g_{(0)}, g_{(n)})$ outside U. (This is of course obvious for analytic data $(g_{(0)}, g_{(n)})$ on ∂M).

It then follows that, for $\{\widetilde{g}_i\}$ as above, the weak uniform bound (3.9) cannot hold in general. The metrics must degenerate in a small neighborhood Ω of $U \subset \partial M$, for "most" choices of $g_{(n)}$, given any fixed choice of $\gamma = g_{(0)}$ on ∂M.

We now contrast this situation with the situation for *globally* defined Poincaré-Einstein metrics. For emphasis, for the result below we require that (M, g) is globally conformally compact, i.e. M is the interior of a compact manifold with boundary, and g is complete and globally defined on M.

Theorem 3.4. (Control near Boundary) *Let (M^{n+1}, g) be a globally conformally compact Poincaré-Einstein metric, with n odd, so that dim M is even. Suppose that g is C^2 conformally compact, with $C^{m,\alpha}$ boundary metric γ, with $m > n$ and $m \geq 6$ if $n = 3$. Then there exists a neighborhood $\Omega = [0, \rho_0] \times \partial M$ of ∂M, depending only on the boundary data $(\partial M, \gamma)$ such that*

$$||\widetilde{g}||_{C^{m,\alpha}(\Omega)} \leq C, \tag{3.11}$$

in some compactification $\widetilde{g}$.

The bound (3.11) implies that the boundary map Π is proper near conformal infinity, in the sense that if one has a fixed boundary metric γ, or compact set of boundary metrics $\gamma \in \Gamma$, then the set of Poincaré-Einstein metrics with boundary metric γ, (or $\gamma \in \Gamma$), is compact, as far as their behavior in Ω is concerned; any sequence has a convergent subsequence on a fixed domain Ω, where Ω only depends on the boundary data.

Proof: This result is proved for $n = 3$ in [5], and the proof for arbitrary n odd is very similar. Thus, we refer to [5] for much of the proof, and only discuss those situations where the proof needs to be modified in higher dimensions.

There are several steps in the proof. First, let $\bar{g}$ be the geodesic compactification of g determined by γ, and let τ be the distance to the cutlocus of the normal exponential map from $(\partial M, \gamma)$ into $(M, \bar{g})$. Here of course g is any Poincaré-Einstein metric on M with boundary metric γ, (or $\gamma \in \Gamma$). The first (and most important) step is to prove that there is a constant $\tau_0 > 0$, depending only on n and γ (or Γ) such that

$$\tau(x) \geq \tau_0. \tag{3.12}$$

The estimate (3.12) already implies for instance that the topology of M cannot become non-trivial too close to the boundary ∂M. The proof of (3.12) in [5, Prop.4.5] holds with only minor and essentially obvious changes in all even dimensions, given the local stability result, Theorem 3.2. As noted in [5, Remark 2.4], one should use the renormalized action in place of the renormalized volume or L^2 norm of the Weyl curvature. Also, the classification of $\mathbb{R}^n$-invariant solutions as AdS toral black holes is given in [9], (again the proof of this holds in all dimensions).

Next, let $\zeta(x) = \zeta^{n,\alpha}(x)$ be the $C^{n,\alpha}$ harmonic radius of $(M, \bar{g})$ at x, for a fixed $\alpha < 1$. The next claim, (cf. [5, Prop.4.4]) is that there is a constant ζ_0, depending only on n and γ, such that

$$\zeta(x) \geq \zeta_0 \tau(x). \tag{3.13}$$

(The proof in [5, Prop.4.4] uses the L^p curvature radius, but the proof works equally well for the much stronger $C^{n,\alpha}$ harmonic radius).

The proof of (3.13) is by contradiction. If (3.13) does not hold, then there exist $x_i \in (M_i, g_i)$ such that $\xi(x_i) << \tau(x_i)$. Choose x_i to realize the minimum of the ratio ξ/τ. One then takes a blow-up limit of the rescalings $g_i' = \zeta(x_i)^{-2}\bar{g}_i$ based at x_i. Since $\zeta'(x_i) = 1$, $\zeta'(y_i) \geq \frac{1}{2}$ for y_i within bounded g_i'-distance to x_i. It follows that in a subsequence, one has convergence to a complete limit (N, g', x). The local stability result, Theorem 3.2, implies that the convergence to the limit is in the (strong) $C^{n,\alpha}$ topology. The radius ζ is continuous in this topology, and hence the limit (N, g') cannot be flat, since $\zeta'(x) = 1$. Here one must also use the non-collapse or volume comparison estimates in [5, Lemma3.8ff]. Thus, to obtain a contradiction, it suffices to prove that the limit (N, g') must be flat. To do this, one distinguishes the following two situations:

I. $dist_{g_i'}(x_i, \partial M_i) \leq D$, for some $D < \infty$. In this case, the limit N has a boundary $(\partial N, \gamma')$. Since this limit is the blow-up of $(\partial M_i, \gamma_i)$, it is clear that $(\partial N, \gamma')$ is flat $(\mathbb{R}^n, \delta)$, where δ is the flat metric. (Here we use of course the fact that Γ is compact). Moreover,

∂N is totally geodesic in N. As in [5], (N, g') is Ricci-flat, $Ric_{g'} = 0$. The proof that N is actually flat in [5] used the fact that N contains a line; when $n = 3$, i.e. in dimension 4, this implies N is flat. This of course does not hold in higher dimensions. Instead, one can argue as follows. Since N is Ricci-flat and has flat and totally geodesic boundary $\mathbb{R}^n$, the reflection double of N across $\mathbb{R}^n$ is a weak C^1 solution of the Einstein equations $Ric_{g'} = 0$ on $\mathbb{R}^{n+1}$. Elliptic regularity implies that g' is then real-analytic across $\mathbb{R}^n$. It then follows easily from the Cauchy-Kovalevsky theorem that (N, g') is flat. This gives the required contradiction in this case.

II. $dist_{g_i'}(x_i, \partial M_i) \to \infty$ as $i \to \infty$. For this case, we give a different and simpler proof than that in [5, Prop.4.4, Case II].

