Boundary Regularity for the Ricci Equation, Geometric Convergence, and Gel'fand's Inverse Boundary Problem

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Abstract

This paper explores and ties together three themes. The first is to establish regularity of a metric tensor, on a manifold with boundary, on which there are given Ricci curvature bounds, on the manifold and its boundary, and a Lipschitz bound on the mean curvature of the boundary. The second is to establish geometric convergence of a (sub)sequence of manifolds with boundary with such geometrical bounds and also an upper bound on the diameter and a lower bound on injectivity and boundary injectivity radius, making use of the first part. The third theme involves the uniqueness and conditional stability of an inverse problem proposed by Gel'fand, making essential use of the results of the first two parts.

1. Introduction

The goals of this paper are to establish regularity, up to the boundary, of the metric tensor of a Riemannian manifold with boundary, under Ricci curvature bounds and control of the boundary's mean curvature; to apply this to results on Gromov compactness and geometric convergence in the category of manifolds with boundary; and then to apply these results to the study of an inverse boundary spectral problem introduced by I. Gel'fand.

Regularity of the metric tensor away from the boundary has been studied and used in a number of papers, starting with [DTK]. The tack has been to construct local harmonic coordinates and use the fact that, in such harmonic coordinates, the Ricci tensor has the form

$$\Delta g_{\ell m} - B_{\ell m}(g, \nabla g) = -2\,\mathrm{Ric}_{\ell m}\,. \tag{1.0.1}$$

Here Δ is the Laplace-Beltrami operator, applied *componentwise* to the components of the metric tensor, and $B_{\ell m}$ is a quadratic form in ∇g_{ij}, with coefficients that are smooth functions of g_{ij} as long as the metric tensor satisfies a bound $C_1|\eta|^2 \leq g_{jk}(x)\eta^j\eta^k \leq C_2|\eta|^2$, with $0 < C_1 \leq C_2 < \infty$. If one is given information on the

Ricci tensor, one can regard (1.0.1) as an elliptic PDE for the metric tensor, and obtain information on its components, in harmonic coordinates.

The notion of compactness of a family of Riemannian manifolds and of geometric convergence issues from work of J. Cheeger [Ch] and M. Gromov (cf. [Gr], the revised and translated version of his 1981 work). The role of harmonic coordinates in the study of such geometric convergence has been exploited in a number of papers. It was used in [P] and in [GW] to obtain a compactness result, assuming a bound on the Riemann tensor, and some other geometric quantities. In [An1] there was a successful treatment of compactness given a sup norm bound on the Ricci tensor, an upper bound on the diameter, and a lower bound on the injectivity radius, for a family of compact Riemannian manifolds of a fixed dimension. Convergence was shown to hold, for a subsequence, in the C^r-topology, for any $r < 2$. (A definition of geometric convergence is recalled in §3.)

One of our motivations to extend the scope of these results to the category of manifolds with boundary arises naturally in the study of a class of inverse problems. In these problems, one wants to determine the coefficients of some partial differential equation in a bounded region via measurements of solutions to the PDE at the boundary. Such problems arise in various areas, including geophysics, medical imaging, and nondestructive testing. One problem, formulated by I. Gel'fand [Ge], consists of finding the shape of a compact manifold $\overline{M}$ with boundary ∂M and the metric tensor on it from the spectral data on ∂M. Namely, if R_λ is the resolvent of the Neumann Laplacian Δ^N on M, the Gel'fand data consists of the restriction of the integral kernel $R_\lambda(x, y)$ of the resolvent to $x, y \in \partial M$, as λ varies over the resolvent set of Δ^N. Another formulation of Gel'fand's inverse problem will be given in §4.

For such an inverse problem, the first issue to investigate is uniqueness. In the context of C^∞ metric tensors, this was established for the Gel'fand problem in [BK1] taking into account the unique continuation in [Ta]. See also [Bz], [NSU], [Be1] and [Nv] for the isotropic inverse problems. As we will explain below it is important to obtain uniqueness with much less regular coefficients.

Once uniqueness results have been obtained, one has to face up to the issue of *ill posedness* of the inverse problem. That is, one can make large changes in $\overline{M}$ that have only small effects on boundary data obtained from examining the boundary behavior of the resolvent kernel mentioned above. For example, given $(\overline{M}, g)$, one could take an auxiliary manifold X, without boundary, of the same dimension as $\overline{M}$, remove a small ball from X and from the interior of $\overline{M}$, and connect these manifolds by a thin tube. One is faced with the task of *stabilizing* this ill posed inverse problem. One ingredient in this process involves having some *a priori* knowledge of the quantities one is trying to determine, typically expressed in terms of a priori bounds on these quantities in certain norms.

An early result in this direction for the Gel'fand problem was given in [Al] by G. Alessandrini, who obtained *conditional stability* for the operator $\operatorname{div} \varepsilon \operatorname{grad}$ in a bounded domain in Euclidean space, where ε is a positive function (scalar conductivity), assumed to be bounded in some Sobolev space $H^s(M)$, with $s > 0$. See also [StU] for a related result for an anisotropic metric tensor close to Euclidean. Despite these successes, there is a clear need for coordinate-invariant constraints.

In the case of trying to determine an unknown Riemannian manifold with boundary $\overline{M}$, from boundary spectral data, it is natural to make a priori hypotheses on geometrical properties of $\overline{M}$. Furthermore, if one must make such a priori hypotheses, it is desirable to get by with as weak a set of hypotheses as possible. There is then a tension between the desire to make weak a priori hypotheses and the need to establish uniqueness results. (For preliminary results in this direction see [K2L], [Ka].)

Here we impose a priori sup norm bounds on the Ricci tensor of $\overline{M}$, and of ∂M. This, together with a Lipschitz norm bound on the mean curvature of $\partial M \hookrightarrow \overline{M}$, is shown in §2 to imply certain regularity, up to the boundary, of the metric tensor of $\overline{M}$, when one is in "boundary harmonic coordinates" (defined in §2). To be precise, we obtain regularity in the Zygmund space $C^2_*(\overline{M})$, a degree of regularity better than C^r for any $r < 2$ and just slightly worse than C^2. This result has the following important advantage over a $C^{2-\varepsilon}$ estimate. The Hamiltonian vector field associated with the metric tensor has components with a log-Lipschitz modulus of continuity. Hence, by Osgood's theorem, it generates a uniquely defined geodesic flow, on the interior of $\overline{M}$, and also for geodesics issuing transversally from ∂M. This property will be very important in §4. (We note that in the context of differential geometry Zygmund-type spaces go back to the habilitation thesis of B. Riemann.)

In §3 we obtain a compactness result for families of compact Riemannian manifolds, of dimension n, with boundary, for which there are fixed bounds on the sup norms of Ric_M and $\mathrm{Ric}_{\partial M}$, on the Lipschitz norm of the mean curvature of ∂M, and on the diameter, and fixed lower bounds on the injectivity and boundary injectivity radius. We show that a sequence of such Riemannian manifolds has a subsequence, converging in the C^r-topology, for all $r < 2$, whose limit $(\overline{M}, g)$ has metric tensor in $C^2_*(\overline{M})$.

In §4 we study Gel'fand's inverse boundary problem, recast in the form of an inverse boundary spectral problem. We show that, having boundary spectral data, we can recognize whether a given function $h \in C(\partial M)$ has the form $h(z) = r_x(z) = \mathrm{dist}(x, z)$, for some $x \in \overline{M}$, all $z \in \partial M$, thus recovering the image in $C(\overline{M})$ of $\overline{M}$ under the boundary distance representation. Such a representation, whose use was initiated in [Ku] and [KuL], plays an important role in the uniqueness proof, but for it to work we need to know that geodesics from points in ∂M, pointing normal to the boundary, are uniquely defined. As noted above, this holds when the metric tensor in in $C^2_*(\overline{M})$, and we obtain a uniqueness result in this category. This fits in perfectly with the compactness result of §3, to yield a result on stabilization of this inverse problem.

Section 5 is devoted to the proof of several elliptic regularity results, of an apparently non-standard nature, needed for some of the finer results of §2.

Remark. A number of classes of function spaces arise naturally in our analysis. These include spaces $C^r(\overline{M})$, mentioned above. Here, if $r = k + \sigma,\ k \in \mathbb{Z}^+,\ \sigma \in (0,1)$, $C^r(\overline{M})$ consists of functions whose derivatives of order k satisfy a Hölder condition, with exponent σ. The Zygmund spaces $C^r_*(\overline{M})$ coincide with $C^r(\overline{M})$ for $r \in (0,\infty) \setminus \mathbb{Z}^+$, and form a complex interpolation scale. We also encounter L^p-Sobolev spaces, $H^{s,p}(M)$ and Besov spaces $B^s_{p,p}(\partial M)$, and $\mathrm{bmo}(M)$, the localized space of functions of bounded mean oscillation. Basic material on these spaces can be found in [Tr1] and Chapters 2, 3 of [Tr2], in Chapter 13 of [T1], and in Chapter 1 of [T2].

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2. Boundary Regularity for the Ricci Equation

In this section we establish the key results on local regularity at the boundary of a metric tensor on which there are Ricci curvature bounds and a Lipschitz bound on the mean curvature. Our set-up is the following.

Let $\mathcal{B}$ be a ball about $0 \in \mathbb{R}^n$, $\Omega = \mathcal{B} \cap \{x : x^n > 0\}$. Let $\Sigma = \mathcal{B} \cap \{x : x^n = 0\}$ and set $\overline{\Omega} = \Omega \cup \Sigma$. Let g be a metric tensor on $\overline{\Omega}$, and denote by h its restriction to Σ. We make the following hypotheses:

$$g_{jk} \in H^{1,p}(\Omega), \quad \text{for some} \quad p > n, \tag{2.0.1}$$

$$h_{jk} \in H^{1,2}(\Sigma), \quad 1 \le j,k \le n-1, \tag{2.0.2}$$

$$\mathrm{Ric}^{\Omega} \in L^\infty(\Omega), \tag{2.0.3}$$

$$\mathrm{Ric}^{\Sigma} \in L^\infty(\Sigma), \tag{2.0.4}$$

$$H \in \mathrm{Lip}(\Sigma), \tag{2.0.5}$$

Here H denotes the mean curvature of $\Sigma \subset \overline{\Omega}$, i.e., $H = \mathrm{Tr}\ A/(n-1)$, where A is the Weingarten map, a section of $\mathrm{End}(T\Sigma)$. Our goal is to establish the following result.

Theorem 2.1. *Under the hypotheses (2.0.1)–(2.0.5), given $z \in \Sigma$, there exist local harmonic coordinates on a neighborhood $\overline{U}$ of z in $\overline{\Omega}$ with respect to which*

$$g_{jk} \in C^2_*(\overline{U}). \tag{2.0.6}$$

Here $C^2_*(\overline{U})$ is a Zygmund space, as mentioned in §1. The harmonic coordinates for which (2.0.6) holds are arbitrary coordinates $(u^1, \dots, u^n)$ satisfying $\Delta u^j = 0$ on a chart not intersecting Σ. On a neighborhood of a point in Σ, these coordinates are "boundary harmonic coordinates," which are defined as follows. We require $(u^1, \dots, u^n)$ to be defined and regular of class at least C^1 on a neighborhood of z in $\overline{\Omega}$, and $\Delta u^j = 0$. We require that $v^j = u^j|_\Sigma$ be harmonic on Σ, i.e., annihilated by the Laplace-Beltrami operator of Σ with its induced metric tensor. We require u^n to vanish on Σ, and we require $(u^1, \dots, u^n)$ to map a neighborhood of z in $\overline{\Omega}$ diffeomorphically onto $\overline{\Omega}$.

Let us note that the hypotheses (2.0.1)–(2.0.2) imply that various curvature tensors are well defined. If (g_{jk}) is the $n \times n$ matrix representation of a metric tensor in a coordinate system, (g^{jk}) its matrix inverse, the connection 1-form Γ is given by

$$\Gamma^a{}_{bj} = \frac{1}{2} g^{am}(\partial_j g_{bm} + \partial_b g_{jm} - \partial_m g_{bj}). \tag{2.0.7}$$

The Riemann tensor is then given by

$$\mathcal{R} = d\Gamma + \Gamma \wedge \Gamma. \tag{2.0.8}$$

It is a matrix valued 2-form with components $R^a{}_{bjk}$. We see that

$$\begin{aligned} g_{jk} \in C(\overline{\Omega}) \cap H^{1,2}(\Omega) &\Longrightarrow \Gamma \in L^2(\Omega),\ R^a{}_{bjk} \in H^{-1,2}(\Omega) + L^1(\Omega) \\ &\Longrightarrow \mathrm{Ric}_{bk} \in H^{-1,2}(\Omega) + L^1(\Omega). \end{aligned} \tag{2.0.9}$$

The hypothesis (2.0.1) is stronger than the hypothesis in (2.0.9). It implies $g_{jk} \in C^r(\overline{\Omega})$ for some $r > 0$, so (2.0.9) is applicable both to g_{jk} on Ω and, in view of (2.0.2), to h_{jk} on Σ. Furthermore,

$$g_{jk} \in H^{1,p}(\Omega),\ p > n \Longrightarrow R^a{}_{bjk},\ \mathrm{Ric}_{bk},\ \mathrm{Ric}^j{}_k,\ S \in H^{-1,p}(\Omega), \tag{2.0.10}$$

where $\mathrm{Ric}^j{}_k = g^{jb}\,\mathrm{Ric}_{bk}$ and $S = \mathrm{Ric}^j{}_j$ is the scalar curvature of Ω. We mention parenthetically that one can use the fact that pointwise multiplication gives a map

$$H^{1,2} \times H^{-1,2} \longrightarrow H^{-1,p'}, \quad \forall\ p' < \frac{n}{n-1}, \tag{2.0.11}$$

to obtain

$$g_{jk} \in C(\overline{\Omega}) \cap H^{1,2}(\Omega) \Longrightarrow \mathrm{Ric}^j{}_k,\ S \in H^{-1,p'}(\Omega). \tag{2.0.12}$$

However, we will not make use of (2.0.12) here.

We next consider the implication of (2.0.1) for the Weingarten map associated to $\Sigma \hookrightarrow \overline{\Omega}$. The unit normal N to Σ is a vector field with coefficients

$$N^j = \frac{1}{\sqrt{g^{nn}}}\, g^{jn}\big|_{\Sigma}, \tag{2.0.13}$$

which by the trace theorem belongs to the Besov space $B^{1-1/p}_{p,p}(\Sigma)$. It follows that the Weingarten map has the property

$$A \in B^{-1/p}_{p,p}(\Sigma), \tag{2.0.14}$$

as a consequence of (2.0.1). Thus we have a priori that $H \in B^{-1/p}_{p,p}(\Sigma)$, and the hypothesis (2.0.5) strengthens this condition on H, in a fashion that is natural for the desired conclusion of Theorem 2.1.

Our approach to the proof of Theorem 2.1 is to obtain the result as a regularity result for an elliptic boundary problem. We use the PDE (1.0.1) (the "Ricci equation") for the components of the metric tensor, in boundary harmonic coordinates, and use Dirichlet boundary conditions on some components of g_{jk} and Neumann boundary conditions on complementary components; see (2.1.8) and (2.1.16)–(2.1.17) for a more precise description.

We will approach the proof of Theorem 2.1 in stages. In §2.1 we prove that the conclusion (2.0.6) holds when the hypothesis (2.0.1) is strengthened to $g_{jk} \in C^{1+s}(\overline{\Omega})$, for some $s > 0$. In §2.2 we replace (2.0.1) by the hypothesis that $g_{jk} \in H^{1,p}(\Omega)$ for some $p > 2n$. In §2.3 we prove the full strength version of Theorem 2.1. These stages serve to isolate three rather different types of arguments, each of which is needed to prove Theorem 2.1, but which are perhaps more digestible when presented separately. Section 2.4 has some complementary results on the degree of regularity of the harmonic coordinates mentioned in Theorem 2.1. In section 2.5 we demonstrate the non-branching of geodesics for metric tensors satisfying (2.0.6), including geodesics starting at a boundary point, in a direction transversal to the boundary. We also discuss examples of branching geodesics, for metric tensors only mildly less regular than those of (2.0.6), extending some examples of [Ha].

The version of Theorem 2.1 established in §2.1 is already useful for the results of §§3–4, and the reader particularly interested in §§3–4 could skip §§2.2–2.3, on first reading. However, the hypothesis (2.0.1) has a "natural" quality that we believe makes the additional effort required to work with it worthwhile. The arguments in §§2.2–2.3 require several elliptic regularity results that do not seem to be standard in the literature that we know, and their proofs are collected later, in §5.

§2.1: First regularity result

Here we prove that the conclusion of Theorem 2.1 holds when the hypotheses (2.0.1)–(2.0.2) are strengthened a bit.

Proposition 2.1.1. *In the setting of Theorem 2.1, replace hypotheses (2.0.1)–(2.0.2) by*

$$g_{jk} \in C^{1+s}(\overline{\Omega}), \quad \textit{for some} \quad s \in (0,1), \tag{2.1.1}$$

and retain hypotheses (2.0.3)–(2.0.5). Then the conclusion (2.0.6) holds.

To begin our demonstration, let h denote the metric tensor induced on Σ, with respect to which (2.0.4) holds. By (2.1.1), $h_{jk} \in C^{1+s}(\Sigma)$, so there exist local harmonic coordinates $v^1, \dots, v^{n-1}$ on a neighborhood $\mathcal{O}$ of z in Σ. Now we can find harmonic $u^1, \dots, u^{n-1}$ on a neighborhood of z in $\overline{\Omega}$ such that $u^j = v^j$ on $\mathcal{O}$. Also we can find u^n, harmonic in $\overline{\Omega}$, with $u^n|_\Sigma = 0$ and arrange that $\partial_{x^n} u^n(z) \neq 0$. We will have

$$u^1, \dots, u^n \in C^{2+s}(\overline{U}). \tag{2.1.2}$$

We have $du^1(z), \dots, du^n(z)$ linearly independent, so, after perhaps further shrinking $\overline{U}$ we have a harmonic coordinate chart on $\overline{U}$, a set we relabel as $\overline{\Omega}$. As mentioned below the statement of Theorem 2.1, this is what we call a set of *boundary harmonic coordinates*. In these new coordinates, (2.1.1) and (2.0.3)–(2.0.5) are preserved.

Now in harmonic coordinates the metric tensor satisfies the elliptic PDE

$$\Delta g_{\ell m} = F_{\ell m}, \tag{2.1.3}$$

where Δ acts componentwise on $g_{\ell m}$, as

$$\Delta u = g^{-1/2} \partial_j (g^{1/2} g^{jk} \partial_k u), \quad g = \det(g_{jk}), \tag{2.1.4}$$

and

$$F_{\ell m} = B_{\ell m}(g, \nabla g) - 2 \operatorname{Ric}^{\Omega}_{\ell m}. \tag{2.1.5}$$

Here $B_{\ell m}$ is a quadratic form in ∇g with coefficients that are rational functions of g_{jk}. Thus, from (2.1.1) and (2.0.3) we have

$$F_{\ell m} \in L^\infty(\Omega), \tag{2.1.6}$$

and the coefficients of Δ have the same degree of regularity as g_{jk} in (2.1.1).

Now, if $j, k \leq n-1$, then well known local regularity results on Σ following from (2.0.4) give

$$g_{jk}\big|_\Sigma = h_{jk} \in H^{2,p}(\Sigma), \quad \forall\, p < \infty, \tag{2.1.7}$$

but in fact there is the following refinement of (2.1.7), established in Proposition III.10.2 of [T2]:

$$g_{jk}\big|_\Sigma = h_{jk} \in \mathfrak{h}^{2,\infty}, \quad 1 \leq j, k \leq n-1. \tag{2.1.8}$$

Here $\mathfrak{h}^{2,\infty}$ denotes the bmo-Sobolev space of functions whose derivatives of order ≤ 2 belong to bmo, the localized space of functions of bounded mean oscillation. We establish the following (after perhaps shrinking $\overline{\Omega}$ to a smaller neighborhood of z).

Lemma 2.1.2. *Under our working hypotheses we have, in the harmonic coordinate system* $(u^1,\dots,u^n)$,

$$g_{jk}\in C^2_*(\overline{\Omega}),\quad 1\le j,k\le n-1. \tag{2.1.9}$$

Proof. First, extend $F_{\ell m}$ by 0 on $\mathcal{B}\setminus\Omega$ and solve $\widetilde{\Delta} w_{\ell m}=F_{\ell m}$ on a neighborhood of 0 in $\mathcal{B}$, where we obtain $\widetilde{\Delta}$ in the form (2.1.4) with g^{jk} extended across Σ to $\tilde{g}^{jk}\in C^{1+s}(\mathcal{B})$. Local elliptic regularity results imply

$$w_{\ell m}\in \mathfrak{h}^{2,\infty}(\mathcal{B})\subset C^2_*(\mathcal{B}). \tag{2.1.10}$$

It follows that $w_{\ell m}|_\Sigma\in C^2_*(\Sigma)$ and, via (2.1.8),

$$g_{jk}-w_{jk}\big|_\Sigma=b_{jk}\in C^2_*(\Sigma),\quad j,k\le n-1, \tag{2.1.11}$$

while

$$\Delta(g_{jk}-w_{jk})=0\ \text{ on }\ \Omega. \tag{2.1.12}$$

Given our assumed regularity of the coefficients of Δ, standard Schauder results give

$$b_{jk}\in C^r(\Sigma)\Longrightarrow g_{jk}-w_{jk}\in C^r(\overline{\Omega}),\quad 1<r<2,\ 2<r<2+s. \tag{2.1.13}$$

Actually, the case $1<r<2$ is perhaps not so classical, but see [Mo1], Theorem 7.3 or [GT], Corollaries 8.35–8.36. From here, an interpolation argument gives

$$b_{jk}\in C^2_*(\Sigma)\Longrightarrow g_{jk}-w_{jk}\in C^2_*(\overline{\Omega}). \tag{2.1.14}$$

See [T1], Chapter 13, §8, particularly (8.37), for interpolation in this context. This establishes (2.1.9).

To continue, following [An2], we switch over to PDE for $g^{\ell m}$. Parallel to (2.1.3), we have

$$\Delta g^{\ell m}=B^{\ell m}(g,\nabla g)+2(\mathrm{Ric}^\Omega)^{\ell m}=F^{\ell m}, \tag{2.1.15}$$

and (2.1.1) and (2.0.3) give $F^{\ell m}\in L^\infty(\Omega)$. We take $m=n$ and proceed to derive Neumann-type boundary conditions for the components $g^{\ell n},\ 1\le\ell\le n$. In fact, as we will show,

$$Ng^{nn}=-2(n-1)Hg^{nn},\quad \text{on }\ \Sigma, \tag{2.1.16}$$

and, for $1\le\ell\le n-1$,

$$Ng^{\ell n}=-(n-1)Hg^{\ell n}+\frac{1}{2}\frac{1}{\sqrt{g^{nn}}}\,g^{\ell k}\partial_k g^{nn},\quad \text{on }\ \Sigma. \tag{2.1.17}$$

Here H is the mean curvature of Σ, which we assume satisfies (2.0.5), and N is the unit normal field to Σ, pointing inside Ω.

To compute (2.1.16)–(2.1.17), we use

$$g^{\ell m} = \langle \nabla u^\ell, \nabla u^m \rangle, \quad N = \frac{\nabla u^n}{|\nabla u^n|} = \frac{1}{\sqrt{g^{nn}}} \nabla u^n, \tag{2.1.18}$$

and

$$N g^{\ell n} = \langle \nabla_N \nabla u^\ell, \nabla u^n \rangle + \langle \nabla u^\ell, \nabla_N \nabla u^n \rangle. \tag{2.1.19}$$

We also use the fact that u^ℓ is harmonic on Ω and $u^\ell\big|_\Sigma = v^\ell$ is harmonic on Σ (0 if $\ell = n$).

Note that if $\{e_j : 1 \le j \le n-1\}$ is an orthonormal frame on Σ and X a vector field on $\overline{\Omega}$ (say both having coefficients in $C^1(\overline{\Omega})$) then

$$\operatorname{div} X\big|_\Sigma = \sum_{j=1}^{n-1} \langle \nabla_{e_j} X, e_j \rangle + \langle \nabla_N X, N \rangle. \tag{2.1.20}$$

In particular, for $X_\ell = \nabla u^\ell$, we have $\operatorname{div} X_\ell = \Delta u^\ell = 0$, so the first term on the right side of (2.1.19) is equal to $-\sqrt{g^{nn}}$ times

$$\sum_{j=1}^{n-1} \langle \nabla_{e_j} X_\ell, e_j \rangle. \tag{2.1.21}$$

Let us set

$$X_\ell = X_\ell^N + X_\ell^T, \quad X_\ell^N = \langle X_\ell, N \rangle N = \varphi N, \quad X_\ell^T = \nabla v^\ell, \tag{2.1.22}$$

with X_ℓ^T tangent to $\partial\Omega$ and $\varphi = g^{ln}/\sqrt{g^{nn}}$. Since $\sum_j \langle \nabla_{e_j} \nabla v^\ell, e_j \rangle = \operatorname{div} \nabla v^\ell = \Delta v^\ell = 0$, we have (2.1.21) equal to

$$\begin{aligned} \sum_{j=1}^{n-1} \langle \nabla_{e_j} (\varphi N), e_j \rangle &= \varphi \sum_j \langle \nabla_{e_j} N, e_j \rangle \\ &= \varphi \sum \langle A e_j, e_j \rangle \\ &= (n-1) H \, \frac{g^{\ell n}}{\sqrt{g^{nn}}}, \end{aligned} \tag{2.1.23}$$

so the first term on the right side of (2.1.19) is equal to $-(n-1)Hg^{\ell n}$. The case $\ell = n$ gives (2.1.16), since the two summands in (2.1.19) are then the same.

To continue when $\ell \neq n$, we note that

$$\langle \nabla u^\ell, \nabla_N \nabla u^n \rangle = \langle N, \nabla_{X_\ell} \nabla u^n \rangle, \tag{2.1.24}$$

with $X_\ell = \nabla u^\ell$. In fact, generally a 1-form η satisfies

$$d\eta(X,Y) = \langle \nabla_X \eta, Y \rangle - \langle \nabla_Y \eta, X \rangle,$$

and applying this to $\eta = du^n$ and $X = X_\ell$, $Y = X_n$ gives (2.1.24). Now

$$X_\ell g^{nn} = 2\langle \nabla_{X_\ell} \nabla u^n, \nabla u^n \rangle,$$

so (2.1.24) is equal to

$$\frac{1}{2\sqrt{g^{nn}}} \langle X_\ell, \nabla g^{nn} \rangle = \frac{1}{2\sqrt{g^{nn}}} g^{\ell k} \partial_k g^{nn}, \tag{2.1.25}$$

which gives (2.1.17).

Having (2.1.15)–(2.1.17), we can establish further regularity of the functions $g^{\ell n}$.

Lemma 2.1.3. *In the harmonic coordinate system $(u^1, \dots, u^n)$, we have*

$$g^{\ell n} \in C^2_*(\overline{\Omega}), \quad 1 \leq \ell \leq n. \tag{2.1.26}$$

Proof. As in Lemma 2.1.2, we extend g^{jk} to $\tilde{g}^{jk} \in C^{1+s}(\mathcal{B})$ and extend the right side of (2.1.15) by 0 and produce a solution

$$w^{\ell n} \in \mathfrak{h}^{2,\infty}(\mathcal{B}) \subset C^2_*(\mathcal{B}) \tag{2.1.27}$$

to $\Delta w^{\ell m} = F^{\ell m}$. Then $g^{\ell n} - w^{\ell n}$ satisfies

$$\Delta(g^{\ell n} - w^{\ell n}) = 0, \quad \text{on } \Omega, \tag{2.1.28}$$

and we have

$$N(g^{nn} - w^{nn})\big|_\Sigma \in C^1_*(\Sigma), \tag{2.1.29}$$

by (2.1.16) and (2.1.27), plus the regularity $N \in C^{1+s}(\Sigma)$. As in Lemma 2.1.2, we can apply an interpolation argument to Schauder-type estimates (see §5.3) and get

$$g^{nn} - w^{nn} \in C^2_*(\overline{\Omega}), \tag{2.1.30}$$

and hence (2.1.26) holds for $\ell = n$. Having this, we get from (2.1.17) that

$$N(g^{\ell n} - w^{\ell n})\big|_\Sigma \in C^1_*(\Sigma), \tag{2.1.31}$$

which gives $g^{\ell n} - w^{\ell n} \in C^2_*(\overline{\Omega})$, so (2.1.26) holds for all ℓ.

The final step is to verify that Lemmas 2.1.2 and 2.1.3 yield regularity of $g_{\ell n}$.

Lemma 2.1.4. *In the setting of Lemmas 2.1.2–2.1.3,*

$$g_{n\ell} = g_{\ell n} \in C^2_*(\overline{\Omega}). \tag{2.1.32}$$

Proof. Let $g = \det(g_{jk})$ and let $A_{\ell m}$ be the determinant of the $(n-1)\times(n-1)$ matrix formed by omitting column ℓ and row m from the matrix (g_{jk}). Then

$$g^{jk} = \frac{(-1)^{j+k}}{g} A_{jk}. \tag{2.1.33}$$

By Lemma 2.1.2, $A_{nn} \in C^2_*(\overline{\Omega})$. Applying Lemma 2.1.3 to g^{nn} (which is > 0) we have

$$g = A_{nn}/g^{nn} \in C^2_*(\overline{\Omega}). \tag{2.1.34}$$

Then it follows that

$$A_{\ell n} = A_{n\ell} = (-1)^{n+\ell} g\, g^{n\ell} \in C^2_*(\overline{\Omega}), \quad 1 \le \ell \le n. \tag{2.1.35}$$

Another way of putting this is the following. Let

$$h_{jk} = g_{jk}, \quad 1 \le j,k \le n-1; \quad h = \det(h_{jk}) \tag{2.1.36}$$

and (h^{jk}) be the matrix inverse to (h_{jk}). Then $A_{\ell n}, \ell \le n-1$, can be written in the form

$$A_{\ell n} = (-1)^{n-1+\ell} g_{jn}\, h h^{j\ell}. \tag{2.1.37}$$

Now the regularity and positive-definiteness of $(h_{jk})_{1\le j,k\le n-1}$ applied to (2.1.37) yield

$$(g_{n1}, \dots, g_{n\,n-1}) \in C^2_*(\overline{\Omega}). \tag{2.1.38}$$

Finally, the identity

$$g_{jn} g^{jn} = 1, \tag{2.1.39}$$

the regularity of g^{jn} in (2.1.26) and of g_{jn} for $j \le n-1$ in (2.1.38), plus the fact that $g^{nn} > 0$, yield

$$g_{nn} \in C^2_*(\overline{\Omega}), \tag{2.1.40}$$

proving the lemma, and completing the proof of Proposition 2.1.1.

§2.2: First improvement

In this section and the next we obtain regularity with a weaker a priori hypothesis than (2.1.1). As we noted above, the results of §2.1 suffice for the applications in §3, but these improvements are quite natural (if not trivial to implement) and surely have the potential for applications elsewhere.

Here we do strengthen the hypothesis (2.0.1) to some degree. Namely we assume:

$$g_{jk} \in H^{1,p}(\Omega), \quad p > 2n, \tag{2.2.1}$$

We retain hypothesis (2.0.2), i.e.,

$$h_{jk} \in H^{1,2}(\Sigma), \quad 1 \le j,k \le n-1. \tag{2.2.2}$$

Here Ω and Σ are as in §1 and $h_{jk} = g_{jk}|_\Sigma$, for $1 \le j,k \le n-1$. Note that (2.2.1) implies $g_{jk} \in C^r(\overline{\Omega})$ with $r = 1 - n/p > 0$, and hence $h_{jk} \in C^r(\Sigma)$.

Our strategy is to show that the hypotheses (2.2.1)–(2.2.2) together with (2.0.3)–(2.0.5) imply $g_{jk} \in C^{1+s}(\overline{\Omega})$ for some $s > 0$, so Proposition 2.1.1 applies. In fact, we will expand the scope of the investigation here, and establish this conclusion under the following hypotheses, which are weaker than (2.0.3)–(2.0.5):

$$\mathrm{Ric}^\Omega \in L^{p_1}(\Omega), \quad p_1 > n, \tag{2.2.3}$$

$$\mathrm{Ric}^\Sigma \in L^{p_2}(\Sigma), \quad p_2 > n-1, \tag{2.2.4}$$

$$H \in C^s(\Sigma), \quad s > 0. \tag{2.2.5}$$

Our goal in this section is to prove:

Proposition 2.2.1. *Under the hypotheses (2.2.1)–(2.2.5), given $z \in \Sigma$, there exist local harmonic coordinates on a neighborhood $\overline{U}$ of z in $\overline{\Omega}$ with respect to which*

$$g_{jk} \in C^{1+s}(\overline{U}), \tag{2.2.6}$$

for some $s > 0$.

As before, we begin by constructing local harmonic coordinates $v^1, \dots, v^{n-1}$ on a neighborhood $\mathcal{O}$ of z in Σ. Knowing that $h_{jk} \in C^r(\Sigma)$, we can do this, and making use also of hypothesis (2.2.2) we have

$$v^j \in C^{1+r}(\mathcal{O}) \cap H^{2,2}(\mathcal{O}), \tag{2.2.7}$$

by Proposition 9.4 in Chapter III of [T2]. It follows that (2.2.2) persists in this new coordinate system. As a consequence of the fact that $h_{jk} \in B^{1-1/p}_{p,p}(\Sigma)$, we also have

$$v^j \in B^{2-1/p}_{p,p}(\mathcal{O}). \tag{2.2.8}$$

This result is established in §5.1.

Next we find harmonic functions $u^1, \dots, u^{n-1}$ on a neighborhood $\overline{U}$ of z in $\overline{\Omega}$ such that $u^j = v^j$ on $\mathcal{O}$ and we find u^n, harmonic in $\overline{\Omega}$, with $u^n|_\Sigma = 0$ and arrange that $\partial_{x^n} u^n(z) \neq 0$. Given (2.2.7)–(2.2.8) and the hypothesis (2.2.1), we claim that

$$u^1, \dots, u^n \in C^{1+r}(\overline{U}) \cap H^{2,p}(U). \tag{2.2.9}$$

The fact that $u^j \in C^{1+r}(\overline{U})$ follows from Corollaries 8.35–8.36 of [GT], or [Mo1], Theorem 7.3. The fact that $u^j \in H^{2,p}(U)$ is established in §5.2. We have $du^1(z), \dots, du^n(z)$ linearly independent, so after perhaps further shrinking $\overline{U}$ we have a harmonic coordinate chart on $\overline{U}$, which we relabel $\overline{\Omega}$. In these new coordinates, (2.2.1)–(2.2.5) are preserved.

In fact, now that we have switched to harmonic coordinates, we can improve (2.2.2), making use of (2.2.4). It follows from Proposition 10.1 in Chapter III of [T2] that

$$h_{jk} \in H^{2,p_2}(\Sigma), \quad 1 \leq j, k \leq n-1. \tag{2.2.10}$$

In particular,

$$h_{jk} \in C^{1+\sigma}(s), \quad \text{for some} \quad s > 0. \tag{2.2.11}$$

We may as well suppose $s \in (0, r)$. Now we can prove:

Lemma 2.2.2. *In the harmonic coordinate system* $(u^1, \dots, u^n)$,

$$g_{\ell m} \in C^{1+s}(\overline{\Omega}), \quad 1 \leq \ell, m \leq n-1. \tag{2.2.12}$$

Proof. We know $g_{\ell m}$ solves the Dirichlet problem

$$\Delta g_{\ell m} = F_{\ell m}, \quad g_{\ell m}\big|_\Sigma = h_{\ell m}, \quad 1 \leq \ell, m \leq n-1, \tag{2.2.13}$$

where

$$F_{\ell m} = B_{\ell m}(g, \nabla g) - 2\,\mathrm{Ric}^{\Omega}_{\ell m}. \tag{2.2.14}$$

From (2.2.1) and (2.2.3) we have

$$F_{\ell m} \in L^{q_1}(\Omega), \quad q_1 = \min(p/2, p_1) > n, \tag{2.2.15}$$

and the coefficients $a^{jk} = g^{1/2} g^{jk}$ of Δ are known to belong to $H^{1,p}(\Omega) \subset C^r(\overline{\Omega})$.

The next step in the proof is by now familiar. Extend $F_{\ell m}$ by 0 on $\mathcal{B} \setminus \Omega$ and solve $\Delta v_{\ell m} = F_{\ell m}$ on a neighborhood $\mathcal{V}$ of z in $\overline{\Omega}$ with $v_{\ell m} \in C^{1+s}(\mathcal{V})$, $s > 0$. Then $w_{\ell m} = g_{\ell m} - v_{\ell m}$ solves

$$\Delta w_{\ell m} = 0 \ \text{ on } \ \Omega, \quad w_{\ell m}\big|_\Sigma = h_{\ell m} - v_{\ell m}\big|_\Sigma \in C^{1+s}(\Sigma), \tag{2.2.16}$$

and the previously cited results of [Mo1] and [GT] yield $w_{\ell m} \in C^{1+s}(\overline{\Omega})$, hence (2.2.12).

It remains to show that

$$g_{\ell n} = g_{n\ell} \in C^{1+s}(\overline{\Omega}), \quad 1 \le \ell \le n, \tag{2.2.17}$$

with $s > 0$. In fact, if we show that

$$g^{\ell n} = g^{n\ell} \in C^{1+s}(\overline{\Omega}), \quad 1 \le \ell \le n, \tag{2.2.18}$$

then an argument parallel to the proof of Lemma 2.1.4 yields (2.2.17).

As before, we have

$$\Delta g^{\ell n} = B^{\ell n}(g, \nabla g) + 2(\mathrm{Ric}^{\Omega})^{\ell n} = F^{\ell n}. \tag{2.2.19}$$

As in (2.2.15), we have

$$F^{\ell n} \in L^{q_1}(\Omega), \quad q_1 = \min(p/2, p_1) > n. \tag{2.2.20}$$

However, this time it is not so straightforward to produce the Neumann-type boundary conditions (2.1.16)–(2.1.17).