Let $d_i(x) = dist_{g_i'}(x_i, \partial M_i)$. It follows easily from Case I above that

$$\zeta_i(y_i) \geq (1 - \delta)d_i(y_i),$$

for y_i within bounded distance to $(\partial M, g_i')$, with $\delta \to 1$ as $i \to \infty$. This just corresponds to the statement that the geometry becomes flat near ∂M_i with respect to g_i', which has been proved in Case I. Now by hypothesis, at x_i, $\zeta_i(x_i)/d_i(x_i) \to 0$, (since the ratio is scale-invariant and $\zeta_i(x_i) = 1$ in the scale g_i'). Therefore, by continuity, there are points y_i such that $\zeta(y_i) = \frac{1}{2}d(y_i)$, with $\zeta(z_i) \geq \frac{1}{2}d(z_i)$, for all z_i such that $d(z_i) \leq d(y_i)$. One now works in the scale $\hat{g}_i = \zeta_i(y_i)^{-2}g_i$ where $\hat{\zeta}_i(y_i) = 1$ and hence $dist_{\hat{g}_i}(y_i, \partial M) = 2$. The proof is now completed just as in Case I. Thus, one may pass to a limit $(N, \hat{g}, y)$. On the one hand, the limit $(N, \hat{g})$ is not flat, since, by Theorem 3.2 the convergence to the limit is in $C^{n,\alpha}$ and ζ is continuous in this topology, so that $\hat{\zeta}(y) = 1$. As before, $(N, \hat{g})$ has flat and totally geodesic boundary, and the same proof as in Case I implies that $(N, \hat{g})$ is flat, giving again a contradiction.

Taken together, (3.12) and (3.13) imply that $\zeta(x) \geq \tau_1 > 0$, for all x in a neighborhood of ∂M of fixed size in $(M, \bar{g})$. The bound (3.11) is then a consequence of the local stability result, Theorem 3.2. ■

An odd dimensional analogue of Theorem 3.4 is unknown, and it would be very interesting to know if a suitable version of it holds. The exact formulation would of course have to be modified somewhat, due to the logarithmic terms. Using Kichenassamy's result [34], Javaheri [33] has proved an analogue of Theorem 3.4 in odd dimensions in the context of analytic boundary metrics.

4. Behavior away from the Boundary.

At least in even dimensions, the analysis in §3 shows that the global behavior of the boundary map Π depends only on the behavior of Einstein metrics in the interior, a fixed distance away from the boundary, (depending only on the boundary metric), in a geodesic compactification. Thus, the issue of whether Π is proper becomes a question on the behavior of Einstein metrics in the interior, i.e. on compact sets, away from infinity; the structure near infinity is uniformly controlled by the data at infinity.

Thus, in effect, one is dealing with the behavior of Einstein metrics on compact manifolds (with boundary). Presumably, the degeneration of such metrics has the same general features as the degeneration of Einstein metrics on compact manifolds without boundary. A detailed study of degenerations of Einstein metrics on compact 4-manifolds was first carried out in [3]. Since there is no general theory of such degenerations in higher dimensions, we restrict in this section to dimension 4.

Let $\{g_i\}$ be a sequence of Poincaré-Einstein metrics on a fixed 4-manifold M, with conformal infinities $\{\gamma_i\} \subset \Gamma$, where Γ is a compact set in $\mathcal{C}^{m,\alpha}$. There are three possibilities for the behavior of $\{g_i\}$, in subsequences; cf. [5] for a more detailed discussion.

I. *Convergence*: A subsequence of $\{g_i\}$ converges, modulo diffeomorphisms, to a limit Poincaré-Einstein metric g on M, with boundary metric $\gamma \in \Gamma$. There is a compactification $\widetilde{g}_i = \rho^2 g_i$ of g_i such that the subsequence $\{\widetilde{g}_i\}$ converges in the $C^{m,\alpha}$ topology on $\bar{M}$.

II. *Orbifolds*: A subsequence of $\{g_i\}$ converges, modulo diffeomorphisms, to a limit Poincaré-Einstein orbifold-singular metric g on M, with boundary metric $\gamma \in \Gamma$. The singular metric g is a smooth metric on an orbifold V, and M is a smooth resolution of V. There are only a finite number of singularities, each the vertex of a cone on a spherical space form. Away from the singularities, the convergence is smooth, as in I. The subsequence (M, g_i) converges to (V, g) in the Gromov-Hausdorff topology, [30].

III. *Cusps*: A subsequence of $\{g_i\}$ converges, modulo diffeomorphisms, to a limit Poincaré-Einstein metric with cusps g on a connected manifold N, with boundary metric $\gamma \in \Gamma$, possibly with a finite number of orbifold singularities.

More precisely, the limit (N, g) has conformal infinity $(\partial M, \gamma)$, but has in addition a collection of complete, finite volume ends. Thus there is a compact hypersurface $H \subset N$, disconnecting N into two non-compact regions, the outside and inside; the outside region contains conformal infinity, and so has the same number of components as ∂M, while the inside region is connected and the metric g is complete and of finite volume.

If x_i are base points in (M, g_i) within bounded distance to ∂M in the geodesic compactification $\bar{g}_i$, then the sequence (M, g_i, x_i) converges in the *pointed* Gromov-Hausdorff topology, (cf. [30]), to the limit (N, g, x). The convergence is smooth, in the sense of I, away from any orbifold singular points, and uniform on compact sets in $\bar{N} = N \cup \partial N$, where $\partial N = \partial M$. Note that if one chooses other base points y_i in (M, g_i) with $dist_{\bar{g}_i}(y_i, \partial M) \to \infty$, then (M, g_i, y_i) may limit on other complete, finite volume manifolds (N', g', y); see the discussion regarding the case of surfaces below. However, since they play no role in the analysis here, we ignore these other components of the limit.

If the boundary map Π is to be proper, one must show that only the convergence case above occurs, i.e. rule out the possible formation of orbifold and cusp limits. We discuss these in turn.

The orbifold limits are topological, in the sense that essential 2-cycles in M not coming from ∂M must be collapsed to points under $\{g_i\}$. Thus, for example, if one has a surjection

$$H_2(\partial M, \mathbb{R}) \to H_2(M, \mathbb{R}) \to 0, \tag{4.1}$$

for instance $M = B^4$, then orbifold limits cannot occur, cf. [5]. The condition (4.1) however is not necessary, and there are 4-manifolds not satisfying (4.1) which do not admit any orbifold degenerations. A list of some examples is given in [5, §6]. Also, it appears that the families of self-dual Poincaré-Einstein metrics constructed by Calderbank-Singer [17], which are natural analogues of the Gibbons-Hawking metrics, do not admit orbifold degenerations, [18]. In fact, at this time, it seems that the only known example of orbifold degenerations is that of the Taub-Bolt curve of metrics on $T(S^2)$ which degenerate to the orbifold $C(\mathbb{RP}^3)$, cf. [5]; I am grateful to Michael Singer for pointing this out.