Consider (2.1.16). The right side is well defined; we have $Hg^{nn}|_{\Sigma} \in C^s(\Sigma)$, for some $s > 0$. As for N, the unit normal field to Σ is also Hölder continuous of class C^r. But applying N to $g^{nn} \in H^{1,p}(\Omega)$ does not yield an object that can be evaluated on Σ. One has the same problem with the left side of (2.1.17), and the right side of (2.1.17) is also problematic.

Our next goal is to show that a weak formulation of the Neumann boundary condition is applicable. Generally, the weak formulation of

$$\Delta w = F, \quad Nw\big|_{\Sigma} = G \tag{2.2.21}$$

is that for all test functions ψ, i.e., all $\psi \in C^\infty(\overline{\Omega})$ with compact support (intersecting Σ but not the rest of $\partial\Omega$),

$$\int_{\Omega} \langle \nabla w, \nabla \psi \rangle \, dV = -\int_{\Omega} F\psi \, dV - \int_{\Sigma} G\psi \, dS. \tag{2.2.22}$$

Here dV is the volume element on Ω and dS the area element on Σ, both determined by the metric tensor in the usual fashion. Note that the left side of (2.2.22) is well defined for all test functions ψ whenever $\nabla w \in L^1(\Omega)$ and the right side of (2.2.22) is well defined whenever $F \in L^1(\Omega)$ and $G \in L^1(\Sigma)$.

Lemma 2.2.3. *The function $w = g^{nn}$ satisfies (2.2.22), with*

$$F = F^{nn} \in L^{q_1}(\Omega), \quad q_1 > n, \qquad G = -2(n-1)Hg^{nn}\big|_\Sigma \in C^r(\Sigma), \quad r > 0. \tag{2.2.23}$$

Hence the result that $g^{nn} \in H^{1,p}(\Omega)$ for some $p > 2n$ is improved to

$$g^{nn} \in C^{1+s}(\overline{\Omega}), \quad s > 0. \tag{2.2.24}$$

That (2.2.22) holds in this context seems a natural generalization of (2.1.16), but the proof, given below, requires some work. Given that (2.2.22) holds, the regularity result (2.2.24) follows from the results §5.3.

Once Lemma 2.2.3 is established, we see that the right side of (2.1.17) is well defined, and we can formulate:

Lemma 2.2.4. *For $1 \le \ell \le n-1$, the function $w = g^{\ell n}$ satisfies (2.2.22), with*

$$F = F^{\ell n} \in L^{q_1}(\Omega), \quad q_1 > n, \qquad G = -(n-1)Hg^{\ell n}\big|_\Sigma + \frac{1}{2}\frac{1}{\sqrt{g^{nn}}}\, g^{\ell k}\partial_k g^{nn}\big|_\Sigma \in C^r(\Sigma), \quad r > 0. \tag{2.2.25}$$

Hence the result $g^{\ell n} \in H^{1,p}(\Omega)$, for some $p > 2n$, is improved to

$$g^{\ell n} \in C^{1+s}(\overline{\Omega}), \quad s > 0. \tag{2.2.26}$$

To set up the proof of Lemmas 2.2.3–2.3.4, let $\Omega_c = \{x \in \Omega : u^n(x) > c\} \subset \Omega$, for small $c > 0$, and let $\Sigma_c = \{x \in \Omega : u^n(x) = c\}$. Since $u^n \in C^{1+s}$ the surfaces Σ_c are uniformly C^{1+s}-smooth. Parallel to (2.2.10), we have from (2.2.3) that

$$g_{jk} \in H^{2,p_1}_{\mathrm{loc}}(\Omega), \tag{2.2.27}$$

so $Ng^{\ell n}|_{\Sigma_c}$ is well defined for small $c > 0$. Calculations parallel to (2.1.18)–(2.1.23) give

$$Ng^{nn}\big|_{\Sigma_c} = -2(n-1)H_c g^{nn}, \tag{2.2.28}$$

where H_c denotes the mean curvature of Σ_c, i.e., $(n-1)H_c = \operatorname{Tr} A_c$, where A_c denotes the Weingarten map of Σ_c. Note that, for X, Y tangent to Σ_c,

$$\langle A_c X, Y\rangle = \langle \nabla_X N, Y\rangle, \quad N = \frac{\nabla u^n}{|\nabla u^n|}, \quad (\nabla u^n)^j = g^{jn}. \tag{2.2.29}$$

The assumption (2.2.1) implies

$$N \in H^{1,p}(\Omega), \quad N\big|_{\Sigma_c} \in B^{1-1/p}_{p,p}(\Sigma_c), \quad A_c \in B^{-1/p}_{p,p}(\Sigma_c). \tag{2.2.30}$$

For a fixed $c > 0$, from (2.2.27), we have $N|_{\Sigma_c} \in B^{2-1/p_1}_{p_1,p_1}(\Sigma_c)$, hence $A_c \in B^{1-1/p_1}_{p_1,p_1}(\Sigma_c)$, but (2.2.30) holds uniformly as $c \to 0$, and if Σ_c is identified with Σ via the $(u^1, \cdots, u^{n-1})$-coordinates, we have A_c continuous in c as $c \to 0$, in the space $B^{-1/p}_{p,p}(\Sigma)$.

Now pick a test function ψ. From (2.2.28) we have, for each (small) $c > 0$,

$$\int_{\Omega_c} \langle \nabla g^{nn}, \nabla \psi \rangle \, dV = -\int_{\Omega_c} F^{nn} \psi \, dV - \int_{\Sigma_c} G_c \psi \, dS, \tag{2.2.31}$$

where F^{nn}, G_c are as in (2.2.23) with Σ_c instead of Σ. We let $c \to 0$. Since we already have $\nabla g^{nn} \in L^2(\Omega)$, the left side of (2.2.31) converges to

$$\int_{\Omega} \langle \nabla g^{nn}, \nabla \psi \rangle \, dV, \tag{2.2.32}$$

and the first term on the right side of (2.2.31) converges to

$$-\int_{\Omega} F^{nn} \psi \, dV. \tag{2.2.33}$$

Finally, from (2.2.30) we have that $H_c \to H_0$ in $B^{-1/p}_{p,p}(\Sigma)$, with $p > 2n$, which, given our knowledge at this point that $g^{jk} \in H^{1,p}(\Omega)$, is more than enough to imply that the last term in (2.2.31) converges to

$$-\int_{\Sigma} G \psi \, dS. \tag{2.2.34}$$

This proves Lemma 2.2.3.

To proceed, we have, for $1 \le \ell \le n-1$, $c > 0$, by a calculation parallel to (2.1.18)–(2.1.25),

$$\begin{aligned} N g^{\ell n}\big|_{\Sigma_c} = & -(n-1) H_c g^{\ell n} - \sqrt{g^{nn}} \Delta_{\Sigma_c} u^\ell \big|_{\Sigma_c} \\ & + \frac{1}{2} \frac{1}{\sqrt{g^{nn}}} \, g^{\ell k} \partial_k g^{nn} \big|_{\Sigma_c}. \end{aligned} \tag{2.2.35}$$

Here Δ_{Σ_c} is the Laplace operator on the surface Σ_c, with its induced Riemannian metric tensor. Hence, given a test function ψ, we have, for each $c > 0$,

$$\int_{\Omega_c} \langle \nabla g^{\ell n}, \nabla \psi \rangle \, dV = -\int_{\Omega_c} F^{\ell n} \psi \, dV - \int_{\Sigma_c} G \psi \, dS, \tag{2.2.36}$$

with $F^{\ell n}$ as in (2.2.25) and G_c given by the right side of (2.2.35). Again we let $c \to 0$, and since we know $\nabla g^{\ell n} \in L^2(\Omega)$, the left side of (2.2.36) converges to

$$\int_\Omega \langle \nabla g^{\ell n}, \nabla \psi \rangle \, dV. \tag{2.2.37}$$

Again the first term on the right side of (2.2.36) converges to

$$-\int_\Omega F^{\ell n} \psi \, dV. \tag{2.2.38}$$

The last term in (2.2.36) is equal to

$$\begin{aligned} &(n-1) \int_{\Sigma_c} H_c g^{\ell n} \psi \, dS + \int_{\Sigma_c} \sqrt{g^{nn}} \big(\Delta_{\Sigma_c} u^\ell\big|_{\Sigma_c}\big) \psi \, dS \\ &- \frac{1}{2} \int_{\Sigma_c} \frac{1}{\sqrt{g^{nn}}} g^{\ell k} \partial_k g^{nn} \psi \, dS. \end{aligned} \tag{2.2.39}$$

As $c \to 0$, the first term in (2.2.39) converges to

$$(n-1) \int_\Sigma H g^{\ell n} \psi \, dS, \tag{2.2.40}$$

by the same arguments as above (via (2.2.30)). Next (making use of (2.2.1)) we have, for some $p > 2n$,

$$u^\ell \in H^{2,p}(\Omega), \quad u^\ell\big|_{\Sigma_c} \in B^{2-1/p}_{p,p}(\Sigma_c), \quad \Delta_{\Sigma_c} u^\ell\big|_{\Sigma_c} \in B^{-1/p}_{p,p}(\Sigma_c), \tag{2.2.41}$$

with uniform bounds and convergence as $c \to 0$, so the second term in (2.2.39) converges to 0 as $c \to 0$.

Finally, since we already have $g^{nn} \in C^{1+s}(\overline{\Omega})$, plus $g^{\ell k} \in H^{1,p}(\Omega)$, the convergence of the last term in (2.2.39) as $c \to 0$ follows, so Lemma 2.2.4 is proven.

§2.3: Proof of Theorem 2.1

In this section we finish the proof of Theorem 2.1 assuming that $n < p \le 2n$. Going further, we extend Proposition 2.2.1 as follows.

Theorem 2.3.1. *Replace hypothesis (2.2.1) by*

$$g_{jk} \in H^{1,p}(\Omega), \quad n < p \le 2n, \tag{2.3.1}$$

and retain hypotheses (2.2.2)–(2.2.5). Then, given $z \in \Sigma$, there exist local harmonic coordinates on a neighborhood $\overline{U}$ of z in $\overline{\Omega}$ with respect to which

$$g_{jk} \in C^{1+s}(\overline{U}) \tag{2.3.2}$$

for some $s > 0$.

To begin, we have local harmonic coordinates $v^1, \dots, v^{n-1}$ on a neighborhood $\mathcal{O}$ of z in Σ satisfying (2.2.7)–(2.2.8) and then local harmonic coordinates $u^1, \dots, u^n$ as in §2.2, satisfying (2.2.9), and in these new coordinates (2.3.1) and (2.2.2)–(2.2.5) are preserved. We also continue to have (2.2.10)–(2.2.11). We next establish the variant of Lemma 2.2.2 that holds in this context.

Lemma 2.3.2. *In the harmonic coordinate system $(u^1, \dots, u^n)$, we have*

$$g_{\ell m} \in H^{1,r_1}(\Omega), \quad 1 \le \ell, m \le n-1, \tag{2.3.3}$$

with

$$r_1 = \frac{q_1}{1 - q_1/n}, \quad \textit{for any } \frac{n}{2} < q_1 < \frac{p}{2}. \tag{2.3.4}$$

Proof. We continue to have (2.2.13)–(2.2.15), except that in (2.2.15) no longer have $q_1 > n$ (rather $n/2 < q_1 < n$). As for the regularity of $g_{\ell m}|_\Sigma$, we continue to have (2.2.11). Extend $F_{\ell m}$ by 0 on $\mathcal{B} \setminus \Omega$ and solve $\Delta v_{\ell m} = F_{\ell m}$ on a neighborhood $\mathcal{V}$ of z in $\mathcal{B}$. This time we can say $v_{\ell m} \in H^{1,r_1}(\mathcal{V})$, with r_1 as in (2.3.4). Since $L^{q_1} \subset H^{-1,r_1}$, this follows from Proposition 1.10 in [T2], Chapter III. Then $w_{\ell m} = g_{\ell m} - v_{\ell m}$ satisfies (2.2.16), so as before we have $w_{\ell m} \in H^{1,r_1}(\overline{\Omega})$, and this gives (2.3.3). Note that

$$q_1 > \frac{n}{2} \Longrightarrow r_1 > 2q_1. \tag{2.3.5}$$

To proceed, we note that the results (2.2.27)–(2.2.34) hold under our relaxed hypotheses on p, so (2.2.23) holds, except we have q_1 as in (2.3.4) instead of $q_1 > n$. With this, we can prove the following.

Lemma 2.3.3. *For all $\varepsilon > 0$, we have*

$$g^{nn} \in H^{1,r_1-\varepsilon}(\Omega). \tag{2.3.6}$$

Proof. We first extend F^{nn} by 0 on $\mathcal{B} \setminus \Omega$ and solve

$$\Delta g_0^{nn} = F^{nn}, \quad g_0^{nn} \in H^{1,r_1}(\mathcal{V}), \tag{2.3.7}$$

on a neighborhood $\mathcal{V}$ of z in $\mathcal{B}$, as in the proof of Lemma 2.3.2. In fact, by Proposition 5.2.2, we have

$$g_0^{nn} \in H^{2,q_1}(\mathcal{V}), \tag{2.3.8}$$

since $1/p + 1/p \le 1/q_1$ in this case. Of course (2.3.8) implies (2.3.7), but it also implies

$$\nabla g_0^{nn}\big|_\Sigma \in B^{1-1/q_1}_{q_1,q_1}(\Sigma) \subset L^{s_1-\varepsilon}(\Sigma), \quad s_1 = \frac{(n-1)q_1}{n-q_1}, \tag{2.3.9}$$

the latter inclusion holding for all $\varepsilon > 0$.

Hence $g^{nn} = g_0^{nn} + g_1^{nn}$ where

$$\Delta g_1^{nn} = 0, \quad N g_1^{nn} = G - N g_0^{nn}. \tag{2.3.10}$$

Here $G = -2(n-1)Hg^{nn}|_\Sigma$, as in (2.2.23), so we know

$$G \in C^r(\Sigma), \quad N g_0^{nn} \in L^{s_1-\varepsilon}(\Sigma), \tag{2.3.11}$$

for all $\varepsilon > 0$. It follows from Proposition 5.5.2 that

$$g_1^{nn} \in H^{1,r_1-\varepsilon}(\Omega), \quad \forall\, \varepsilon > 0, \tag{2.3.12}$$

which gives (2.3.6). For use below we also record the non-tangential maximal function estimate

$$(\nabla g_1^{nn})^* \in L^{s_1-\varepsilon}(\Sigma), \tag{2.3.13}$$

also established in §5.5. The meaning of the left side of (2.3.13) is the following. First, we have ∇g_1^{nn} continuous on the *interior* of Ω. Next, for $x \in \Sigma$,

$$(\nabla g_1^{nn})^*(x) = \sup_{y \in \Gamma_x} |\nabla g_1^{nn}(y)|,$$

where $\Gamma_x = \{y \in \Omega : d(y,x) \le 2\, d(y,\Sigma)\}$.

Lemma 2.3.4. *For $1 \le \ell \le n-1$, and for all $\varepsilon > 0$, we have*

$$g^{\ell n} \in H^{1,r_1-\varepsilon}(\Omega). \tag{2.3.14}$$

Proof. Using (2.3.8) and (2.3.13), we can extend the analysis in (2.2.35)–(2.2.41), to conclude that $g^{\ell n}$ is a weak solution to

$$\Delta g^{\ell n} = F^{\ell n}, \quad N g^{\ell n} = G, \tag{2.3.15}$$

with G as in (2.2.25), i.e.,

$$G=-(n-1)Hg^{\ell n}\big|_{\Sigma}+\frac{1}{2}\frac{1}{\sqrt{g^{nn}}}g^{\ell k}\partial_k g^{nn}\big|_{\Sigma}. \tag{2.3.16}$$

From what we know so far, we have

$$G\in L^{s_1-\varepsilon}(\Sigma),\quad \forall\,\varepsilon>0. \tag{2.3.17}$$

As in the proof of Lemma 2.3.3, we can then write

$$g^{\ell n}=g_0^{\ell n}+g_1^{\ell n},\quad g_0^{\ell n}\in H^{2,q_1}(\Omega), \tag{2.3.18}$$

with

$$\Delta g_1^{\ell n}=0,\quad N g_1^{\ell n}=G-N g_0^{\ell n}\in L^{s_1-\varepsilon}(\Sigma), \tag{2.3.19}$$

which by an analysis parallel to that of (2.3.10)–(2.3.11) gives

$$g_1^{\ell n}\in H^{1,r_1-\varepsilon}(\Omega) \tag{2.3.20}$$

and proves (2.3.14).

Now an argument parallel to the proof of Lemma 2.1.4 gives

$$g_{jk}\in H^{1,r_1-\varepsilon}(\Omega),\quad \forall\,\varepsilon>0, \tag{2.3.21}$$

for all $j,k\le n$. Now, for ε small and q_1 close to $p/2$,

$$r_1-\varepsilon>p, \tag{2.3.22}$$

an improvement over the hypothesis (2.3.1), as long as $p\le 2n$. Thus replacing (2.3.1) by (2.3.22) and iterating this argument a finite number of times, we establish that actually (2.2.1) holds.

This proves Theorem 2.3.1. It also reduces Theorem 2.1 to Proposition 2.1.1, and hence proves Theorem 2.1.

Remark. Theorem 2.1 remains valid if we change conditions (2.0.3)–(2.0.5) into

$$\mathrm{Ric}^{\Omega}\in \mathrm{bmo}(\Omega),\quad \mathrm{Ric}^{\Sigma}\in \mathrm{bmo}(\Sigma),\quad H\in C^1_*(\Sigma). \tag{2.3.23}$$

§2.4: Complements on coordinates

Under the hypotheses of Theorem 2.1, we know there are local boundary harmonic coordinates with respect to which $g_{jk} \in C^2_*(\overline{\Omega})$. Here we show that, with respect to such coordinates, any *other* boundary harmonic coordinates are smooth of class $C^3_*(\overline{\Omega})$, so $\overline{\Omega}$ has the structure of a C^3_*-manifold. Since the interior behavior is simpler to establish than the behavior at the boundary, we confine our analysis to the following result.

Proposition 2.4.1. *Assume the metric tensor on $\overline{\Omega}$ is of class $C^2_*(\overline{\Omega})$. Let $u \in H^{1,2}(\Omega)$ solve*

$$\Delta u = f \in C^1_*(\overline{\Omega}), \quad u\big|_{\partial\Omega} = h \in C^3_*(\partial\Omega). \tag{2.4.1}$$

Then $u \in C^3_*(\overline{\Omega})$.