Remark 4.1. In this context, it is worth noting that the manifold theorem, Theorem 2.1, holds also for orbifolds. Thus, let V be an $(n+1)$-dimensional orbifold with boundary, in the sense that V is a smooth manifold away from from finitely many singular points in the interior, each having a neighborhood homeomorphic to the cone on a spherical space form. (Note that this definition of orbifold is much more restrictive than the general definition due to Thurston). Let $\mathcal{E}^{m,\alpha}(V)$ be the moduli space of orbifold smooth Poincaré-Einstein metrics on V, defined as in §2. Then Theorem 2.1 holds for $\mathcal{E}^{m,\alpha}(V)$; the proof is exactly the same. In fact, all the discussion and results above, from §2 to the classification I-III above, holds equally well for Poincaré-Einstein orbifold metrics.

As will be seen in the following, it would be very interesting to understand to what extent Theorem 2.1 generalizes to metrics with other singularities (e.g. cusps) on a compact manifold with boundary, cf. also [37] for some further results in this direction.

Although the situation of orbifold degenerations still needs to be better understood in general, the issue of cusp formation is much more serious and much less well-understood. As seen in Example 2.3(II), there are at least some situations where cusp degenerations can occur. There are no known relations between the possibility of cusp formation and the topology of M, (as is the case with orbifold degenerations); this is a fundamental and very interesting open problem, which exists also for Einstein metrics on compact manifolds. It would also be useful to obtain more detailed information about the geometry of cusp ends.

Instead of trying to find situations where orbifold and cusp formation can be ruled out, (as in (4.1) for example), one can take a different perspective. Namely, these are the only possible degenerations of Poincaré-Einstein metrics with controlled conformal infinity and so it is natural to consider an enlarged space of Poincaré-Einstein metrics which includes these limits.

Thus, let $\bar{\mathcal{E}}$ be the completion of the moduli space $\mathcal{E}$ of Poincaré-Einstein metrics with respect to the pointed Gromov-Hausdorff topology; the base points x are chosen so that

$$dist_{\bar{g}}(x, \partial M) = 1, \tag{4.2}$$

for example. The discussion concerning I-III above shows that metrics in $\bar{\mathcal{E}}$ with controlled conformal infinity $\gamma \in \Gamma$ are compact in this topology; any sequence in $\bar{\mathcal{E}}$ has a convergent subsequence to a limit in $\bar{\mathcal{E}}$ with conformal infinity $[\gamma] \in \Gamma$. Note that this topology on $\bar{\mathcal{E}}$ is quite different than the (unpointed) Gromov-Hausdorff topology; if $\{g_i\}$ is a sequence in $\mathcal{E}$ converging to a cusp metric in $\bar{\mathcal{E}}$, then $dist_{GH}(g_i, g_0) \to \infty$, for any fixed $g_0 \in \mathcal{E}$. This is because $diam_{g_i} M \to \infty$, and the diameter is continuous in the Gromov-Hausdorff topology. In particular, although the Gromov-Hausdorff topology is a metric topology on $\mathcal{E}$, this is not known for the pointed Gromov-Hausdorff topology on $\bar{\mathcal{E}}$.

Let $\partial\mathcal{E} = \bar{\mathcal{E}} \setminus \mathcal{E}$, so that $\partial\mathcal{E}$ consists of orbifold and cusped Poincaré-Einstein metrics, obtained as limits of smooth Poincaré-Einstein metrics on M. If $g_i \in \mathcal{E}$ converges to $g \in \partial\mathcal{E}$, then for any fixed R, the metrics g_i on $B_{x_i}(R)$ converge smoothly to the limit metric g on $B_x(R)$, away from any orbifold singular points; here x_i are base points satisfying (4.2) and $x_i \to x$. Briefly, away from orbifold singular points, one has smooth convergence on compact subsets. Further, the compactified metrics $\bar{g}_i$ converge in $C^{m,\alpha}$ to $\bar{g}$ up the boundary ∂M. (As always, the smooth convergence is understood to be modulo diffeomorphisms). Note that the closure of $\mathcal{E}$ in the Gromov-Hausdorff topology consists of $\mathcal{E}$ together with orbifold-singular Poincaré-Einstein metrics obtained as limits.

Now one has an extension $\bar{\Pi}$ of Π to $\bar{\mathcal{E}}$, and

$$\bar{\Pi} : \bar{\mathcal{E}} \to \mathcal{C} \tag{4.3}$$

is continuous, cf. [5]. Moreover, by construction, $\bar{\Pi}$ is proper.

If $\bar{\mathcal{E}}$ has roughly the structure of a manifold, then as is the case with $\mathcal{E}^0$ before, one can define a degree deg $\bar{\Pi}$ associated with each component of $\bar{\mathcal{E}}$ and

$$\deg \bar{\Pi} = \deg \Pi.$$

So deg $\bar{\Pi} \neq 0$ implies at least that almost every choice of conformal class in $\mathcal{C}$ is the conformal infinity of a smooth Poincaré-Einstein metric on M.

Unfortunately, very little is known about the structure of $\bar{\mathcal{E}}$, even regarding its point set topology. As a first step, the following conjecture seems very plausible:

Conjecture 4.2. For any component $\mathcal{E}_0$ of $\mathcal{E}$, $\bar{\Pi}(\partial\mathcal{E}_0)$ has empty interior in $\mathcal{C}$.

The intuition leading to Conjecture 4.2 is that Π has Fredholm index 0. As a simple illustration, let $E = \{(x,y,z) \in \mathbb{R}^3 : z \in (0,1)\}$. Then $\bar{E} = E \cup \partial E$, $\partial E = \{z = 1\}$ is a manifold with boundary and the projection map $\pi : E \to \mathbb{R}^2$, $\pi(x,y,z) = (x,y,0)$ has the property that π maps ∂E onto $\pi(E)$. Of course Conjecture 4.2 fails on this example; however, the index of the map π is one.

Similarly, if $\Pi(\partial\mathcal{E}_0)$ did have non-trivial interior $\mathcal{V} \subset \mathcal{C}$, and if $\overline{\mathcal{E}}_0$ is reasonably well-behaved, one would expect there are curves σ in $\overline{\mathcal{E}}_0$ on which Π is constant, i.e. for all $\gamma \in \mathcal{U}$, there exists $\sigma_\gamma(t) \subset \overline{\mathcal{E}}_0$, with $\sigma_\gamma(t) \subset \mathcal{E}_0$ for $t > 0$ and $\sigma_\gamma(0) \in \partial\mathcal{E}_0$, such that $\Pi \circ \sigma_\gamma = \gamma$. Hence, if γ is a regular value of Π, then index $\Pi \geq 1$, which is impossible.

As will be seen in §5, (cf. Example 5.2), Conjecture 4.2 is probably false if $\mathcal{E}_0$ is replaced by $\mathcal{E}$. As a toy model where this conjecture fails, (with index $\pi = 0$), let E_i be the collection of planes in $\mathbb{R}^3$ given by $E_i = \{z = 1 - \frac{1}{i}\}$. Now connect these planes by a collection of tubes or wormholes, deleting the corresponding discs in $\{E_i\}$ and let E be the resulting connected space. Then $\bar{E} = E \cup E_\infty$, where $E_\infty = \{z = 1\}$. As above, let $\pi : E \to \mathbb{R}^2$ be the projection onto $E_0 = \{z = 0\}$. One may then choose the connecting tubes so that π is continuous and surjective on E, on $\bar{E}$, and on $\partial E = E_\infty$. By choosing the tubes to become arbitrarily small and dense near ∂E, one may arrange that E is uniformly locally path connected.

One would not expect that $\mathcal{E}_0$ or $\overline{\mathcal{E}}_0$ has such a complicated structure. Instead, it seems more likely that both $\partial\mathcal{E}_0$ and $\Pi(\partial\mathcal{E}_0)$ should be lower-dimensional in the spaces $\bar{\mathcal{E}}$ and $\mathcal{C}$ respectively. If codim $\partial\mathcal{E}_0 = 1$ in $\bar{\mathcal{E}}_0$, then $\partial\mathcal{E}_0$ acts as a topological boundary and so $\bar{\mathcal{E}}_0$ does not have the structure of a manifold; at best it is a manifold with boundary. In this case, it will be difficult to define a suitable degree. On the other hand, if codim $\partial\mathcal{E}_0 > 1$, then the metric boundary $\partial\mathcal{E}_0$ is not topological and one expects that $\bar{\mathcal{E}}_0$ behaves sufficiently well to allow one to define a degree deg $\bar{\Pi}$ on $\bar{\mathcal{E}}_0$. It would of course be very interesting to make progress on these speculative remarks.

An interesting alternate path is to try extend the map Φ in (2.6)-(2.7) to singular metrics which effectively model orbifold singular and cusp metrics on the manifold M. This is perhaps easier in the orbifold case, since the behavior of the Einstein metrics g_i converging to an orbifold limit g is quite well-understood. If Φ can be extended to such an enlarged space, consisting of smooth and singular metrics on M modelling orbifolds and cusps, such that Φ is still a smooth mapping, with Fredholm linearization L, then the same proof as Theorem 2.1 will show that $\bar{\mathcal{E}}$ is a smooth manifold. For a discussion of orbifold singular metrics on a manifold M, (as opposed to smooth metrics on an orbifold associated to M), cf. [3].

With regard to the work to follow in §5, it is worthwhile to describe in some detail the simplest situation where curves of Einstein metrics form cusps, i.e. the case of hyperbolic metrics on surfaces. Thus, let (Σ, g_Σ) be any complete conformally compact Riemann surface with non-empty boundary of constant negative curvature, normalized so that

$$K_\Sigma = Ric_\Sigma = -1, \tag{4.4}$$

and with $\pi_1(\Sigma) \neq 0$. Topologically, Σ is S^2 with at least two discs removed, or a surface of genus $g \geq 1$, with at least one disc removed. The boundary $\partial\Sigma$ is a union of q circles, $\partial\Sigma = \cup_1^q S^1$, with $q \geq 1$. In the free homotopy class of each end E_i of Σ, one has a unique closed geodesic σ_i, $1 \leq i \leq q$, of length $\alpha_i > 0$.

Let $\mathcal{M}$ be the moduli space of such metrics satisfying (4.4). There are several definitions of the moduli space, depending on the choice of the action of the diffeomorphism group on $\partial\Sigma$. To obtain a finite dimensional space, $\mathcal{M}$ is considered as the space of all conformally compact metrics satisfying (4.4) divided out by the action of all diffeomorphisms of $\bar{\Sigma} = \Sigma \cup \partial\Sigma$

mapping $\partial\Sigma$ onto itself. It is well-known, cf. [1] for example, that $\mathcal{M}$ is a smooth orbifold, of dimension

$$m = dim_{\mathbb{R}}\mathcal{M} = 6g - 6 + 3q, \tag{4.5}$$

(with $m = 2$ if $g = 0$ and $q = 2$). The boundary map

$$\Pi : \mathcal{M} \to \mathcal{C}/\mathrm{Diff} \tag{4.6}$$

is a constant map, since S^1 has a unique conformal structure up to diffeomorphism. Thus one has index $\Pi = m > 0$.

The boundary $\partial\mathcal{M}$ of the moduli space $\mathcal{M}$ with respect to the Deligne-Mumford compactification consists of Riemann surfaces with nodes or punctures; this coincides with the boundary in the pointed Gromov-Hausdorff topology, where $\partial\mathcal{M}$ is represented by complete hyperbolic metrics which have cusp ends, obtained by shrinking a collection of disjoint closed geodesics in Σ to 0 length. Note that such geodesics may or may not include geodesics from the collection $\{\sigma_i\}$. Thus $\partial\mathcal{M}$ is stratified by the moduli spaces of Riemann surfaces of lower genus, and a positive number of punctures; the strata are of dimension

$$d = 6g - 6 + 3q - 2p,$$

where p is the number of cusp ends, (punctures). In particular, $\partial\mathcal{M}$ has codimension 2 in $\overline{\mathcal{M}}$. The closure $\overline{\mathcal{E}} = \mathcal{M} \cup \partial\mathcal{M}$ has the structure of a real-analytic variety, cf. [1]. The boundary map extends to $\overline{\mathcal{E}}$, and it is still the constant map.