Proof. Classical Schauder estimates readily give $u \in C^{2+s}_*(\overline{\Omega})$ for all $s < 1$, as in (2.1.2); we just need to go a little further. The key to success is to abandon our former practice (which worked so well) of writing the Laplace-Beltrami operator Δ in divergence form, and instead write it in non-divergence form:

$$g^{jk}\partial_j\partial_k u = f_1, \quad u\big|_{\partial\Omega} = h, \tag{2.4.2}$$

where

$$f_1 = f - g^{-1/2}\partial_j(g^{1/2}g^{jk})(\partial_k u).$$

The hypothesis on g_{jk}, plus the current handle we have on u, gives $g^{-1/2}\partial_j(g^{1/2}g^{jk})$ $(\partial_k u) \in C^1_*(\overline{\Omega})$, hence $f_1 \in C^1_*(\overline{\Omega})$, under the hypothesis on f in (2.4.1). Now the hypothesis on g^{jk} is strong enough for standard Schauder estimates to apply, yielding

$$f_1 \in C^r(\overline{\Omega}), \quad h \in C^{r+2}(\overline{\Omega}) \Rightarrow u \in C^{r+2}(\overline{\Omega}), \quad \text{for } r \in (0,1) \cup (1,2). \tag{2.4.3}$$

Then an interpolation argument gives the conclusion stated in Proposition 2.4.1.

Remark. Under the hypotheses of Theorem 2.1, we also have

$$g_{jk} \in H^{2,p}(\Omega), \quad \forall\, p < \infty, \tag{2.4.4}$$

in boundary harmonic coordinates. Given this, we have, in parallel with (2.4.1),

$$\Delta u = f \in H^{1,q}(\Omega),\ u\big|_{\partial\Omega} = h \in B^{3-1/p}_{p,p}(\partial\Omega) \Longrightarrow u \in H^{3,q}(\Omega), \tag{2.4.5}$$

for $q \in (1, \infty)$. The proofs are simple variants of arguments presented above.

§2.5: Non-branching (and branching) of geodesics

Suppose (g_{jk}) is a metric tensor on an open set $\mathcal{V} \subset \mathbb{R}^n$. The geodesic equation can be written in Hamiltonian form as

$$\dot{x}^j = \frac{\partial}{\partial \xi_j} G(x,\xi), \quad \dot{\xi}_j = -\frac{\partial}{\partial x^j} G(x,\xi), \tag{2.5.1}$$

where

$$G(x,\xi) = \frac{1}{2} g^{jk}(x) \xi_j \xi_k. \tag{2.5.2}$$

In other words, the geodesic flow is the flow on $T^*\mathcal{V} = \mathcal{V} \times \mathbb{R}^n$ generated by

$$X(x,\xi) = \sum_j \frac{\partial G}{\partial \xi_j} \frac{\partial}{\partial x^j} - \frac{\partial G}{\partial x^j} \frac{\partial}{\partial \xi_j}. \tag{2.5.3}$$

Osgood's theorem states that a vector field generates a uniquely defined flow provided its coefficients have a modulus of continuity $\omega(t)$ satisfying

$$\int_0^{1/2} \frac{dt}{\omega(t)} = \infty. \tag{2.5.4}$$

See, e.g., Chapter 1 of [T1] for a proof. An example for which (2.5.4) holds is

$$\omega(t) = t \log \frac{1}{t}, \tag{2.5.5}$$

which is just a bit rougher than the Lipschitz modulus of continuity. In fact, one has

$$f \in C^1_*(U) \Longrightarrow |f(x+y) - f(x)| \le C \|f\|_{C^1_*} \omega(|y|), \tag{2.5.6}$$

with $\omega(t)$ given by (2.5.5). See, e.g., [T2], Chapter I, for a proof of this classical result. This applies to (2.5.1)–(2.5.3) provided $g_{jk} \in C^2_*(\mathcal{V})$, so we have the following.

Proposition 2.5.1. *Let $(g_{jk}) \in C^2_*(\mathcal{V})$ be a metric tensor on an open set $\mathcal{V} \subset \mathbb{R}^n$. Then the geodesic flow is locally uniquely defined.*

Corollary 2.5.2. *With $\overline{\Omega}$ as in Theorem 2.1, suppose $g_{jk} \in C^2_*(\overline{\Omega})$. Then the geodesic flow is locally uniquely defined when applied to any initial point (z,ξ) with $z \in \Sigma$ and $v = v(\xi) \in T_z\overline{\Omega}$ pointing inside Ω, transversal to Σ.*

Here ξ and v are related by $v^j = g^{jk}(z)\xi_k$. To prove the corollary, let $\mathcal{V}$ be a collar neighborhood of $\overline{\Omega}$. One can extend g_{jk} to $g_{jk} \in C^2_*(\mathcal{V})$, and we have that the geodesic flow is locally uniquely defined on $T^*\mathcal{V}$. If the initial point is (z,ξ), as described above, the geodesic has tangent vector v at z, so under our hypotheses the geodesic initially moves into Ω, for small positive t. This behavior is hence independent of the chosen extension of g_{jk} to $\mathcal{V}$.

We now discuss some examples of metric tensors only slightly rougher than treated in Proposition 2.5.1, for which there is branching of geodesics. These examples are variants of some produced by P. Hartman in [Ha]. As in [Ha], we take

$$ds^2 = h(v)(du^2 + dv^2). \tag{2.5.7}$$

As noted there, curves of the form $v = v(u)$ are (variable speed) geodesics provided

$$2\frac{d^2v}{du^2} = \Big(1 + \Big(\frac{dv}{du}\Big)^2\Big) H'(v), \quad H(v) = \log h(v). \tag{2.5.8}$$

Multiplying by dv/du yields

$$\frac{d}{du}\Big(\frac{dv}{du}\Big)^2 = \Big(1 + \Big(\frac{dv}{du}\Big)^2\Big) H'(v)\,\frac{dv}{du}, \tag{2.5.9}$$

hence

$$\frac{d}{du}\log(1 + v_u^2) = \frac{d}{du} H(v(u)). \tag{2.5.10}$$

Thus $v = v(u)$ solves (2.5.8) provided $dv/du \neq 0$ and

$$h(v(u)) = 1 + \Big(\frac{dv}{du}\Big)^2. \tag{2.5.11}$$

If, however,

$$h(0) = 1, \quad h'(0) = 0, \tag{2.5.12}$$

then $v \equiv 0$ solves (2.5.8), so if we produce another solution to (2.5.8) such that $v(0) = 0,\ v'(0) = 0$, we will have branching of geodesics.

For the first class of examples, we pick $k \in \{1, 2, 3, \dots\}$ and construct $h(v)$ such that

$$v(u) = u^{2k+1} \tag{2.5.13}$$

solves (2.5.8). Indeed, by (2.5.11), this happens when

$$h(v) = 1 + (2k+1)^2 |v|^{4k/(2k+1)}. \tag{2.5.14}$$

In this case the metric tensor has one derivative in $C^{(2k-1)/(2k+1)}$. The case $k = 1$ was explicitly mentioned in [Ha].

For another family of examples, we take $k \in \{1, 2, 3, \dots\}$ and construct $h(v)$ such that

$$v(u) = e^{-1/|u|^k} \operatorname{sgn} u \tag{2.5.15}$$

solves (2.5.8). This time we have

$$u = \frac{\operatorname{sgn} v}{|\log|v||^{1/k}}, \quad \left(\frac{dv}{du}\right)^2 = \frac{k^2}{u^{2k+2}} e^{-2/|u|^k}, \tag{2.5.16}$$

and hence (2.5.11) gives

$$h(v) = 1 + k^2 |\log|v||^{2+2/k} v^2. \tag{2.5.17}$$

For these examples, the first derivative of the metric tensor has a modulus of continuity only slightly worse than log-Lipschitz. Also the metric tensor has two derivatives in L^p for all $p < \infty$. Hence the curvature belongs to L^p for all $p < \infty$.

3. Geometric convergence for manifolds with boundary

A sequence $(\overline{M}_k, g_k)$ of compact Riemannian manifolds with boundary ∂M_k is said to converge in the C^r-topology (given $0 < r < \infty$) to a compact Riemannian manifold $(\overline{M}, g)$ provided that g is a C^r metric tensor on $\overline{M}$ and, for k sufficiently large, there exist diffeomorphisms $F_k : \overline{M} \to \overline{M}_k$ such that $F_k^* g_k$ converges to g in the C^r-topology. (Necessarily $F_k : \partial M \to \partial M_k$.) In this section we will identify classes of Riemannian manifolds with boundary that are pre-compact in the C^r-topology, for any given $r < 2$.

We work with families of Riemannian manifolds with boundary of the following sort. Fix the dimension, n. Given $R_0, i_0, S_0, d_0 \in (0, \infty)$, denote by $\mathcal{M}(R_0, i_0, S_0, d_0)$ the class of compact, connected, n-dimensional Riemannian manifolds with boundary $(\overline{M}, g)$, with smooth metric tensor, with the following four properties:

$$\| \operatorname{Ric}_M \|_{L^\infty(M)} \le R_0, \quad \| \operatorname{Ric}_{\partial M} \|_{L^\infty(\partial M)} \le R_0, \tag{3.0.1}$$

where Ric denotes the Ricci tensor.

$$i_M \ge i_0, \quad i_{\partial M} \ge i_0, \quad i_b \ge 2 i_0. \tag{3.0.2}$$

Here i_M denotes the injectivity radius of $\overline{M}$, $i_{\partial M}$ that of ∂M, and i_b the boundary injectivity radius of $\overline{M}$.

$$\|H\|_{\operatorname{Lip}(\partial M)} \le S_0, \tag{3.0.3}$$

where H is the mean curvature of ∂M in $\overline{M}$.

$$\text{diam}\,(\overline{M}, g) \le d_0. \tag{3.0.4}$$

We recall the concept of boundary injectivity radius, i_b. It is the optimal quantity with the following property. Namely, there is a collar neighborhood $\mathcal{C}$ of ∂M in $\overline{M}$ and a (unique) function $f \in C^2(\mathcal{C})$ such that $f\big|_{\partial M} = 0, \quad |\nabla f| \equiv 1, \quad f(\mathcal{C}) \supset [0, i_b)$. With this, local coordinates $(v^1, \dots, v^{n-1})$ on an open set in ∂M can be continued inside, as constant on the integral curves of ∇f, to produce, along with $v^n = f$, a set of "boundary normal coordinates."

To further clarify the first part of (3.0.2), we mean that

$$\text{Exp}_p : B_\rho(0) \to M, \tag{3.0.5}$$

where $B_\rho(0) = \{v \in T_pM : g(v,v) < \rho^2\}$, is a diffeomorphism for $\rho = i_0$ if $\text{dist}(p, \partial M) \ge i_0$ and it is a diffeomorphism for $\rho = \text{dist}(p, \partial M)$ if $\text{dist}(p, \partial M) \le i_0$.

The main goal in this section is to prove the following.

Theorem 3.1. *Given $R_0, i_0, S_0, d_0 \in (0, \infty)$, $\mathcal{M}(R_0, i_0, S_0, d_0)$ is precompact in the C^r-topology for each $r < 2$. In particular, any sequence $(\overline{M}_k, g_k)$ in $\mathcal{M}(R_0, i_0, S_0, d_0)$ has a subsequence that converges in the C^r-topology to a limit $(\overline{M}, g)$. Furthermore, the metric tensor g belongs to $C^2_*(\overline{M})$.*

Such a result was established in [An1] in the category of compact manifolds without boundary; subsequently there have been expositions in [HH] and in [Pe]. Our proof of Theorem 3.1 follows the structure of the argument in [An1], with necessary modifications to treat the case of nonempty boundary. In this regard the boundary regularity results of §2.1 play a major role. The C^2_* part of the conclusion is also more precise than that noted in earlier results. This precision will be of major value in the application of Theorem 3.1 to results on inverse boundary spectral problems presented in §4.

The proof of Theorem 3.1 involves a blow-up argument that takes us outside the category of compact manifolds, and it is useful to have the following notion of pointed convergence of a sequence $(\overline{M}_k, g_k, p_k)$, with distinguished points $p_k \in \overline{M}_k$. We say $(\overline{M}_k, g_k, p_k)$ converges to $(\overline{M}, g, p)$ in the pointed C^r-topology provided the following holds. For large k there exist $\rho_k < \sigma_k, \rho_k \nearrow \infty$, and compact $\overline{\Omega}_k \subset \overline{M}_k$ and $\overline{\mathcal{V}}_k \subset \overline{M}$, such that

$$B_{\rho_k}(p_k) \subset \overline{\Omega}_k \subset B_{\sigma_k}(p_k), \quad B_{\rho_k}(p) \subset \overline{\mathcal{V}}_k \subset B_{\sigma_k}(p),$$

and diffeomorphisms

$$F_k : \overline{\mathcal{V}}_k \to \overline{\Omega}_k, \quad F_k : \overline{\mathcal{V}}_k \cap \partial M \to \overline{\Omega}_k \cap \partial M_k,$$

such that $F_k^* g_k$ converges to g in the C^r-topology (on each compact subset of $\overline{M}$) and $F_k^{-1}(p_k) \to p$. We assume $\overline{M}$ is connected, and these hypotheses imply $(\overline{M}, g)$ must be complete.

§3.1: Basic convergence results

We begin by describing a slight variant of a well known "abstract" convergence result. As before, fix the dimension n. Given $s = \ell + \sigma$ ($\ell \in \mathbb{Z}^+,\ 0 < \sigma < 1$), $\rho \in (0, \infty),\ Q \in (1, 2)$, let $\mathcal{N}(s, \rho, Q)$ denote the class of connected, n-dimensional Riemannian manifolds $(\overline{M}, g)$ with boundary, with the following properties. Take $p \in \overline{M}$.

(i) If $\operatorname{dist}(p, \partial M) > \rho$, there is a neighborhood U of p in M^{int} and a coordinate chart $\varphi : B_{\rho/2}(0) \to U$, such that, in these coordinates

$$Q^{-2}|\eta|^2 \le g_{jk}(x)\eta^j\eta^k \le Q^2|\eta|^2, \tag{3.1.1}$$

and

$$\rho^s \sum_{|\beta|=\ell} \sup |x-y|^{-\sigma} \left|\partial^\beta g_{jk}(x) - \partial^\beta g_{jk}(y)\right| \le Q - 1. \tag{3.1.2}$$

(ii) If $\operatorname{dist}(p, \partial M) \le \rho$, there is a neighborhood U of p in $\overline{M}$ and a coordinate chart $\varphi : B^+_{4\rho}(0) \to U$ such that $\{x^n = 0\}$ maps to ∂M and (3.1.1)–(3.1.2) hold in these coordinates.

Let $\mathcal{N}_*(s, \rho, Q)$ denote the class of pointed manifolds $(\overline{M}, g, p)$, with $p \in \overline{M}$, satisfying these properties.

The following compactness result goes back to [Ch]; a detailed proof is given on pp. 293–296 of [Pe] for the case of manifolds without boundary, but no essential changes are required for the case of manifolds with boundary.

Theorem 3.1.1. *Given $s, \rho \in (0, \infty),\ Q \in (1, 2)$, the class $\mathcal{N}_*(s, \rho, Q)$ is compact in $\mathcal{N}_*(s', \rho, Q)$ in the pointed $C^{s'}$-topology, for all $s' < s$.*

As for convergence without specifying base points, we can define $\mathcal{N}(s, \rho, Q, d_0)$ to consist of $(\overline{M}, g) \in \mathcal{N}(s, \rho, Q)$ satisfying also $\operatorname{diam}(\overline{M}, g) \le d_0$, and conclude that $\mathcal{N}(s, \rho, Q, d_0)$ is compact in $\mathcal{N}(s', \rho, Q, d_0)$ for all $s' < s$.

To apply Theorem 3.1.1 to our situation, we will show that, given $R_0, i_0, S_0, d_0 \in (0, \infty)$, and given $Q \in (1, 2),\ s \in (1, 2)$, there exists $\rho > 0$ such that

$$\mathcal{M}(R_0, i_0, S_0, d_0) \subset \mathcal{N}(s, \rho, Q). \tag{3.1.3}$$

In fact, we will establish a result that is more precise, in two respects.

For one, we will produce harmonic coordinates, in case (i), and boundary harmonic coordinates, in case (ii). Doing this brings in the notion of C^s-harmonic radius, introduced in [An1] in the context of manifolds without boundary. To be precise, if $s = \ell + \sigma$, as above, a number $r_h^s = r_h^s(p, g, Q)$ is called the C^s-harmonic radius of $(\overline{M}, g)$ at $p \in \overline{M}$ provided it is the optimal quantity with the following property. Take any $\rho < r_h^s$. Then, in case (i), there exist harmonic coordinates $\varphi^{-1} : U \to B_{\rho/2}(0)$ such that (3.1.1)–(3.1.2) hold on $B_{\rho/2}(0)$. In case (ii), there exist harmonic coordinates $\varphi^{-1} : U \to B_{4\rho}^{+}(0)$, such that (3.1.1)–(3.1.2) hold on $B_{4\rho}^{+}(0)$. If $\overline{M}$ is compact, we define the C^s-harmonic radius of $(\overline{M}, g)$ by

$$r_h^s(\overline{M}, g, Q) = \inf_{p \in \overline{M}} r_h^s(p, g, Q). \tag{3.1.4}$$

The containment (3.1.3) follows from the fact that, given $R_0, i_0, S_0, d_0 \in (0, \infty)$ and $s \in (1, 2)$, $Q \in (1, 2)$, there is a lower bound on the C^s-harmonic radius of $(\overline{M}, g) \in \mathcal{M}(R_0, i_0, S_0, d_0)$.

The result we will establish is more precise in one more respect; we bring in the notion of C_*^2-harmonic radius. Namely, a number $r_h = r_h(p, g, Q)$ is called the C_*^2-harmonic radius of $(\overline{M}, g)$ at $p \in \overline{M}$ given the circumstances described above (in the definition of C^s-harmonic radius) but with (3.1.2) replaced by

$$\rho^2 \sum_{|\beta|=1} \sup |x - y|^{-1} \big|\partial^\beta g_{jk}(x) + \partial^\beta g_{jk}(y) - 2\partial^\beta g_{jk}\big((x+y)/2\big)\big| \le Q - 1. \tag{3.1.5}$$

Then the C_*^2-harmonic radius of $(\overline{M}, g)$ is defined by

$$r_h(\overline{M}, g, Q) = \inf_{p \in \overline{M}} r_h(p, g, Q). \tag{3.1.6}$$

We note the following elementary but important scaling property of harmonic radius, valid for $\lambda \in (0, \infty)$:

$$r_h(\overline{M}, \lambda^2 g, Q) = \lambda r_h(\overline{M}, g, Q), \quad r_h^s(\overline{M}, \lambda^2 g, Q) = \lambda r_h^s(\overline{M}, g, Q). \tag{3.1.7}$$

Our next goal is to show that, given $R_0, i_0, S_0, d_0 \in (0, \infty)$, there is a lower bound on the C_*^2-harmonic radius of $(\overline{M}, g) \in \mathcal{M}(R_0, i_0, S_0, d_0)$.

§3.2: Harmonic radius estimate

As advertised, our goal here is to prove the following.