One would like to have a similar concrete description of cusp formation on some class of examples in higher dimensions. However, no such examples are known. Observe that conformally compact metrics are not closed under products; also the product of a compact metric and conformally compact metric is not conformally compact. In §5, we discuss a construction of families of conformally compact metrics forming cusps, based on the model of this behavior for surfaces. However, perhaps surprisingly, this does not lead to examples of Einstein metrics.

The following remains a simple but basic open question: does there exist a curve of Poincaré-Einstein metrics on M^{n+1}, $n \geq 3$, which converges to a Poincaré-Einstein metric with cusps?

Remark 4.3. The classification of degenerations in I-III above is special to dimension 4 and very little is known in such generality in higher dimensions. However, in the presence of symmetry, the equations for Einstein metrics on higher dimensional manifolds can be reduced to the Einstein equations coupled to other fields in lower dimensions, via the well-known Kaluza-Klein procedure. In this regard, note that Theorem 2.4 implies that symmetries of a boundary metric γ are automatically inherited by any Einstein metric (N, g) filling $(\partial N, \gamma)$.

For example, suppose the compact group G acts freely and isometrically on a Poincaré-Einstein metric (N^{n+1}, g_N). Let $M = N/G$ be the orbit space of this action; then the metric g_N may be written in the form

$$g_N = \pi^* g_M + \langle\theta, \theta\rangle, \tag{4.7}$$

where $\pi : N \to M$ is the projection onto the orbit space, θ is a connection 1-form on N with values in the Lie algebra $\mathcal{L}(G)$ and $\langle,\rangle$ is a family of left-invariant metrics on $\mathcal{L}(G)$ parametrized by $x \in M$.

The Einstein equations (2.2) for g_N become the Einstein equations for g_M coupled to the gauge field θ and form $\langle,\rangle$. When dim $M = 4$, one can then consider whether the results above for the Einstein equations generalize to the Einstein equations coupled to various extra fields. This has been worked out in detail by Javaheri [33] for the case that $G = S^1$

and the action is static, so that the metric (4.7) has the form of a warped product; the equations on M^4 then take the form of the Einstein equations coupled to a scalar field. Note that already in this case, the Fefferman-Graham expansion on M^4 has logarithmic terms, due to the extra scalar field.

5. Discussion on Cusp and Orbifold Formation.

In this section, we show that it is not very easy to find continuous curves of Poincaré-Einstein metrics on a fixed manifold which limit on Poincaré-Einstein metrics with cusps. This gives some evidence, not particularly strong at the moment, but nevertheless suggesting that cusps may not form in components $\mathcal{E}_0$ of $\mathcal{E}$. Although the main focus of this section is on cusp formation, it will be seen that similar results often apply to orbifold formation.

Let (N, g_0) be a Poincaré-Einstein metric with cusps. As in §2, let $\mathbb{S}_2^{m,\alpha}$ be the space of symmetric bilinear forms on N which are bounded in $C^{m,\alpha}$ with respect to g_0 and decay in $C^{m,\alpha}$ at conformal infinity on the order of ρ^2. The map $\Phi = \Phi^{g_0}$ is then defined as in (2.6)-(2.7).

Conjecture 5.1. *Let $(N,\ g_0)$ be a Poincaré-Einstein with cusps, and suppose that*

$$K_N \leq 0, \tag{5.1}$$

i.e. g_0 has non-positive curvature. Then the map Φ^{g_0} is a submersion at g_0, and the boundary map Π taking Poincaré-Einstein metrics with cusps on N to conformal classes $\mathcal{C}$ is a diffeomorphism in a neighborhood of g_0.

Conjecture 5.1 stands in stark contrast to Conjecture 4.2; they almost contradict each other. In fact, they would contradict each other if one knew that Conjecture 4.2 holds and that every $\gamma \in \mathrm{Im}\,\Pi$ near γ_0 from Proposition 5.1 is the limit of a sequence $\gamma_i = \Pi(g_i)$ with g_i in a connected component $\mathcal{E}_0$. If this were the case, it would follow that cusps satisfying (5.1) cannot form as limits within $\mathcal{E}_0$. Exactly the same remarks apply to orbifolds in place of cusps, since Conjecture 5.1 does hold for orbifold Poincaré-Einstein metrics on an orbifold V.

In this context, it is worth considering some concrete examples:

Example 5.2. Let g_C be the standard hyperbolic cusp metric on $N = \mathbb{R} \times T^n$ given by

$$g_C = dr^2 + e^{2r}\gamma_{T^n}, \tag{5.2}$$

where γ_{T^n} is any flat metric on the torus T^n. Clearly (N, g_C) satisfies (5.1), so Conjecture 5.1 would imply that Π is a local diffeomorphism near g_C, i.e. any boundary metric γ near a flat metric γ_{T^n} on T^n is the conformal infinity of a complete Poincaré-Einstein cusp metric on $\mathbb{R} \times T^n$.

On the other hand, as discussed in Example 2.3(II), there is an infinite sequence of conformally compact twisted toral black hole metrics g_i on $M = \mathbb{R}^2 \times T^{n-1}$. The metrics g_i converge to g_C in the pointed Gromov-Hausdorff topology, cf. [4]. These metrics all lie in distinct components $\mathcal{E}_i$ of $\mathcal{E}(M)$. If $\Pi_i : \mathcal{E}_i \to \mathcal{C}$ is the boundary map, then we conjecture that for all i large, Π_i is surjective onto a fixed neighborhood $\mathcal{V}$ of $\Pi(g_i) = (T^n, \gamma_{T^n})$.

In this case, every conformal class $[\gamma] \in \mathcal{V}$, for some open set $\mathcal{V} \subset \mathcal{C}$ containing g_{T^n} is the conformal infinity of an infinite sequence of Poincaré-Einstein metrics on M, limiting on a Poincaré-Einstein cusp metric on N. This indicates that Conjecture 4.2 is false if the assumption that $\mathcal{E}_0$ is connected is dropped.

We point out that exactly the same discussion holds with $\mathbb{R} \times T^n$ replaced by any conformally compact hyperbolic manifold N, with a collection of cusp ends. As shown in [23], the cusp ends can be Dehn filled with solid tori to produce Poincaré-Einstein metrics (M_i, g_i)

with a fixed conformal infinity. In this case, instead of having infinitely many components $\mathcal{E}_i = \mathcal{E}_i(M)$ of $\mathcal{E}$ on a fixed manifold M, one has a collection of components $\mathcal{E}_i = \mathcal{E}(M_i)$ on infinitely many topologically distinct manifolds M_i, with common boundary ∂N.

The discussion above presents some speculative evidence that cusps do not form within $\bar{\mathcal{E}}_0$, for any component $\mathcal{E}_0$ of $\mathcal{E}$. Next, we present a construction of (connected) families of conformally compact metrics which are very close to being Einstein, and which do limit on cusps. This seems to be the simplest possible construction of such metrics, since it is based on the formation of cusps on surfaces. However, we argue that rather surprisingly, it is unlikely that these metrics can be perturbed to nearby Poincaré-Einstein metrics.

To begin the construction, return to the static AdS black hole metrics (2.11) on $\mathbb{R}^2 \times N$, with $k = -1$. In this situation, g_m is well-defined for negative values of m; in fact, g_m is well-defined for $m \in [m_-, \infty)$, where

$$m_- = -\frac{1}{n-2}\left(\frac{n-2}{n}\right)^{n/2}, \quad \text{with } r_+ = \left(\frac{n-2}{n}\right)^{1/2}. \tag{5.3}$$

For the extremal value m_- of m, $V(r_+) = V'(r_+) = 0$, and a simple calculation, (cf. (5.6) below) shows that the horizon $\{r = r_+\}$ occurs at infinite distance to any given point in $(\mathbb{R}^2 \times N, g_{m_-})$; the horizon in this case is called degenerate, (with zero surface gravity). Note that $\beta(m_-) = \infty$, so that the θ-circles are in fact lines $\mathbb{R}$. As m decreases to m_-, the horizon diverges to infinity, (in the opposite direction from the conformal infinity), while the length of the θ-circles expands to ∞. Thus, the metric g_{m_-} is a complete metric on the manifold $\mathbb{R}\times\mathbb{R}\times N = \mathbb{R}^2 \times N$, but is no longer conformally compact; the conformal infinity is $\mathbb{R} \times (N, g_N)$.

However, one may divide the infinite θ-factor $\mathbb{R}$ of the metric g_{m_-} by $\mathbb{Z}$ to obtain a complete metric g_E on $C \times N = \mathbb{R} \times S^1 \times N$ of the form

$$g_E = V^{-1}dr^2 + Vd\theta^2 + r^2 g_N, \tag{5.4}$$

where $V(r) = -1 + r^2 - \frac{2m_-}{r^{n-2}}$, with m_- and r_+ given by (5.3). The length β of the θ-parameter in (5.4) is now arbitrary. The metric g_E is called an extreme black hole metric, and is Poincaré-Einstein with a single cusp end; g_E has a smooth conformal compactification to the boundary metric $S^1(\beta) \times (N, g_N)$.

To understand the behavior of the metric g_E in the cusp region, we convert to geodesic coordinates. Let $ds = V^{-1/2}dr$, so that, (up to an additive constant),

$$s = \int_{r_+ + 1} V^{-1/2}(r)dr. \tag{5.5}$$

Thus, as $r \to r_+$, $s \to -\infty$, while as $r \to \infty$, $s \to \infty$. The metric (5.4) takes the form

$$g_E = ds^2 + V(s)d\theta^2 + r^2(s)g_N, \tag{5.6}$$

and the integral curves of ∇s are geodesics. A simple calculation shows that

$$V(s) = ne^{2\sqrt{n}s}(1 + \varepsilon(s)), \quad \text{as } s \to -\infty, \tag{5.7}$$

where $\varepsilon(s) \to 0$ as $s \to -\infty$.

As $s \to -\infty$, the length of the θ-circles of course goes to 0, i.e. one has a collapse. However, the collapse can be unwrapped by passing to large covering spaces of the S^1 factor; on any sequence of base points x_i with $s(x_i) \to -\infty$, one may choose coverings so that the length of the S^1 factor at x_i is approximately 1. One may then pass to a pointed limit to obtain the metric

$$g_\infty = ds^2 + e^{2\sqrt{n}s}d\theta^2 + r_+^2 g_N. \tag{5.8}$$

The metric (5.8) is a product of the constant curvature metric on the cusp $C = \mathbb{R} \times S^1$ with a rescaling of the metric g_N on N. By (2.10), the Ricci curvature of $r_+^2 g_N$ equals $-r_+^{-2}(n-2) = -n g_N$, while the curvature of the cusp metric is also $-n$. The metric (5.8) (of course) has Ricci curvature $Ric_{g_\infty} = -n g_\infty$. The limit (5.8) is unique, up to rescalings of the length of the S^1 factor.

The curvature of the extremal metric g_E converges to that of g_∞ exponentially fast in s. Straightforward computation from the estimates above shows that

$$(5.9) \qquad ||R_{g_E} - R_{g_\infty}|| \leq Ce^{\sqrt{n}s}, \quad \text{as} \quad s \to -\infty,$$

where the norm is the L^∞ norm. The same estimate holds for all covariant derivatives of these curvatures.

We now make g_E conformally compact, by closing off the cusp end. To do this, glue $\Sigma \times N$, where Σ is any hyperbolic surface with an open expanding end, onto the cusp end of g_E. For simplicity, assume that Σ has a single end; it is easy to generalize the construction below to any finite number of ends.

Thus, let (Σ, g_Σ) be any conformally compact Riemann surface with connected, non-empty boundary of constant negative curvature, normalized so that

$$(5.10) \qquad K_\Sigma = Ric_\Sigma = -n,$$

and with $\pi_1(\Sigma) \neq 0$. Let σ be the unique closed geodesic in the free homotopy class of the end E of Σ, and let α be the length of σ. We will only consider metrics g_Σ in the moduli space $\mathcal{M}$, discussed in §4, for which

$$(5.11) \qquad \alpha \leq \varepsilon,$$

where ε is fixed and sufficiently small. Let $\mathcal{M}_\varepsilon$ be the domain in $\mathcal{M}$ satisfying (5.11).

Let $T(\tau)$ be the tubular neighborhood of radius τ about σ; coordinates may be introduced in this region so that the metric on $T(\tau)$ has the form

$$(5.12) \qquad g_\Sigma = dt^2 + \cosh^2(\sqrt{n}t)d\theta^2,$$

where the length of θ is α and $t \in (-\tau, \tau)$. For α (arbitrarily) small, the expression (5.12) is valid for τ (arbitrarily) large.