Theorem 3.2.1. *Let R_0, i_0, S_0, and d_0 be given, in $(0, \infty)$, and let $Q \in (1, 2)$ be given. Then there exists $r_{\mathcal{M}} = r_{\mathcal{M}}(R_0, i_0, S_0, d_0, Q) > 0$ such that*

$$r_h(\overline{M}, g, Q) \ge r_{\mathcal{M}}, \quad \forall\ (\overline{M}, g) \in \mathcal{M}(R_0, i_0, S_0, d_0). \tag{3.2.1}$$

The proof will be by contradiction. Suppose there exist $(\overline{M}_k, \tilde{g}_k) \in \mathcal{M}(R_0, i_0, S_0, d_0)$ such that

$$r_h(\overline{M}_k, \tilde{g}_k, Q) = \varepsilon_k \to 0. \tag{3.2.2}$$

Let us scale the metric $\tilde{g}_k$ to $g_k = \varepsilon_k^{-2}\tilde{g}_k$, and consider the scaled Riemannian manifolds $(\overline{M}_k, g_k)$. Then

$$r_h(\overline{M}_k, g_k, Q) = 1, \tag{3.2.3}$$

while, for the rescaled metric,

$$\|\operatorname{Ric}_k\|_{L^\infty(M)} \to 0, \quad \|\operatorname{Ric}_{\partial M_k}\|_{L^\infty(\partial M_k)} \to 0, \quad \|H_k\|_{\mathrm{Lip}^1(\partial M_k)} \to 0, \tag{3.2.4}$$

and

$$i_k \to \infty, \quad i_{\partial M_k} \to \infty, \quad i_{b,M_k} \to \infty, \tag{3.2.5}$$

as $k \to \infty$. We will show that conditions (3.2.3)–(3.2.5) lead to a contradiction.

To see this, pick $p_k \in M_k$ such that

$$r_h(p_k, g_k, Q) = 1. \tag{3.2.6}$$

By Theorem 3.1.1, there is a subsequence, which for simplicity we will also denote $(\overline{M}_k, g_k, p_k)$ which converges to a pointed manifold $(\overline{M}, g, p)$ in the C^r-topology, where $r \in (1,2)$ is arbitrary. With $\tau_k = \operatorname{dist}_k(x, \partial M_k)$, where dist_k is the distance on $(\overline{M}_k, g_k)$, there are two possibilities:

(i) $\tau_k(p_k) \to \infty$.
Then we claim (M, g) is isometric to $\mathbb{R}^n$, with its standard flat metric.

(ii) $\tau_k(p_k) \le K < \infty$.
Then we claim $(\overline{M}, g)$ is isometric to $\overline{\mathbb{R}}^n_+$, with its standard flat metric.

We demonstrate these claims in Lemma 3.2.2 below. For now we assume this, and proceed with a further analysis of these two cases.

Case (i). This case is treated in [An1]. We recall the argument here, in a slightly varied form (also borrowing from [HH]), both to establish a slightly stronger conclusion and to set the stage to examine Case (ii).

We have neighborhoods U_k of p_k in M_k, identified with $B_5 = \{x \in \mathbb{R}^n : |x| \le 5\}$, with $p_k = 0$, and the metric tensors $g_k \to \delta$ in C^r-norm on B_5, where δ is the standard Euclidean metric tensor on $\mathbb{R}^n$. Taking x^ν $(1 \le \nu \le n)$ to be the standard Cartesian coordinates on $\mathbb{R}^n$, we solve

$$\Delta_k u_k^\nu = 0 \ \text{ on } \ B_5, \quad u_k^\nu = x^\nu \ \text{ on } \ \partial B_5, \tag{3.2.7}$$

where Δ_k is the Laplace operator with respect to the metric tensor g_k:

$$\Delta_k u = g_{(k)}^{-1/2} \partial_i \big(g_{(k)}^{1/2} g_{(k)}^{ij} \partial_j u \big). \tag{3.2.8}$$

Note that

$$\Delta_k (u_k^\nu - x^\nu) = f_k^\nu \ \text{ on } \ B_5, \quad u_k^\nu - x^\nu = 0 \ \text{ on } \ \partial B_5, \tag{3.2.9}$$

where

$$f_k^\nu = -g_{(k)}^{-1/2} \partial_i (g_{(k)}^{1/2} g_{(k)}^{i\nu}), \tag{3.2.10}$$

for which we have

$$\| f_k^\nu \|_{C^{r-1}(B_5)} \to 0, \quad k \to \infty. \tag{3.2.11}$$

In view of this and the uniform C^r estimates on the coefficients in the elliptic differential operators (3.2.8), elliptic regularity gives

$$u_k^\nu - x^\nu \to 0 \ \text{ in } \ C^{r+1}(B_5), \quad k \to \infty. \tag{3.2.12}$$

Hence, for all large k, $(u_k^1, \dots, u_k^n)$ form a coordinate system on U_k, and if now $g_{ij}^{(k)}$ denotes the components of the metric tensor g_k in this coordinate system, then

$$\| g_{ij}^{(k)} - \delta_{ij} \|_{C^r(B_4)} \to 0, \quad k \to \infty. \tag{3.2.13}$$

The last step is to obtain an analogue of (3.2.13) in a stronger norm, using the Ricci equation, which implies

$$\Delta_k (g_{ij}^{(k)} - \delta_{ij}) = F_{ij}^{(k)}, \tag{3.2.14}$$

where Δ_k is as in (3.2.8), acting componentwise, and

$$F_{ij}^{(k)} = B_{ij}(g_{\ell m}, \nabla g_{\ell m}) - 2 \, \mathrm{Ric}_{ij}^{(k)} . \tag{3.2.15}$$

Here B_{ij} is a quadratic form in $\nabla g_{\ell m}$. In view of (3.2.4) and (3.2.13) we have $\| F_{ij}^{(k)} \|_{L^\infty(B_4)} \to 0$. Hence local elliptic regularity results on (3.2.14) (plus another appeal to (3.2.13)) give

$$\| g_{ij}^{(k)} - \delta_{ij} \|_{C_*^2(B_3)} \to 0, \quad k \to \infty, \tag{3.2.16}$$

and even the stronger result

$$\| g_{ij}^{(k)} - \delta_{ij} \|_{\mathfrak{h}^{2,\infty}(B_3)} \to 0, \quad k \to \infty. \tag{3.2.17}$$

Hence we have $r_h(p_k, g_k, Q) \geq 3$ for large k, contradicting (3.2.6).

REMARK. Clearly there is nothing special about "3" in (3.2.16). For any $L \in (1, \infty)$ one obtains (3.2.16) with $C^2_*(B_3)$ replaced by $C^2_*(B_L)$.

CASE (II). In this case, for large k, there is a unique $q_k \in \partial M_k$ closest to p_k. In addition to the pointed convergence $(\overline{M}_k, g_k, p_k) \to (\overline{M}, g, p)$, we also have $(\partial M_k, g_k, q_k) \to (\partial M, g, q)$, and $\operatorname{dist}_k(p_k, q_k) \to \operatorname{dist}(p, q)$. This time $(\overline{M}, \partial M, g) = (\overline{\mathbb{R}}^n_+, \mathbb{R}^{n-1}, \delta)$, with $\mathbb{R}^{n-1} = \{x^n = 0\}$, and we can identify q with $0 \in \mathbb{R}^{n-1} \subset \mathbb{R}^n$. Fix $L \geq 2\operatorname{dist}(p, q) + 4$.

In this case we have neighborhoods U_k of q_k in $\overline{M}_k$, identified with

$$B^+_{L+5} = \{x \in \overline{\mathbb{R}}^n_+ : |x| \leq L + 5\},$$

with $q_k = 0$, and the metric tensors $g_k \to \delta$ in C^r-norm on B^+_{L+5}. Note that $\{x \in M_k : \operatorname{dist}_k(x, p_k) \leq 2\} \subset B^+_{L+5}$, for large k.

With x^ν $(1 \leq \nu \leq n)$ the standard Cartesian coordinates on $\overline{\mathbb{R}}^n_+$ ($x^n = 0$ defining $\partial\overline{\mathbb{R}}^n_+$) we first solve, for $1 \leq \nu \leq n - 1$,

$$\Delta_{\partial M_k} v^\nu_k = 0 \quad \text{on} \quad \widetilde{B}_{L+5}, \quad v^\nu_k = x^\nu \quad \text{on} \quad \partial\widetilde{B}_{L+5}, \tag{3.2.18}$$

where

$$\widetilde{B}_{L+5} = \{x \in \mathbb{R}^{n-1} : |x| \leq L + 5\}.$$

Parallel to (3.2.12), we have

$$\|v^\nu_k - x^\nu\|_{C^{r+1}(\widetilde{B}_{L+5})} \to 0, \quad k \to 0, \quad 1 \leq \nu \leq n - 1. \tag{3.2.19}$$

Next solve

$$\Delta_k u^\nu_k = 0 \quad \text{on} \quad B^+_{L+5}, \quad u^\nu_k\big|_{\widetilde{B}_{L+5}} = v^\nu_k, \quad u^\nu_k\big|_{\partial^+ B_{L+5}} = x^\nu, \tag{3.2.20}$$

where

$$\partial^+ B_{L+5} = \partial(B^+_{L+5}) \setminus \widetilde{B}_{L+5}.$$

Parallel to (3.2.9)–(3.2.11), we have

$$\Delta_k(u^\nu_k - x^\nu) = f^\nu_k \quad \text{on} \quad B^+_{L+5}, \quad u^\nu_k - x^\nu = \varphi^\nu_k \quad \text{on} \quad \partial B^+_{L+5}, \tag{3.2.21}$$

with

$$\|f^\nu_k\|_{C^{r-1}(B^+_{L+5})} \to 0, \tag{3.2.22}$$

and

$$\|\varphi_k^\nu\|_{L^\infty(\partial B^+_{L+5})} \to 0, \quad \|\varphi_k^\nu\|_{C^{r+1}(\widetilde{B}_{L+5})} \to 0. \tag{3.2.23}$$

This gives first a global estimate

$$\|u_k^\nu - x^\nu\|_{L^\infty(B^+_{L+5})} \to 0, \tag{3.2.24}$$

and then a "local" estimate (away from the corners of B^+_{L+5})

$$\|u_k^\nu - x^\nu\|_{C^{r+1}(B^+_{L+4})} \to 0, \tag{3.2.24}$$

as $k \to \infty$, for $1 \le \nu \le n-1$.

To construct u_k^n, we solve

$$\Delta_k u_k^n = 0 \quad \text{on} \quad B^+_{L+5}, \quad u_k^n = x^n \quad \text{on} \quad \partial B^+_{L+5}. \tag{3.2.26}$$

Parallel to (3.2.25) we have

$$\|u_k^n - x^n\|_{C^{r+1}(B^+_{L+4})} \to 0, \quad k \to \infty. \tag{3.2.27}$$

Hence, for large k, $(u_k^1, \dots, u_k^n)$ form a coordinate system on B^+_{L+4}, with $\{u_k^n = 0\}$ defining the face $\{x^n = 0\}$. Parallel to (3.2.13), if $g_{ij}^{(k)}$ denotes the components of the metric tensor g_k in this coordinate system, we have

$$\|g_{ij}^{(k)} - \delta_{ij}\|_{C^r(\partial B^+_{L+3})} \to 0, \quad k \to \infty, \tag{3.2.28}$$

for any $r < 2$, $1 \le i, j \le n$.

As in Case (i), the last step is to obtain an analogue of (3.2.28) in a stronger norm. First, if we set $h_{ij}^{(k)} = g_{ij}^{(k)}\big|_{\widetilde{B}_{L+5}}$, $1 \le i, j \le n-1$, then the argument in Case (i) applies to yield

$$\|h_{ij}^{(k)} - \delta_{ij}\|_{\mathfrak{h}^{2,\infty}(\widetilde{B}_{L+2})} \to 0, \quad k \to \infty. \tag{3.2.29}$$

Now estimates such as derived in §2 apply to $g_{ij}^{(k)}$, solving (3.2.14)–(3.2.15) and the boundary condition (3.2.29), for $1 \le i, j \le n-1$, to yield

$$\|g_{ij}^{(k)} - \delta_{ij}\|_{C^2_*(B^+_{L+1})} \to 0, \quad k \to \infty, \tag{3.2.30}$$

for $1 \le i, j \le n-1$. Next, we use

$$\Delta_k(g^{\ell n}_{(k)} - \delta^{\ell n}) = F^{\ell n}_{(k)} = B^{\ell n}(g_{(k)}, \nabla g_{(k)}) + 2\,\mathrm{Ric}^{\ell n}_{(k)}, \tag{3.2.31}$$

with boundary conditions

$$
\begin{aligned}
&N(g^{nn}_{(k)} - \delta^{nn}) = -2(n-1)H_k g^{nn}_{(k)}, \\
(3.2.32)\qquad &N(g^{\ell n}_{(k)} - \delta^{\ell n}) = -(n-1)H_k g^{\ell n}_{(k)} + \frac{1}{2}\frac{1}{\sqrt{g^{nn}_{(k)}}} g^{\ell i}_{(k)} \partial_i g^{nn}_{(k)}, \quad 1 \le \ell \le n-1,
\end{aligned}
$$

on $\widetilde{B}_{L+5}$. The results of §2 together with hypotheses in (3.2.4) and the analogue of (3.2.28) for $\|g^{ij}_{(k)} - \delta^{ij}\|_{C^r(B^+_{L+3})}$, yield first

$$
\|g^{nn}_{(k)} - \delta^{nn}\|_{C^2_*(B^+_L)} \to 0, \quad k \to \infty, \tag{3.2.33}
$$

and then

$$
\|g^{\ell n}_{(k)} - \delta^{\ell n}\|_{C^2_*(B^+_L)} \to 0, \quad k \to \infty, \quad \ell < n. \tag{3.2.34}
$$

Finally an argument parallel to that in Lemma 2.1.3 gives (3.2.30) for all $i, j \le n$.

Hence we have $r_h(p_k, g_k, Q) \ge 2$ for large k, contradicting (3.2.6). This finishes the proof of Theorem 3.2.1, modulo the proof of the following lemma.

Lemma 3.2.2. *Assume that a sequence of Riemannian manifolds $(\overline{M}_k, g_k, p_k)$ satisfies the conditions of Theorem 3.2.1 and $(\overline{M}_k, g_k, p_k) \to (\overline{M}, g, p)$ in the C^r-topology, $r \in (1,2)$. Assume that*

$$
\|\operatorname{Ric}_k\|_{L^\infty(M_k)} \to 0, \quad \|\operatorname{Ric}_{\partial M_k}\|_{L^\infty(\partial M_k)} \to 0, \quad \|H_k\|_{L^\infty(\partial M_k)} \to 0, \tag{3.2.35}
$$

and

$$
i_{M_k} \to \infty, \quad i_{\partial M_k} \to \infty, \quad i_{b, M_k} \to \infty, \tag{3.2.36}
$$

as $k \to \infty$. Then $(\overline{M}, g)$ is isometric to either $\mathbb{R}^n$ or $\overline{\mathbb{R}}^n_+$.

Proof. We consider separately the cases $\tau_k(p_k) \to \infty$ and $\tau_k(p_k) \le K < \infty$.

CASE (I). Suppose $\tau_k(p_k) \to \infty$.

This case is effectively treated in [An1]. We recall briefly the argument, since it also plays a role in Case (ii). In this case the hypotheses imply

$$
\operatorname{Ric}_M = 0, \tag{3.2.37}
$$

weakly. Hence the metric tensor g is smooth in local harmonic coordinates. Also any unit speed geodesic $\gamma(t)$ such that $\gamma(0) = p$ is defined in M for all $t \in \mathbb{R}$. Take $T \in (0, \infty)$. We claim γ is the shortest path from $\gamma(-T)$ to $\gamma(T)$.

Consider $B_{6T}(p) = \{x \in M : \operatorname{dist}(x, p) < 6T\}$. The hypotheses imply that there exists k_0 such that for all $k \ge k_0$ there are open sets in M_k identified with $B_{6T}(p)$

via diffeomorphisms, such that $p_k \to p$ and $g_k \to g$ in $C^r(B_{6T}(p))$. Also, for each $x \in B_{2T}(p)$, $i(x, g_k) \geq 3T$.

Since (M, g) is smooth, there is some $c_0 > 0$ such that $i(p, g) \geq c_0$. Say $\gamma(c_0) = q$. We have unit speed geodesics γ_k on (M_k, g_k), defined for $|t| \leq 2T$, such that $\gamma_k(0) = p$, $\gamma_k(c_k) = q$, $c_k \to c_0$. By Arzela's theorem there is a subsequence $\gamma_k \to \sigma$, uniformly on $t \in [-2T, 2T]$. We see that σ is a geodesic on (M, g), $\sigma(0) = p$, $\sigma(c_0) = q$. Hence $\sigma = \gamma$. (It follows that the entire sequence $\gamma_k \to \gamma$, not just a subsequence.) Since $i(x, g_k) \geq 3T$ for $x \in B_{2T}(p)$, we can deduce that

$$\operatorname{dist}_k(\gamma_k(-T), \gamma_k(T)) = 2T, \tag{3.2.38}$$

for $k \geq k_0$. Since $\gamma_k \to \gamma$ uniformly on $[-T, T]$ and $g_k \to g$ in C^0, the left side of (3.2.38) converges to $\operatorname{dist}(\gamma(-T), \gamma(T))$.

Thus (M, g) is complete and Ricci flat and each geodesic through p is globally length minimizing. It follows from the Cheeger-Gromoll splitting theorem [CG] that (M, g) is isometric to standard flat $\mathbb{R}^n$.

Case (ii). Assume $\tau_k(p_k) \leq K < \infty$.

Let $q_k \in \partial M_k$ be the nearest point on ∂M_k to p_k. By (3.2.36) q_k is uniquely defined (for large k). Also $(\partial M_k, g_k, q_k)$ converges in the C^r topology to $(\partial M, g, q)$, and $\operatorname{dist}(p, q) = \lim \tau_k(p_k)$. In this case the hypotheses yield

$$\operatorname{Ric}_M = 0, \quad \operatorname{Ric}_{\partial M} = 0, \quad H\big|_{\partial M} = 0. \tag{3.2.39}$$

Hence g is smooth on $\overline{M}$ in local harmonic coordinates.

We will make strong use of two equations. One is:

$$\Delta_k \tau_k = H_k \ \text{ on } \ \{\tau_k = c\}. \tag{3.2.40}$$

Here, in slight contrast to notation in §2, we set H_k equal to the trace of the Weingarten map, i.e., to $n-1$ times the mean curvature of the surface $\{\tau_k = c\}$. The other is (in boundary normal coordinates (z, τ_k), $z \in \partial M$):

$$\partial_\tau H_k = -\operatorname{Tr} A_k^2 - \operatorname{Ric}_{(k)}(\partial_\tau, \partial_\tau), \tag{3.2.41}$$

where A_k is the Weingarten map of the surface $\{\tau_k = c\}$. See [Pe], §§2.3–2.4.

Under our hypotheses, there exist $\varepsilon_k \to 0$ such that

$$\|\operatorname{Ric}_{(k)}(\partial_\tau, \partial_\tau)\|_{L^\infty} \leq \varepsilon_k, \quad \|H_k(0)\|_{L^\infty} \leq \varepsilon_k. \tag{3.2.42}$$

Hence

$$\partial_\tau H_k \leq -\frac{1}{n-1} H_k^2 + \varepsilon_k. \tag{3.2.43}$$

In particular, $H_k(\tau, z) \le \mu_k(\tau)$ for $0 \le \tau \le i_{b,M_k}$, where

$$\partial_\tau \mu_k = -\frac{1}{n-1}\mu_k^2 + \varepsilon_k, \quad \mu_k(0) = \varepsilon_k. \tag{3.2.44}$$

Hence

$$H_k(\tau, z) \le \max(\sqrt{(n-1)\varepsilon_k}, \varepsilon_k), \tag{3.2.45}$$

since (3.2.44) forces $\mu_k'(\tau) \le 0$ before $\mu_k(\tau)$ can be larger than the right side of (3.2.45).