Next, on the product $\Sigma \times N$, form the product metric

$$(5.13) \qquad g_D = g_\Sigma + r_+^2 g_N.$$

The metric g_D is Einstein, of Ricci curvature $-n$.

We now truncate the two metrics g_D and g_E and glue them together. To begin, topologically, set

$$M = \Sigma \times N.$$

While the product metric g_D on M is Einstein, it is not conformally compact. This metric has one end E of the form $C \times N$, where C is an expanding cusp. In the τ-tubular neighbhorhood of the geodesic $\sigma \subset E \subset \Sigma$, the metric has the form (5.12). Choose R large (to be determined below), and let D_R be the region in $\Sigma \times N$ where $t \leq R$, in the end E. Thus, $\partial D_R = S^1(L^-) \times (r_+^2 g_N)$ where

$$(5.14) \qquad L^- = \cosh(\sqrt{n}R)\alpha.$$

Next, take the conformally compact extreme metric g_E on $C \times N$, and truncate it to the region E^R where $s \geq -R$. The length of the boundary circle ∂C is then

$$(5.15) \qquad L^+ = V^{1/2}(-R)\beta.$$

To perform the glueing, we require that the lengths of the circles agree, $L^- = L^+$. Since the length β is fixed, given the length α, this imposes the relation

$$\alpha = \{\cosh(\sqrt{n}R)\}^{-1}V^{1/2}(-R)\beta. \tag{5.16}$$

The parameter t in (5.12) is related with the parameter s in (5.6) by setting $t - R = s + R$.

Given these choices, one may easily construct a conformally compact approximate Einstein metric $\widetilde{g}$ on M by attaching the truncated metrics D_R and E^R along their boundaries and smoothing the seam in a neighborhood U of radius 1 of the boundaries. The metric $(M, \widetilde{g})$ is smoothly conformally compact, and Einstein outside the glueing region U. Conformal infinity is given by the conformal class $(\partial M, [\gamma_0(\beta)])$, where $\partial M = S^1 \times N$, and $\gamma_0(\beta) = \beta^2 d\theta^2 + g_N$, with $\theta \in [0,1]$.

Let $\Phi = \Phi^{\widetilde{g}}$ be as in (2.6)-(2.7). Then by construction, $\Phi^{\widetilde{g}}(\widetilde{g}) = 0$ outside U. An elementary computation, using (5.9) and simple estimates for the 2^{nd} fundamental forms of the boundaries of (D_R, g_D) and (E^R, g_E) gives the estimate

$$|\Phi^{\widetilde{g}}(\widetilde{g})| \leq Ce^{-\sqrt{n}R}, \tag{5.17}$$

inside U, where C is independent of R.

This gives the construction of the approximate solutions $\widetilde{g}$. In fact the construction gives a smooth moduli space $\widetilde{\mathcal{M}}_\varepsilon$ of approximate solutions on M, naturally diffeomorphic to the moduli space $\mathcal{M}_\varepsilon$ on Σ. One has a natural boundary map

$$\widetilde{\Pi} : \widetilde{\mathcal{M}}_\varepsilon \to \mathcal{C}, \tag{5.18}$$

which is the constant map to the conformal class $[\gamma_0(\beta)]$; note that dim $\widetilde{\mathcal{M}} = m = 6g - 3$.

From (5.17), for ε sufficiently small, or equivalently R sufficiently large, one would expect that it should be possible to perturb the metrics $\widetilde{g}$ to exact Einstein metrics g on M, i.e. perturb $\widetilde{g}$ to g satisfying $\Phi^{\widetilde{g}}(g) = 0$. Following the proof of Theorem 2.1, it is not difficult to show that $\Phi^{\widetilde{g}}$ is a submersion at $\widetilde{g}$, so that the image of $\Phi^{\widetilde{g}}$ contains an open ball $B(\mu) \subset \mathcal{C}$ about $\Phi^{\widetilde{g}}(\widetilde{g})$. However, one needs to obtain a lower bound on the radius μ of such a ball, independent of R, or at least prove that

$$\mu >> e^{-\sqrt{n}R}, \tag{5.19}$$

for R large.

Now it is clear that the linearized operator L in (2.8) cannot be uniformly invertible at $\widetilde{g} \in \widetilde{\mathcal{M}}$. In fact, there is an approximate kernel $\widetilde{K}$ of L acting on forms in $\mathbb{S}_2^{m,\alpha}$, induced by forms κ tangent to the moduli space $\mathcal{M}$ of hyperbolic metrics on Σ and extended to M in a natural way, so that, if $||\widetilde{\kappa}||_{L^\infty} = 1$, then

$$||L(\widetilde{\kappa})||_{L^\infty} \to 0, \tag{5.20}$$

as the glueing radius $R \to \infty$. Note that forms κ tangent to $\mathcal{M}$, (equivalent to holomorphic quadratic differentials on Σ), decay to 0 on the end E outside the closed geodesic σ.

Suppose that the Einstein manifold (N, g_N) in (2.10) has the property that

$$K_N \leq 0, \tag{5.21}$$

i.e. g_N has non-positive curvature. Then it is not particularly difficult to show, although we will not give the details here, that L is uniformly invertible on the orthogonal complement $\widetilde{K}^\perp$ of $\widetilde{K}$ in $\mathbb{S}_2^{m,\alpha}$ with respect to the L^2 metric. One has then dim$\widetilde{K} = m = 6g - 3$.

There are now two methods to try to establish (5.19) and thus obtain a Poincaré-Einstein metric g near $\widetilde{g}$. First, one can try to arrange that

$$\Phi^{\widetilde{g}}(\widetilde{g}) \in \widetilde{K}^\perp, \tag{5.22}$$

possibly by modifying or perturbing $\widetilde{g}$, and iteratively solve $\Phi^{\widetilde{g}}(g) = 0$ within the space $\mathbb{S}_2^{m,\alpha}$. This would lead to the existence of a Poincaré-Einstein metric g with the same boundary metric as $\widetilde{g}$. However this is not possible:

Proposition 5.3. *There is no Einstein metric on M with conformal infinity given by $S^1(\beta) \times (N, g_N)$.*

Proof: This is an immediate consequence of Corollary 2.5. ■

Thus, to obtain an Einstein metric g on M close to $\widetilde{g}$ requires changing the boundary metric of $\widetilde{g}$, i.e. working outside the space $\mathbb{S}_2^{m,\alpha}$. This means one must use the dependence of $\Phi^{\widetilde{g}}$ on the boundary metrics to try to kill the cokernel of L, as in the proof of Theorem 2.1. In turn, this requires proving that the pairing (2.9) is non-degenerate and bounded below. More precisely, for any $\widetilde{\kappa} \in \widetilde{K}$, there must exist a variation $\dot{\gamma}$ of the boundary metric γ_0, with $||\widetilde{\kappa}||_{L^\infty} \leq 1$ and $||\dot{\gamma}||_{L^\infty} \leq 1$, such that

$$|\int_M \langle D\Phi^{\widetilde{g}}(\dot{\gamma}), \widetilde{\kappa} \rangle dV| \geq \mu', \tag{5.23}$$

where μ' also satisfies (5.19).