We next estimate $H_k(\tau, z)$ from below. Pick $\delta > 0$ and suppose

$$H_k(\tau_0, z) \le -\delta, \text{ for some } \tau_0 \le \frac{i_{b,M_k}}{2}, \ z \in \partial M. \tag{3.2.46}$$

For short, we set $H_k(\tau) = H_k(\tau, z)$. If k is large enough that $\delta^2 > 2(n-1)\varepsilon_k$, it follows that $H_k(\tau) < -\delta$ for all $\tau \in [\tau_0, i_{b,M_k})$, and hence

$$\partial_\tau H_k \le -\frac{1}{2(n-1)} H_k^2, \quad \tau \ge \tau_0. \tag{3.2.47}$$

As $H_k(\tau_0) \le -\delta$, this implies

$$H_k(\tau) \le \frac{2(n-1)}{(\tau - \tau_0) - 2(n-1)/\delta}. \tag{3.2.48}$$

But this implies blow-up of $H_k(\tau)$ somewhere on $\tau \in [\tau_0, \tau_0 + 2(n-1)/\delta]$, contradicting the fact that $H_k(\tau)$ is finite on $\tau \in [0, i_{b,M_k})$, which contains $[\tau_0, \tau_0 + 2(n-1)/\delta]$ when $\delta > 4(n-1)/i_{b,M_k}$. This contradiction shows that, for any given $\delta > 0$, (3.2.46) must fail for all sufficiently large k. Hence we have $\delta_k \to 0$ such that

$$|H_k(\tau)| \le \delta_k \ \text{ for } \ 0 \le \tau \le \frac{i_{b,M_k}}{2}. \tag{3.2.49}$$

If $\tau(x) = \operatorname{dist}(x, \partial M)$, then $\tau_k \to \tau$ in C^0 due to $g_k \to g$ in C^r. Using again that $g_k \to g$ in C^r together with (3.2.40), (3.2.49), the boundary condition $\tau_k|_{\partial M_k} = 0$, and Propositions 5.1.1 and 5.2.1, we deduce that, for all $s < 2$, $n/(n-1) \le p < \infty$,

$$\tau_k \to \tau \ \text{ in } \ C^s \cap H^{2,p}(B^+(\rho_k)), \tag{3.2.50}$$

where $\rho_k \to \infty$ for $k \to \infty$. In particular, $i_{b,M} = +\infty$. Applying again (3.2.40), (3.2.49) together with $g_k \to g$ in C^r, we obtain that

$$H_k \to H \quad \text{in } L^p(B^+(\rho_k)), \quad \frac{n}{n-1} \le p < \infty, \tag{3.2.51}$$

i.e.,

$$H(\tau) = 0, \quad \forall\, \tau \geq 0. \tag{3.2.52}$$

Now, parallel to (3.2.41), we have

$$\partial_\tau H + \operatorname{Tr} A^2 = -\operatorname{Ric}_M(\partial_\tau, \partial_\tau) = 0, \tag{3.2.53}$$

where A is the Weingarten map on the surface $\{\tau = c\}$, and hence $\operatorname{Tr} A^2 = 0$, which implies

$$A = 0. \tag{3.2.54}$$

Furthermore, since

$$\partial_\tau g_{ij} = 2A^\ell{}_i\, g_{\ell j} = 0, \tag{3.2.55}$$

we have

$$g_{ij}(\tau, z) = g_{ij}(0, z), \quad z \in \partial M. \tag{3.2.56}$$

By the argument of Case (i), ∂M is isometric to $\mathbb{R}^{n-1}$, so this implies that $\overline{M}$ is isometric to $[0, \infty) \times \partial M = \overline{\mathbb{R}}^n_+$, with its standard flat metric.

This finishes the proof of Lemma 3.2.2, hence of Theorem 3.2.1.

§3.3: Proof of Theorem 3.1

All the work needed to prove Theorem 3.1 has been done, and we need only collect the pieces. Say $(\overline{M}_k, g_k) \in \mathcal{M}(R_0, i_0, S_0, d_0)$. The results of §3.2 show that Theorem 3.1.1 is applicable. Hence, after passing to a subsequence, we have diffeomorphisms $F_k : \overline{M} \to \overline{M}_k$ and a Riemannian metric g on $\overline{M}$ such that $F_k^* g_k \to g$ in $C^r(\overline{M})$ for all $r < 2$. It remains to show that $g \in C^2_*(\overline{M})$.

Note that $F_k^* \operatorname{Ric}_{M_k} = \operatorname{Ric}_{F_k^* g_k}$ has uniformly bounded L^∞ norm, when measured via $F_k^* g_k$, hence when measured via g. Identities of the form (2.0.7)–(2.0.8) imply $F_k^* \operatorname{Ric}_{M_k} \to \operatorname{Ric}_M$ in $H^{-\varepsilon,p}(M)$, for each $\varepsilon > 0$, $p < \infty$. But the observation above implies some subsequence converges weak* in L^∞. It follows that

$$\operatorname{Ric}_M \in L^\infty(M).$$

A similar argument gives $\operatorname{Ric}_{\partial M} \in L^\infty(\partial M)$, and also similarly we obtain for the mean curvature H of $\partial M \hookrightarrow \overline{M}$ that $H \in \operatorname{Lip}(\partial M)$. Thus Theorem 2.1 applies to give $g \in C^2_*(\overline{M})$, in boundary harmonic coordinates. This finishes the proof of Theorem 3.1.

Remark. Invoking the definition of the Gromov-Hausdorff topology (cf. [Gr]) we can show that $\overline{\mathcal{M}(R_0, i_0, S_0, d_0)}$ is compact in the Gromov-Hausdorff topology and C^r-convergence is equivalent to Gromov-Hausdorff convergence on this compact set, for any $r \in [1, 2)$.

§3.4: Convergence of the geodesic flow and implications

Here we establish results on the limiting behavior of geodesic flows under C^1 convergence, and implications for the injectivity radius, that sharpen and generalize some of the results of [Sak]. We work in the following setting. Let $\overline{M}$ be a fixed compact manifold, with a C^3_* coordinate system. Let g, g_k be metric tensors on $\overline{M}$. Assume

$$g,\, g_k \in C^2_*(\overline{M}), \quad g_k \to g \ \text{ in } \ C^1(\overline{M}). \tag{3.4.1}$$

Say these metric tensors define exponential maps

$$\mathrm{Exp}_p : T_p\overline{M} \supset U \to \overline{M}, \quad \mathrm{Exp}_{k,p} : T_p\overline{M} \supset U \to \overline{M}, \tag{3.4.2}$$

where U is a neighborhood of $0 \in T_p\overline{M}$ if $p \in M^{int}$ and U consists of $v \in T_p\overline{M}$ in a neighborhood of 0 that point into M^{int}, transversally to ∂M, if $p \in \partial M$. Here is our first result.

Proposition 3.4.1. *Under the hypotheses listed above, given $v \in U \subset T_p\overline{M}$,*

$$\lim_{k\to\infty} \mathrm{Exp}_{k,p}(v) = \mathrm{Exp}_p(v). \tag{3.4.3}$$

Proof. Let X_k, X denote the vector fields on $T\overline{M}$ generating these flows, essentially the Hamiltonian vector fields associated with g_k and g. Thus the coefficients of these vector fields belong to C^1_* and there is C^0-convergence $X_k \to X$. Orbits of X_k are

$$y_k(t) = (\gamma_k(t), \gamma_k'(t)), \tag{3.4.4}$$

and we have $y_k'(t) = X_k(y_k(t))$ uniformly bounded, hence both γ_k and γ_k' uniformly Lipschitz. It follows from Arzela's theorem that a subsequence converges locally uniformly:

$$\gamma_{k_\nu} \to \sigma, \quad \gamma_{k_\nu}' \to \sigma'. \tag{3.4.5}$$

Clearly (σ, σ') is an orbit of X, and $\sigma(0) = p$, $\sigma'(0) = v$. Since (by Osgood's theorem) the flow generated by X is unique, it follows that $\sigma \equiv \gamma$. Since this is true for any convergent subsequence, the result (3.4.3) follows.

We next discuss the injectivity radius. We need to modify the definition used for (3.0.2), since in the present case $\mathrm{Exp}_p : U \to \overline{M}$ need not be C^1 or even Lipschitz. If $p \in M^{int}$ and $\rho \le \mathrm{dist}(p, \partial M)$, we will say

$$\widetilde{i}(p, g) \ge \rho \tag{3.4.6}$$

provided that, for each unit vector $v \in T_pM^{int}$,

$$|t| < \rho \Longrightarrow \mathrm{dist}(p, \mathrm{Exp}_p(tv)) = |t|, \tag{3.4.7}$$

i.e., provided $\gamma_v(s) = \mathrm{Exp}_p(sv)$ is length-minimizing from p to $\gamma_v(t)$ for $|t| < \rho$. This does imply injectivity:

Proposition 3.4.2. *If* $\widetilde{i}(p,g) \ge \rho$, *then*

$$\mathrm{Exp}_p : B_\rho(0) \longrightarrow M^{int} \quad \text{is one-to-one,} \tag{3.4.8}$$

where $B_\rho(0) = \{v \in T_pM^{int} : g(v,v) < \rho^2\}$.

Proof. If $v, w \in T_pM^{int}$ are unit vectors and $\gamma_v(s) = \gamma_w(t)$ with $|s|, |t| < \rho$, then the condition (3.4.7) forces $s = t$ (maybe after changing the sign of w). If $v \ne w$, these geodesics must intersect non-tangentially at $q = \gamma_v(t) = \gamma_w(t)$, i.e., at a positive angle. Then a standard construction produces, for small $\varepsilon > 0$, a curve from p to $\gamma_v(t+\varepsilon)$ shorter than $|t| + \varepsilon$, contradicting (3.4.7).

We next have the following semicontinuity result.

Proposition 3.4.3. *In the setting of Proposition 3.4.1,*

$$\widetilde{i}(p, g_k) \ge \rho_0 \ \forall \ k \Longrightarrow \widetilde{i}(p,g) \ge \rho_0. \tag{3.4.9}$$

Proof. Given $v \in T_pM^{int}$ such that $g(v,v) = 1$, we set $v_k = v/\sqrt{g_k(v,v)}$, so $g_k(v_k, v_k) = 1$ and $v_k \to v$. We are given that for all k,

$$|t| < \rho_0 \Longrightarrow \mathrm{dist}_k(p, \mathrm{Exp}_{k,p}(tv_k)) = |t|, \tag{3.4.10}$$

where dist_k denotes distance as determined by the metric tensor g_k. Now it is elementary that

$$\mathrm{dist}_k(p,q) \to \mathrm{dist}(p,q), \quad \text{as } k \to \infty, \tag{3.4.11}$$

uniformly for $q \in \overline{M}$. Hence

$$\mathrm{dist}_k(p, \mathrm{Exp}_p(tv)) \to \mathrm{dist}(p, \mathrm{Exp}_p(tv)), \tag{3.4.12}$$

while Proposition 3.4.1 together with (3.4.11) implies

$$\mathrm{dist}_k(p, \mathrm{Exp}_{k,p}(tv_k)) - \mathrm{dist}_k(p, \mathrm{Exp}_p(tv)) \to 0. \tag{3.4.13}$$

so we have

$$|t| < \rho_0 \Longrightarrow \mathrm{dist}(p, \mathrm{Exp}_p(tv)) = |t|, \tag{3.4.14}$$

as desired.

We can also define the following sort of "boundary injectivity radius." We say

$$\widetilde{i}_b(\overline{M}, g) \ge \rho \tag{3.4.15}$$

provided that for each $p \in \partial M$, inward normal $\nu_p \in T_p\overline{M}$ (orthogonal to $T_p(\partial M)$),

$$0 \le t < \rho \Longrightarrow \mathrm{dist}(\mathrm{Exp}_p(t\nu_p), \partial M) = t. \tag{3.4.16}$$

Parallel to Proposition 3.4.2, we have:

Proposition 3.4.4. *If* $\widetilde{i}_b(\overline{M}, g) \geq \rho$*, then*

$$\Phi : \partial M \times [0, \rho) \longrightarrow \overline{M}, \tag{3.4.17}$$

given by

$$\Phi(p, t) = \mathrm{Exp}_p(t\nu_p), \tag{3.4.18}$$

is one-to-one.

Then, parallel to Proposition 3.4.3, we have

Proposition 3.4.5. *In the setting of Proposition 3.4.1,*

$$\widetilde{i}_b(\overline{M}, g_k) \geq \rho_0 \ \forall\ k \Longrightarrow \widetilde{i}_b(\overline{M}, g) \geq \rho_0. \tag{3.4.19}$$

The proofs of these results are very similar to those of their counterparts above. Finally, we have the following significant implication for the geometric convergence obtained in Theorem 3.1.

Corollary 3.4.6. *If* $(\overline{M}, g)$ *is a limit of* $(\overline{M}_k, g_k) \in \mathcal{M}(R_0, i_0, S_0, d_0)$ *as in Theorem 3.1, then*

$$\begin{aligned} \widetilde{i}(p, g) &\geq \min(i_0, \mathrm{dist}(p, \partial M)), \quad \forall\ p \in M^{int}, \\ \widetilde{i}(p, g|_{\partial M}) &\geq i_0, \quad \forall\ p \in \partial M, \\ \widetilde{i}_b(\overline{M}, g) &\geq i_0. \end{aligned} \tag{3.4.20}$$

4. Gel'fand inverse boundary problem

In this section we prove uniqueness and stability and provide a reconstruction procedure for the inverse boundary spectral problem. To fix notations, assume that $(\overline{M}, g, \partial M)$ is a compact, connected manifold, with nonempty boundary, provided with a metric tensor g with some limited smoothness (specified more precisely below). Let Δ^N be the Neumann Laplacian. Denote by

$$0 = \lambda_1 < \lambda_2 \leq \cdots$$

the eigenvalues (counting multiplicity) of $-\Delta^N$ and by

$$\phi_1 = \mathrm{Vol}(M)^{-1/2},\ \phi_2, \ldots,$$

the corresponding orthonormalized eigenfunctions.

The *Gel'fand inverse boundary problem* is the problem of the reconstruction of $(\overline{M}, g)$ from its boundary spectral data, i.e., the collection $(\partial M, \{\lambda_k, \phi_k|_{\partial M}\}_{k=1}^{\infty})$.

REMARK. The data used in the original formulation of the Gel'fand inverse boundary problem [Ge] consists of the trace on ∂M of the resolvent kernel, $R_\lambda(x,y)$ of Δ^N given for all $\lambda \in \mathbb{C} \setminus \operatorname{spec} \Delta^N,\ x, y \in \partial M$. The equivalence of these data and boundary spectral data, and also dynamic inverse boundary data for the corresponding wave, heat and non-stationary Schrödinger equations is proven in [KKL], [KLM], as are analogous results for the Dirichlet problem.

The main results of this section are the uniqueness result below and stability result in §4.3.

Theorem 4.1. *Let $\overline{M}$ be a compact, connected manifold with nonempty boundary and C^2_* metric tensor. Then the boundary spectral data $(\partial M, \{\lambda_k, \phi_k|_{\partial M}\}_{k=1}^{\infty})$ determine the manifold $\overline{M}$ and its metric g uniquely.*

Such a result was established in the C^∞ case in [BK1], taking into account [Ta]; see also [KKL]. Our proof here will incorporate techniques from these papers, plus some additional arguments necessary to handle the reduced smoothness. One essential role played by the C^2_*-hypothesis is that this implies non-branching of geodesics on $\overline{M}$, including geodesics passing transversally from ∂M. In §4.1 we show that $\overline{M}$ is uniquely determined as a topological space. We proceed in §4.2 to show that the differential structure and metric tensor on $\overline{M}$ are uniquely determined. In §4.3 we apply Theorem 4.1 together with Theorem 3.1 to establish a result on the conditional stability of this inverse boundary problem, building on results of [K2L].

§4.1: DETERMINING THE DOMAIN

We start with the introduction of some useful geometric objects. Let $\Gamma \subset \partial M$ be open and take $t \geq 0$. Then we set

$$M(\Gamma, t) = \{x \in M : d(x, \Gamma) \leq t\}, \tag{4.1.1}$$

the domain of influence of Γ at "time" t. This terminology refers to the corresponding wave equation where $M(\Gamma, t)$ is the subdomain of M filled by the time t with waves sent from Γ.

Now let $\underline{\Gamma}$ consist of a finite number $\Gamma_1, \dots, \Gamma_m$ of subsets Γ and $\underline{t}^+,\ \underline{t}^-$ be two m-dimensional vectors with positive entries, $\underline{t}^+ = (t_1^+, \dots, t_m^+)$, $\underline{t}^- = (t_1^-, \dots, t_m^-)$. Then set

$$M(\underline{\Gamma}, \underline{t}^+, \underline{t}^-) = \bigcap_{i=1}^{m} (M(\Gamma_i, t_i^+) \setminus M(\Gamma_i, t_i^-)) \subset M, \tag{4.1.2}$$

and define

$$\mathbf{L}(\underline{\Gamma},\, \underline{t}^+,\, \underline{t}^-) = \mathcal{F}L^2(M(\underline{\Gamma},\, \underline{t}^+,\, \underline{t}^-)) \subset \ell^2. \tag{4.1.3}$$

Here, $\mathcal{F}$ stands for the Fourier transform of functions from $L^2(M)$,

$$\mathcal{F}(u) = \{u_k\}_{k=1}^{\infty} \in \ell^2, \quad u(x) = \sum_{k=1}^{\infty} u_k \phi_k(x), \quad u_k = (u, \phi_k)_{L^2(M)}, \tag{4.1.4}$$

and the subspace $L^2(M(\underline{\Gamma},\, \underline{t}^+,\, \underline{t}^-))$ consists of all functions in $L^2(M)$ with support in the set $M(\underline{\Gamma},\, \underline{t}^+,\, \underline{t}^-)$.

Two basic ingredients for the reconstruction of the manifold $\overline{M}$ are the approximate controllability and Blagovestchenskii's formula.

The controllability result is an implication of Tataru's unique continuation result for the wave equation ([Ta], see also [Ho], [Ta2]). To describe it, consider the wave equation

$$\begin{aligned} &(\partial_t^2 - \Delta) u^f(x,t) = 0 \quad \text{in } M \times \mathbb{R}_+ \\ &u^f|_{t=0} = 0, \quad u_t^f|_{t=0} = 0, \quad Nu^f|_{\partial M \times \mathbb{R}_+} = f \in C_0^1(\Gamma \times (0,T)), \end{aligned} \tag{4.1.5}$$

where N is the exterior unit normal field to ∂M. Using Tataru's theorem, it was shown in Theorem 3.10 of [KKL] that the following holds.

Proposition 4.1.1. *For each $T > 0$, the set $\{u^f(T) : f \in L^2(\Gamma \times (0,T))\}$ is a dense subspace of $L^2(M(\Gamma, T))$.*

(Actually Theorem 3.10 of [KKL] is written to address the Dirichlet boundary condition, but the same argument works for the Neumann boundary condition.)

Blagovestchenskii's formula gives the Fourier coefficients $u_k^f(t)$ of a wave $u^f(\cdot\,,t)$ in terms of the boundary spectral data,

$$u_k^f(t) = \int_0^t \int_{\partial M} f(x,t')\, \frac{\sin\sqrt{\lambda_k}(t-t')}{\sqrt{\lambda_k}} \phi_k(x)\, dS_g\, dt'. \tag{4.1.6}$$

To prove this one starts with $\partial_t^2 (u(t), \phi_k)_{L^2} = (\Delta u(t), \phi_k)_{L^2}$ and applies Green's formula to get an inhomogeneous ODE for $(u(t), \phi_k)_{L^2}$, yielding (4.1.6).

Note that in the formula (4.1.6) there appears the Riemannian volume dS_g of ∂M, which we are not given. However, we are given ∂M as a C^2 manifold, so an arbitrarily chosen volume element element has the form $dS = \kappa\, dS_g$, where κ is C^1-smooth and strictly positive. We can construct the Fourier coefficients of the wave $u^{\kappa f}(x,t)$ for any boundary source f, and despite our lack of knowledge of κ, we do have the following.

Corollary 4.1.2. *Given $\Gamma \subset \partial M$ and $t > 0$, the boundary spectral data determine the subspace*

$$\mathbf{L}(\Gamma, t) = \mathcal{F} L^2(M(\Gamma, t)) \subset \ell^2. \tag{4.1.7}$$

In fact, let $\{f_\nu : \nu \in \mathbb{Z}^+\}$ have dense linear span in $L^2(\Gamma \times (0,T))$. Then $\mathbf{L}(\Gamma, t)$ is the closed linear span in ℓ^2 of $\{\varphi_\nu : \nu \in \mathbb{Z}\}$, where $\varphi_\nu \in \ell^2$ is given by $\varphi_{\nu,k} = u_k^{f_\nu}(t)$.

Thus we can find the orthoprojection $P : \ell^2 \to \mathbf{L}(\Gamma, t)$ to this subspace.

From here, using the elementary identities

$$L^2\Big(\bigcap_i S_i\Big) = \bigcap_i L^2(S_i), \quad L^2(A_i \setminus B_i) = L^2(A_i) \cap L^2(B_i)^\perp, \tag{4.1.8}$$

we deduce that the boundary spectral data uniquely determine the subspaces $\mathbf{L}(\underline{\Gamma}, \underline{t}^+, \underline{t}^-)$ of ℓ^2, for any $\underline{\Gamma}$, $\underline{t}^+$, $\underline{t}^-$ with arbitrary m (compare [KKL] and [Be1]). In particular, for any $\underline{\Gamma}$, $\underline{t}^+$, $\underline{t}^-$ we can see if $\mathbf{L}(\underline{\Gamma}, \underline{t}^+, \underline{t}^-) = \{0\}$ or not. Equivalently, we can see if $M(\underline{\Gamma}, \underline{t}^+, \underline{t}^-)$ contains an open ball or not.

Next, let $h \in C(\partial M)$. We can ask if h is the boundary distance function for some $x \in M$. To this end, we choose points $z_j \in \partial M$, $j = 1, \dots, m$, their small neighborhoods $\Gamma_1, \dots, \Gamma_m$ and numbers $t_j^\pm = h(z_j) \pm 1/m$. When $m \to \infty$, the fact that $\mathbf{L}(\underline{\Gamma}, \underline{t}^+, \underline{t}^-) \neq \{0\}$ for any m determines whether there is a point $x \in M$ such that $h(z) = \operatorname{dist}(z, x)$, $z \in \partial M$. Thus we have shown that the boundary spectral data determine the image in $L^\infty(\partial M)$ of the boundary distance representation R. Here, $R : \overline{M} \to C(\partial M)$ is defined by

$$R(x) = r_x(\cdot), \quad r_x(z) = \operatorname{dist}(x, z), \quad z \in \partial M. \tag{4.1.9}$$

(Compare [KKL] and [Ku1]). Clearly, the map R is Lipschitz continuous. Moreover, under the assumptions of Theorem 4.1 it is injective. To see this, let $r_x = r_y$ and let $z \in \partial M$ be a point of minimum of these functions. Then both x and y lie on the normal geodesic to ∂M starting in z at the same arclength $r_x(z) = r_y(z)$. As the metric $g \in C_*^2(\overline{M})$, it follows from Corollary 2.5.2 that this normal geodesic does not branch. Therefore, $x = y$.

Since $\overline{M}$ is compact, injectivity and continuity imply that R is a homeomorphism, i.e., $R(\overline{M})$ with the distance inherited from $L^\infty(\partial M)$ and $(\overline{M}, g)$ are homeomorphic, and thus $R(\overline{M})$ can be identified with $\overline{M}$ as a topological manifold. We have established the following.

Proposition 4.1.3. *Assume $(\overline{M}_1, g_1)$ and $(\overline{M}_2, g_2)$ satisfy the hypotheses of Theorem 4.1. If they have identical boundary spectral data, including $\partial M_1 = \partial M_2 = X$, as C^2 manifolds, then there is a natural correspondence of $R(\overline{M}_1)$ and $R(\overline{M}_2) \subset C(X)$, producing a uniquely defined homeomorphism*

$$\chi : \overline{M}_1 \longrightarrow \overline{M}_2. \tag{4.1.10}$$

§4.2: Determining the metric

Our next goal is to reconstruct the differential and Riemannian structures on M. To this end, let us return to $\mathbf{L}(\underline{\Gamma},\, \underline{t}^+,\, \underline{t}^-)$ and consider the orthoprojection $P(\underline{\Gamma},\, \underline{t}^+,\, \underline{t}^-)$ of ℓ_2 onto this subspace. Then,

$$(4.2.1) \qquad \big(P(\underline{\Gamma},\, \underline{t}^+,\, \underline{t}^-)e_i, e_j\big)_{\ell^2} = \int_{M(\underline{\Gamma},\, \underline{t}^+,\, \underline{t}^-)} \phi_i(x)\phi_j(x)\, dV_x,$$

where $e_j = (0, \dots, 0, 1, 0, \dots)$ is the sequence having a 1 at the jth place. Also,

$$(4.2.2) \qquad \int_{M(\underline{\Gamma},\, \underline{t}^+,\, \underline{t}^-)} \phi_1(x)^2\, dV_x = \frac{\mathrm{Vol}(M(\underline{\Gamma},\, \underline{t}^+,\, \underline{t}^-))}{\mathrm{Vol}(M)}.$$

Next choose a sequence $(\underline{\Gamma}_k,\, \underline{t}_k^+,\, \underline{t}_k^-)$, $k = 1, 2, \dots$, with $m_k \to \infty$, where m_k is the dimension of $\underline{t}_k^\pm$, so that $M(\underline{\Gamma}_k,\, \underline{t}_k^+,\, \underline{t}_k^-)$ shrinks to $\{x\}$ when $k \to \infty$. Then by formulae (4.2.1)–(4.2.2) we see that

$$(4.2.3) \qquad \lim_{k\to\infty} \big(P(\underline{\Gamma}_k,\, \underline{t}_k^+,\, \underline{t}_k^-)e_1,\, e_j\big) \cdot \big(P(\underline{\Gamma}_k,\, \underline{t}_k^+,\, \underline{t}_k^-)e_1,\, e_1\big)^{-1/2} = \phi_j(x).$$

Thus we can find values of the eigenfunctions $\phi_k(x)$ for all $k = 1, 2, \dots$ and $x \in M$.

To proceed further we need an auxiliary statement about the properties of the eigenfunctions. Let Φ be the space of all finite linear combinations of ϕ_k, $k = 1, 2, \dots$.

Lemma 4.2.1. *Under the assumptions of Theorem 4.1, Φ is dense in the space $\{u \in H^{s,p}(M) : Nu|_{\partial M} = 0\}$ for any $s \in [0, 3)$, $p \in (1, \infty)$. Moreover, if $x \in M^{int}$, there are n indices $k(1), \dots, k(n)$ (depending on x) and a neighborhood U of x such that $\phi_{k(1)}(x), \dots, \phi_{k(n)}(x)$ form a C^3_*-smooth coordinate system in U.*

Proof. Assuming that $g \in C^r$, $r \in (1, 2)$, consider, for any $p \in (1, \infty)$, the Neumann Laplacian, Δ_p^N, with domain

$$(4.2.4) \qquad \mathcal{D}(\Delta_p^N) = \{u \in H^{2,p}(M) : Nu|_{\partial M} = 0\}.$$

Denote by $e^{t\Delta_p^N}$, $t \geq 0$ the corresponding contraction semigroup and by $(-\Delta_p^N)^s$, $s > 0$, the real powers of $-\Delta_p^N$, defined for $s \in (0, 1)$ via subordination. By Stein's Littlewood-Paley theory for symmetric diffusion semigroups (cf. [St]),

$$(4.2.5) \qquad Y^{s,p} = \mathcal{D}((-\Delta_p^N)^{s/2}), \quad s \in [0, \infty),$$

is a complex interpolation scale, in s, for each $p \in (1, \infty)$. In particular,

$$(4.2.6) \qquad \mathcal{D}((-\Delta_p^N)^{s/2}) = H^{s,p}(M), \quad \text{for } 0 \leq s \leq 1.$$

Hence, for $0 \le s \le 1$,

$$\begin{aligned}(4.2.7) \qquad \mathcal{D}((-\Delta_p^N)^{1+s/2}) &= \{u \in \mathcal{D}(-\Delta_p^N) : \Delta u \in \mathcal{D}((\Delta_p^N)^{s/2})\} \\ &= \{u \in H^{2,p}(M) : Nu|_{\partial M} = 0, \text{ and } \Delta u \in H^{s,p}(M)\}.\end{aligned}$$

If we require $g \in C^2_*(M)$, we can use regularity results to obtain that

$$(4.2.8) \qquad \mathcal{D}((-\Delta_p^N)^{1+s/2}) = \{u \in H^{2+s,p}(M) : Nu|_{\partial M} = 0\}, \quad 0 \le s < 1.$$

Since $(1 - \Delta_p^N)^{s/2} : Y^{s,p} \to L^p(M)$ is an isomorphism, acting bijectively on Φ, the desired density of Φ will follow from the density of Φ in $L^p(M)$. We now demonstrate this density.

Suppose $f \in L^q(M)$ (with $q = p'$) and $\langle f, u\rangle = 0$ for all $u \in \Phi$. If $q \ge 2$ then $f \in L^2(M)$ and clearly $f \equiv 0$. So we need only worry about the case $q < 2$. Note that $\langle f, u\rangle = 0$ implies

$$(4.2.9) \qquad \langle e^{t\Delta^N} f, u\rangle = 0, \quad \forall\, t \ge 0,\ u \in \Phi.$$

Now $e^{t\Delta_q^N}$ is a holomorphic semigroup on $L^q(M)$, so for all $t > 0$,

$$(4.2.10) \qquad e^{t\Delta_q^N} f \in \mathcal{D}(\Delta_q^N) \subset H^{2,q}(M) \subset L^{q_2}(M),$$

with $q_2 > q$, by the Sobolev embedding theorem. Iterating this and using the semigroup property gives

$$(4.2.11) \qquad f \in L^q(M) \Longrightarrow e^{t\Delta^N} f \in L^2(M) \quad \forall\, t > 0.$$

Hence (4.2.9) implies $e^{t\Delta^N} f = 0$, for all $t > 0$. But $e^{t\Delta^N} f \to f$ in $L^q(M)$ as $t \searrow 0$, so $f = 0$. This completes the proof of the first statement of the lemma.

To demonstrate the second statement, we first note that since $s < 3$ and $p < \infty$ are arbitrary, it follows from the first part of the lemma that

$$(4.2.12) \qquad C^2_0(M^{int}) \subset \text{closure of } \Phi \text{ in } C^2(\overline{M}).$$

Let now $x \in M^{int}$ and $(x^1, \cdots, x^n)$ be some local coordinates near x. Denote by $T_x : \Phi \to \mathbb{R}^n$ the map,

$$(4.2.13) \qquad T_x(u) = (\partial_1 u(x), \cdots, \partial_n u(x)).$$

It follows from (4.2.12) that $T_x(\Phi) = \mathbb{R}^n$, i.e., there are indices $k(1), \ldots, k(n)$ (depending on x) such that $\nabla\phi_{k(i)}(x)$, $i = 1, \ldots, n$, are linearly independent. Moreover, we know that the eigenfunctions $\phi_k \in C^3_*(M^{int})$. This proves the second statement of the lemma.

Having this, we are in a position to refine our statement about the homeomorphism $\chi : \overline{M}_1 \to \overline{M}_2$ established in Proposition 4.1.3.

Proposition 4.2.2. *Let $(\overline{M}_1, g_1)$ and $(\overline{M}_2, g_2)$ be as in Proposition 4.1.3, with identical boundary spectral data. Then the map χ in (4.1.10) has the property that*

$$\chi : M_1^{int} \longrightarrow M_2^{int} \text{ is a } C^2\text{-diffeomorphism, and } \chi^* g_2 = g_1. \tag{4.2.14}$$

Proof. We use the fact that, if $\{\phi_j\}$ are the normalized eigenfunctions of Δ^N on M_1 and $\{\tilde{\phi}_j\}$ those on M_2, then, as a consequence of (4.2.3),

$$\phi_j(x) = \tilde{\phi}_j(\chi(x)). \tag{4.2.15}$$

Given $p \in M_1^{int}$, there exist indices $k(1), \dots, k(n)$ such that $\tilde{\phi}_{k(1)}, \dots, \tilde{\phi}_{k(n)}$ form a local coordinate system on a neighborhood of $\tilde{p} = \chi(p)$. Then, for x near p, we have

$$x \mapsto (\phi_{k(1)}(x), \dots, \phi_{k(n)}(x)) = \big(\tilde{\phi}_{k(1)}(\chi(x)), \dots, \tilde{\phi}_{k(n)}(\chi(x))\big) \mapsto \chi(x) \tag{4.2.16}$$

a composition of C^2 smooth maps. Thus χ in (4.2.14) is C^2 smooth on a neighborhood of each $p \in M_1^{int}$, hence on M_1^{int}. Interchanging the roles of $\overline{M}_1$ and $\overline{M}_2$, we have the same result for χ^{-1}.

Finally, we show that the metric tensor is uniquely determined. For notational simplicity, just consider $M = M_1$. Let $(x^1, \dots, x^n)$ be a C^3_* coordinate system in a domain $U \subset M^{int}$, e.g., the one obtained earlier from the eigenfunctions. Then, for all $k = 1, 2, \dots,$

$$-g^{ij}(x)\partial_i\partial_j\phi_k(x) - b^i(x)\partial_i\phi_k(x) = \lambda_k\phi_k(x), \quad b^i = g^{-1/2}\partial_j(g^{1/2}g^{ij}), \tag{4.2.17}$$

where all eigenfunctions ϕ_k and, henceforth, their derivatives as well as λ_k are already found. Let us consider equations (4.2.17) as linear equations for $g^{ij}(x) = g^{ji}(x),\ b^i(x)$. Using again (4.2.12), we see that the map $\widetilde{T}_x : \Phi \to \mathbb{R}^{n+n(n+1)/2}$,

$$\widetilde{T}_x(u) = (\partial_i u(x),\, \partial_i\partial_j u(x) :\, i \le j = 1, \cdots, m), \tag{4.2.18}$$

is surjective (compare with (4.2.13)). Thus equations (4.2.17) are uniquely solvable since the 2-jets of the eigenfunctions ϕ_k at x span the whole space $\mathbb{R}^{n+n(n+1)/2}$.

It follows that the diffeomorphism χ in (4.2.14) pulls the metric tensor g_2 back to g_1. The proof of Proposition 4.2.2, and hence of Theorem 4.1, is complete.

We can get some more insight into how the geometry of $(\overline{M}, g)$ is determined by the boundary spectral data, particularly through (4.2.3), by examining further the maps

$$\Psi_k : \overline{M} \longrightarrow \mathbb{R}^k, \quad \Psi_k(x) = (\phi_1(x), \dots, \phi_k(x)). \tag{4.2.19}$$

For simplicity we assume the eigenfunctions are arranged to be real valued. The argument proving Proposition 4.2.3 shows that for each compact $K \subset M^{int}$, there exists k such that Ψ_k restricted to K is an embedding. In fact, we can do better than that, though for no k will Ψ_k be an embedding of $\overline{M}$, since $D\Psi_k(x)$ annihilates the normal to ∂M for each $x \in \partial M$, each k. Note that Lemma 4.2.1 implies the space of restrictions of elements of Φ to ∂M is dense in $C^2(\partial M)$, so we can find $k_0 = k_0(\overline{M}, g)$ such that

$$\Psi_k : \partial M \to \mathbb{R}^k \quad \text{is an embedding, for} \quad k \geq k_0. \tag{4.2.20}$$

We now augment Ψ_k to

$$\Psi_k^\# : \overline{M} \longrightarrow \mathbb{R}^{k+1}, \quad \Psi_k^\#(x) = (\psi_0(x), \phi_1(x), \dots, \phi_k(x)), \tag{4.2.21}$$

where $\psi_0 \in C_*^3(\overline{M})$ is the eigenfunction for the *Dirichlet problem*, with smallest eigenvalue:

$$\Delta\psi_0 = -\mu_0\psi_0, \quad \psi_0\big|_{\partial M} = 0, \tag{4.2.22}$$

normalized by

$$\int\limits_M |\psi_0(x)|^2 \, dV = 1, \tag{4.2.23}$$

and let us insist $\psi_0(x) > 0$ on M^{int}. Hopf's principle implies $N\psi_0(x) \neq 0, \ \forall \ x \in \partial M$. Hence the map (4.2.21) has the property

$$D\Psi_k^\#(x) \quad \text{is injective}, \quad \forall \ x \in \partial M, \ k \geq k_0. \tag{4.2.24}$$

Thus injectivity holds on a collar neighborhood of ∂M in $\overline{M}$. Combined with our previous observation about Ψ_k embedding compact $K \subset M^{int}$, this implies that there exists $k_1 = k_1(\overline{M}, g)$ such that

$$D\Psi_k^\#(x) \quad \text{is injective}, \quad \forall \ x \in \overline{M}, \ k \geq k_1. \tag{4.2.25}$$

Also we have $\Psi_k^\#$ embedding ∂M. Perhaps increasing k_1 to be sure points are completely separated, we have the following result.

Proposition 4.2.3. *Let $\overline{M}$ be as in Theorem 4.2. Then there exists $k_2 = k_2(\overline{M}, g)$ such that*

$$\Psi_k^\# : \overline{M} \longrightarrow \mathbb{R}^{k+1} \quad \textit{is a } C^2\textit{-embedding}, \quad \forall \ k \geq k_2. \tag{4.2.26}$$

Having this, we can improve Proposition 4.2.2, as follows.