We have not been able to verify (5.23), and expect it is not true. First, the choice of $\widetilde{K}$ is not canonical, and to establish (5.23), one needs a precise definition or choice. If $\widetilde{K}$ is defined as forms tangent to the moduli space $\mathcal{M}_\varepsilon$ on Σ, naturally extended to forms on M and which have compact support, then $\kappa = 0$ near infinity. However, by the definition given in §2, the support of $D\Phi^{\widetilde{g}}(\dot{\gamma})$ is near ∂M. Thus, this definition implies that the integrals in (5.23) are all 0.

However one defines $\widetilde{K}$ exactly, (for instance as the space of eigenforms with small eigenvalues of L acting on $\mathbb{S}_2^{m,\alpha}$), the support of any $\widetilde{\kappa} \in \widetilde{K}$ with L^∞ norm 1 will be almost completely contained in the region D from (5.14). Thus, such forms must decay quickly at infinity, and we expect the decay is too fast to give the lower bound (5.23). However, we have not been able to verify this in detail. Instead, we give other, heuristic, arguments which suggest that (5.23) must fail.

First, when (5.21) holds, a straightforward computation shows that the extremal black hole (N, g_E) also has non-positive curvature, cf. [10] for instance. Thus, (5.1) holds and Conjecture 5.1 would imply that any metric γ on the boundary $S^1 \times N$ sufficiently close to $\gamma_0(\beta)$ is the boundary metric of a Poincaré-Einstein g_γ with a cusp end on the manifold $\mathbb{R} \times S^1 \times N$. The metric g_γ is asymptotic to the extreme metric g_E, or equivalently to the metric g_∞ in (5.8) down the cusp end. This is because g_∞ is rigid, in that it admits no bounded infinitesimal Einstein deformations. Consequently, exactly the same construction of the approximate Einstein metrics as above may be carried out with g_γ in place of the extremal black hole metric g_E.

This now gives an infinite dimensional moduli space $\widetilde{\mathcal{E}}$ of approximate Einstein metrics and a corresponding boundary map

$$\widetilde{\Pi} : \widetilde{\mathcal{E}} \to \mathcal{C}. \tag{5.24}$$

The map $\widetilde{\Pi}$ is surjective onto a neighborhood of $\gamma_0(\beta)$ and has fibers $\widetilde{\Pi}^{-1}(\gamma) = \widetilde{\mathcal{M}_\gamma}$ diffeomorphic to $\widetilde{\mathcal{M}}$ as before. Thus, $\widetilde{\Pi}$ is Fredholm of index $m = 6g - 3$. Clearly $\widetilde{\Pi}$ is a submersion, so that the dimension of the kernel of $D_{\widetilde{g}}\widetilde{\Pi}$ is m, at every $\widetilde{g} \in \widetilde{\mathcal{E}}$.

Now if (5.23) holds, (for some choice of $\widetilde{K}$), one expects it should hold equally well on $\widetilde{\mathcal{M}_\gamma}$ in place of $\widetilde{\mathcal{M}}$. This implies the existence of a space of Poincaré-Einstein metrics $\mathcal{E}$ and smooth boundary map

$$\Pi : \mathcal{E} \to \mathcal{C}. \tag{5.25}$$

We abuse notation here slightly and assume that $\mathcal{E}$ consists only of metrics close to the approximate Einstein metrics $\widetilde{g} \in \widetilde{\mathcal{E}}$; thus $\mathcal{E}$ is a connected open subset of the full moduli space.

Both maps $\widetilde{\Pi}$ and Π extend as continuous maps to the completions $\overline{\widetilde{\mathcal{E}}}$ and $\overline{\mathcal{E}}$ of the moduli spaces $\widetilde{\mathcal{E}}$ and $\mathcal{E}$ respectively. The boundary $\partial\widetilde{\mathcal{E}}$ consists of the (original) cusp metrics g_γ constructed as perturbations of the extreme metric g_E; in particular, the metrics in $\partial\widetilde{\mathcal{E}}$ are all Einstein. For metrics $\widetilde{g}$ near $\partial\widetilde{\mathcal{E}}$, the estimate (5.17) holds, with R very large. Hence, one has

$$\partial\widetilde{\mathcal{E}} = \partial\mathcal{E},$$

and Π maps $\partial\mathcal{E}$ onto an open set in $\mathcal{C}$. This of course contradicts Conjecture 4.2. In fact the maps $D_{\widetilde{g}}\widetilde{\Pi}$ and $D_g\Pi$ are both Fredholm, and are arbitrarily close for R sufficiently large. The Fredholm index is constant under small perturbations and since index $D_{\widetilde{g}}\widetilde{\Pi} = 6g - 3$, while index $D_{\widetilde{g}}\widetilde{\Pi} = 0$, one has a contradiction.

Although the arguments above require some further justifications to be made completely rigorous, they strongly suggest that (5.23) does not hold and that metrics in $\widetilde{\mathcal{M}}$ or $\widetilde{\mathcal{E}}$ cannot be perturbed to Poincaré-Einstein metrics on M. Further independent evidence for this is also given by Corollary 2.5, which implies that if the space $\mathcal{E}$, close to $\widetilde{\mathcal{E}}$ exists, then Im Π must miss the infinite-dimensional space of S^1-invariant metrics on ∂M. We see no good reason why this should be the case.

This leads again then to the basic question raised at the end of §4 and discussed above; are there situations where Poincaré-Einstein cusps form as limits of metrics in a component $\mathcal{E}_0$ of $\mathcal{E}$? Resolution of this question would represent major progress on understanding the global existence question for the Dirichlet problem.

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