Corollary 4.2.4. *In the setting of Proposition 4.2.2, we have*

$$\chi : \overline{M}_1 \longrightarrow \overline{M}_2 \quad \text{is a } C^2\text{-diffeomorphism.} \tag{4.2.27}$$

Proof. Apply the construction (4.2.19)–(4.2.26) to obtain embeddings

$$\Psi_k^{\#} : \overline{M}_1 \longrightarrow \mathbb{R}^{k+1}, \quad \widetilde{\Psi}_k^{\#} : \overline{M}_2 \longrightarrow \mathbb{R}^{k+1}, \tag{4.2.28}$$

for k sufficiently large. Here we take $\Psi_k^{\#}$ as in (4.2.21), with $(\overline{M}, g) = (\overline{M}_1, g_1)$, and define

$$\widetilde{\Psi}_k^{\#}(x) = (\tilde{\psi}_0(x), \tilde{\phi}_1(x), \dots, \tilde{\phi}_k(x))$$

on $(\overline{M}_2, g_2)$ in the analogous fashion. Using Proposition 4.2.2, we see that χ pulls back the volume element of (M_2, g_2) to that of (M_1, g_1), and deduce that

$$\psi_0(x) = \tilde{\psi}_0(\chi(x)). \tag{4.2.29}$$

In concert with (4.2.15), this gives

$$\Psi_k^{\#} = \widetilde{\Psi}_k^{\#} \circ \chi. \tag{4.2.30}$$

Since both maps in (4.2.28) are C^2-embeddings (onto the same range), the result (4.2.27) follows from the implicit function theorem.

Remark 1. One can readily strengthen the regularity results on χ, given above, to regularity of class C_*^3. Details can be left to the reader.

Remark 2. The reason we do not use in this section the construction of [KKL] to recover the differential and Riemannian structure of (M, g) is the following. In [KKL] we use as coordinates some distance functions on M. However, it is well known (e.g., [DTK]) that the resulting coordinates, in principle, lose two orders of regularity, i.e., in our case are just C_*^1 regular. Clearly, it is rather inconvenient to work with such coordinates.

Remark 3. There is an analogue of Theorem 4.1 in the case of the Dirichlet boundary spectral data $(\partial M, \{\lambda_k, N\psi_1|_{\partial M}\}_{k=1}^{\infty})$. The proof is actually simpler. However, it requires some modifications because we can no longer find $\mathrm{Vol}(M(\underline{\Gamma}, \underline{t}^+, \underline{t}^-))$. Therefore, instead of $\psi_k(x)$ we find

$$\xi_k(x) = \frac{\psi_k(x)}{\psi_1(x)}, \quad x \in M^{int}, \quad k > 1.$$

However, an analog of Lemma 4.2.1 is valid and we can construct C_*^3-smooth coordinates in the vicinity of any $x \in M^{int}$ using functions ξ_k, $k > 1$. Moreover, the map $\tilde{T}_x$, given by (4.2.18) is surjective on the linear span of the functions ξ_k. Since these functions satisfy equations

$$-g^{ij}(x)\partial_i\partial_j\xi_k(x) - \tilde{b^i}(x)\partial_i\xi_k(x) = (\lambda_k - \lambda_1)\xi_k(x),$$

with the same metric tensor g^{ij} but different $\tilde{b^i}$, we can use these equations to find $g^{ij}(x)$, $x \in M^{int}$. Having found the metric inside M we can return to the analogs of equations (4.2.1), (4.2.2) and reconstruct $\psi_k(x)$, $x \in M^{int}$.

§4.3: Stabilization of the inverse problem

In this section we consider stabilization of inverse problems using geometric convergence results and apply them to the Gel'fand problem. The basic thrust of our argument provides an illustration of a general "stabilization principle for inverse problems," which we can describe abstractly as follows.

Suppose $\mathcal{M}$ is a collection of objects, one element M of which you want to identify via the observation of data $\mathcal{D}(M)$, in some set $\mathcal{B}$ of observable data. Suppose $\mathcal{M}$ and $\mathcal{B}$ have natural topologies, and the map

$$\mathcal{D} : \mathcal{M} \longrightarrow \mathcal{B}$$

has been shown to be continuous. Suppose the uniqueness problem has been solved, so you know this map is one-to-one. However, typically such a map does not have a continuous inverse. This is a standard situation in the study of inverse problems, giving rise to the phenomenon of ill-posedness. This problem is made more acute by the fact that what one measures is not exactly equal to $\mathcal{D}(M)$, but only an approximation to it.

The key to the stabilization, which is useful for a wide variety of inverse problems, requires an *a priori* knowledge that the object M one wants to identify actually belongs to a subset $\mathcal{M}_0$ of $\mathcal{M}$, and that furthermore its closure in $\mathcal{M}$, $\overline{\mathcal{M}}_0$, is *compact.* In that case the restriction

$$\mathcal{D} : \overline{\mathcal{M}}_0 \longrightarrow \mathcal{B}$$

is automatically a *homeomorphism* of $\overline{\mathcal{M}}_0$ onto its range in $\mathcal{B}$. Thus, when trying to identify the desired object M, one minimizes some measure of the difference between the calculated data $\mathcal{D}(M_j)$ and the observed data, while constraining M_j to belong to $\overline{\mathcal{M}}_0$.

Having set up the abstract stabilization principle, we show how it applies to the Gel'fand problem.

To prepare for this discussion, let us set up some notation. Denote by $\mathcal{M}_X(C^2_*)$ the set of compact, connected manifolds $\overline{M}$ with nonempty boundary X, endowed with a metric tensor in $C^2_*(\overline{M})$. Given $(\overline{M}, g) \in \mathcal{M}_X(C^2_*)$, set

$$\mathcal{D}(\overline{M}, g) = \{\lambda_j, \phi_j|_X\}_{j=1}^{\infty}, \tag{4.3.1}$$

the right side denoting the boundary spectral data of $(\overline{M}, g)$. We have

$$\mathcal{D} : \mathcal{M}_X(C^2_*) \longrightarrow \mathcal{B}_X, \tag{4.3.2}$$

where $\mathcal{B}_X$ denotes the set of sequences $\{\mu_j, \psi_j : j \geq 1\}$, with $\mu_j \in \mathbb{R}^+$, $\mu_j \nearrow +\infty$, and $\psi_j \in L^2(X)$, modulo an equivalence relation, which can be described as follows. We say $\{\mu_j, \psi_j\} \sim \{\mu_j, \tilde{\psi}_j\}$ if $\psi_j(x) = \alpha_j \tilde{\psi}_j(x)$ for some $\alpha_j \in \mathbb{C}, |\alpha_j| = 1$. More generally, if $\mu_{k_0} = \cdots = \mu_{k_1}$, we allow

$$\psi_j(x) = \sum_{k=k_0}^{k_1} \alpha_{jk} \tilde{\psi}_k(x), \quad j = k_0, \cdots, k_1, \tag{4.3.3}$$

for a unitary $l \times l$ matrix (α_{jk}), $l = k_1 - k_0 + 1$. The content of Theorem 4.1 is that the map (4.3.2) is one-to-one.

There are natural topologies one can put on the sets in (4.3.2). On $\mathcal{M}_X(C^2_*)$ one has the topology of C^r convergence, for any $r \in (1, 2)$, defined in §3. On $\mathcal{B}_X$ one has a topology described as follows. We describe when $\{\mu_j^\nu, \psi_j^\nu\} \to \{\mu_j, \psi_j\}$, as $\nu \to \infty$. First we require $\mu_j^\nu \to \mu_j$ for each j. Next, if μ_k is simple, i.e., different from μ_{k-1} and μ_{k+1}, we require $\alpha^\nu \psi_k^\nu \to \psi_j$ in $L^2(X)$, for some $\alpha^\nu \in \mathbb{C}$, $|\alpha^\nu| = 1$. More generally, if μ_k has multiplicity ℓ, say $\mu_k = \cdots = \mu_{k+\ell-1}$, we require that there exist unitary $\ell \times \ell$ matrices $(\alpha_{ij}^\nu)_{k \leq i,j \leq k+\ell-1}$ such that

$$\sum_{j=k}^{k+\ell-1} \alpha_{ij}^\nu \psi_j^\nu \longrightarrow \psi_i \ \text{ in } \ L^2(X). \tag{4.3.4}$$

Compare [K2L].

Given these topologies, it follows from standard techniques of perturbation theory (cf. [K]) that $\mathcal{D}$ is continuous in (4.3.2).

Now the map (4.3.2) is by no means invertible. This is a standard situation encountered in the study of inverse problems, giving rise to the phenomenon of ill-posedness. One wants to "stabilize" the inverse problem, showing that certain *a priori* hypotheses on the domain $(\overline{M}, g)$ put it in a subset $K \subset \mathcal{M}_X(C^2_*)$ having the property that $\mathcal{D}^{-1}$ can be shown to act continuously on the image of K. The results of §3 provide a tool to accomplish this.

Recall the class $\mathcal{M}(R_0, i_0, S_0, d_0)$ defined in §3. Given a boundary X, let $\mathcal{M}_X(R_0, i_0, S_0, d_0)$ denote the set of such manifolds with boundary X. It follows from Theorem 3.1 that $\overline{\mathcal{M}_X(R_0, i_0, S_0, d_0)}$ is compact in the C^r topology, for any $r \in (1, 2)$, and is contained in $\mathcal{M}_X(C^2_*)$. We hence give $\overline{\mathcal{M}_X(R_0, i_0, S_0, d_0)}$ the C^r topology, and we see this is independent of r, for $r \in (1, 2)$.

Combined with Theorem 4.1, these observations yield the following conditional stability of the Gel'fand inverse problem.

Theorem 4.3.1. *Given* $R_0, i_0, S_0, d_0 \in (0, \infty)$,

$$\mathcal{D} : \overline{\mathcal{M}_X(R_0, i_0, S_0, d_0)} \longrightarrow \mathcal{B}_X$$

is a homeomorphism of $\overline{\mathcal{M}_X(R_0, i_0, S_0, d_0)}$ *onto its range,* $\mathcal{B}_X(R_0, i_0, S_0, d_0)$*; hence*

$$\mathcal{D}^{-1} : \mathcal{B}_X(R_0, i_0, S_0, d_0) \longrightarrow \overline{\mathcal{M}_X(R_0, i_0, S_0, d_0)}$$

is continuous.

Thus, if $(\overline{M}_k, g_k)$, $(\overline{M}, g) \in \mathcal{M}_X(R_0, i_0, S_0, d_0)$ and the boundary spectral data of $(\overline{M}_k, g_k)$ tend to the boundary spectral data of $(\overline{M}, g)$ in $\mathcal{B}$, then, for large k, $\overline{M}_k$ are diffeomorphic to $\overline{M}$ and $g_k \to g$ in C^r, for all $r < 2$.

5. Auxiliary regularity results

Here we establish a number of elliptic regularity results, needed in the analysis in §2, which we did not find in the literature. These results tend to be variants of known results, but they differ in various key respects. Sometimes it is in the category of function space involved, e.g., coefficients of the PDE in a non-standard space, which nevertheless arose naturally in the Ricci equation analysis. In some cases we can get away with a short argument based on standard results, while in other cases we need to do more work.

In §5.1 we establish a local regularity result for an elliptic PDE with coeficients simultaneously satisfying a Hölder condition and a Besov condition. In §5.2 we establish estimates for the Dirichlet problem when the coefficients simultaneously satisfy a Hölder condition and a Sobolev condition. In §§5.3–5.5 we obtain estimates on weak solutions to Neumann boundary problems, with rough coefficients.

§5.1: Local Besov regularity

The following result establishes (2.2.8).

Proposition 5.1.1. *Assume* $u \in H^{1,2}(\mathcal{O})$ *solves the elliptic PDE*

$$\partial_j a^{jk} \partial_k u = 0 \quad on \quad \mathcal{O}. \tag{5.1.1}$$

Assume

$$a^{jk} \in C^r \cap B^s_{p,p} \tag{5.1.2}$$

with $r, s \in (0,1),\ p \in (1, \infty)$*. (One should assume* $r \le s$*.) Then, locally,*

$$u \in C^{1+r} \cap B^{1+s}_{p,p}. \tag{5.1.3}$$

Proof. That $u \in C^{1+r}$ is well known; we show $u \in B^{1+s}_{p,p}$. The proof is like that of Proposition 9.4 in Chapter III of [T2]. In particular, we use paraproducts, operators of the form $T_a u$, a tool in nonlinear PDE introduced by J.-M. Bony. A sketch of the behavior of paraproducts can be found in Chapter II of [T2].

Set $L^{\#} = \partial_j T_{a^{jk}} \partial_k$ and write (5.1.1) as

$$L^{\#} u = -\partial_j [T_{\partial_k u} a^{jk} + R(a^{jk}, \partial_k u)]. \tag{5.1.4}$$

Here $L^{\#} \in OP\mathcal{B}S^2_{1,1}$ is elliptic and, since $a^{jk} \in C^r$, by Proposition 6.1 in Chapter I of [T2] we have $E \in OPS^{-2}_{1,1}$ such that $EL^{\#} = I + F,\ F \in OPS^{-r}_{1,1}$. Then we have

$$u = -E\partial_j [T_{\partial_k u} a^{jk} + R(a^{jk}, \partial_k u)] - Fu. \tag{5.1.5}$$

Now

$$\begin{aligned} u \in C^{1+r} &\Rightarrow T_{\partial_k u}, R_{\partial_k u} \in OPS^0_{1,1} \\ &\Rightarrow T_{\partial_k u} a^{jk} + R(a^{jk}, \partial_k u) \in B^s_{p,p} \\ &\Rightarrow E\partial_j [T_{\partial_k u} a^{jk} + R(a^{jk}, \partial_k u)] \in B^{1+s}_{p,p}, \end{aligned} \tag{5.1.6}$$

the second implication using the hypothesis that $a^{jk} \in B^s_{p,p}$. Thus (5.1.5) gives

$$u = -Fu \mod B^{1+s}_{p,p}, \tag{5.1.7}$$

and since $F \in OPS^{-r}_{1,1}$ an iteration from $u \in C^{1+r}$ readily yields $u \in B^{1+s}_{p,p}$.

§5.2: L^p-Sobolev estimates for the Dirichlet problem

Let $\overline{\Omega}$ be a smooth, n-dimensional, manifold with boundary, with metric tensor

$$g_{jk} \in C^s(\overline{\Omega}) \cap H^{1,p}(\Omega), \quad s \in (0,1),\ p \in (1,\infty). \tag{5.2.1}$$

We aim to prove the following regularity result.

Proposition 5.2.1. *Let $\mathcal{O} \subset \partial\Omega$ be open. Assume $u \in \mathrm{Lip}(\overline{\Omega})$ satisfies*

$$\Delta u = f \in L^p(\Omega), \quad u\big|_{\mathcal{O}} = 0. \tag{5.2.2}$$

Also assume $p \geq n/(n-1)$. Given $z \in \mathcal{O}$, there exists a neighborhood $\overline{U}$ of z in $\overline{\Omega}$ such that

$$u \in H^{2,p}(U). \tag{5.2.3}$$

Proof. Take $\varphi \in C_0^\infty(\overline{\Omega})$ such that $\varphi \equiv 1$ near z and $\varphi \equiv 0$ on a neighborhood in $\overline{\Omega}$ of $\partial\Omega \setminus \mathcal{O}$. Suppose φ is supported in a coordinate patch. With $a^{jk} = g^{1/2} g^{jk}$, we have

$$\partial_j a^{jk} \partial_k (\varphi u) = \varphi g^{1/2} f + u(\partial_j a^{jk} \partial_k \varphi) + 2a^{jk}(\partial_k \varphi)(\partial_j u), \tag{5.2.4}$$

and if (5.2.1) holds and $u \in \mathrm{Lip}(\overline{\Omega})$, each term on the right side of (5.2.4) belongs to $L^p(\Omega)$. This reduces us to the case $\Omega = \mathbb{R}^n_+$, with $u|_{\partial\Omega} = 0$ and u having compact support in $\overline{\Omega}$, where we relabel φu as u. (This part of the argument still works even if we weaken the hypothesis $u \in \mathrm{Lip}(\overline{\Omega})$ to $u \in L^\infty(\Omega) \cap H^{1,p}(\Omega)$.)

Computing formally, we have for $1 \le \ell \le n-1$ that $u_\ell = \partial_\ell u$ satisfies

$$\partial_j a^{jk} \partial_k u_\ell = \partial_\ell (g^{1/2} f) - \partial_j \big((\partial_\ell a^{jk}) \partial_k u \big). \tag{5.2.5}$$

We have $g^{1/2} f \in L^p(\Omega)$ and $(\partial_\ell a^{jk})(\partial_k u) \in L^p(\Omega)$, under our hypotheses, so the right side of (5.2.5) belongs to $H^{-1,p}(\Omega)$, and $u_\ell|_{\partial\Omega} = 0$.

We claim this implies

$$\partial_\ell u \in H^{1,p}(\Omega), \quad 1 \le \ell \le n-1. \tag{5.2.6}$$

This follows from Theorem 5.5.5' of [Mo2], as long as $p \ge n/(n-1)$. Granted this, we have $\partial_j \partial_k u \in L^p(\Omega)$ for all j, k except $j = k = n$, and the standard trick of using the PDE (5.2.2) to solve for $\partial_n^2 u$ yields $\partial_n^2 u \in L^p(\Omega)$, completing the proof.

Noting some alternative conditions that imply the right sides of (5.2.4) and (5.2.5) belong to $L^p(\Omega)$ and $H^{-1,p}(\Omega)$, respectively, we have the following extension of Proposition 5.2.1.

Proposition 5.2.2. *The conclusion (5.2.3) holds for a solution to (5.2.2) provided*

$$g_{jk} \in C^s(\overline{\Omega}) \cap H^{1,a}(\Omega), \quad u \in L^\infty(\Omega) \cap H^{1,b}(\Omega), \tag{5.2.7}$$

with $s \in (0,1)$, $a, b \in (1, \infty)$, *and*

$$\frac{1}{a} + \frac{1}{b} \le \frac{1}{p}. \tag{5.2.8}$$

Returning to the setting of Proposition 5.2.1, we note the following simple corollary, which is directly applicable to establish (2.2.9).

Corollary 5.2.3. *The conclusion of Proposition 5.2.1 remains valid if (5.2.2) is generalized to*

$$\Delta u = f \in L^p(\Omega), \quad u\big|_{\mathcal{O}} = g \in B^{2-1/p}_{p,p}(\mathcal{O}) \cap \mathrm{Lip}(\mathcal{O}). \tag{5.2.9}$$

Proof. After perhaps shrinking $\mathcal{O}$, we can assume $g = G|_{\mathcal{O}}$ with $G \in H^{2,p}(\Omega) \cap \mathrm{Lip}(\overline{\Omega})$. Then $u - G$ solves

$$\Delta(u - G) = \tilde{f} \in L^p(\Omega), \quad u - G\big|_{\mathcal{O}} = 0, \tag{5.2.10}$$

and Proposition 5.2.1 applies to $u - G$.

§5.3: Regularity for weak solutions to the Neumann problem

Let M be a smooth, compact, connected manifold, of dimension n. Assume M has a Riemannian metric tensor that is Hölder continuous, of class C^r, for some $r \in (0,1)$. Let $\Omega \subset M$ be a connected open set, with boundary of class C^{1+r}. Actually we can assume $\partial\Omega$ is smooth, since a C^{1+r} diffeomorphism can smooth out $\partial\Omega$ while producing a new metric tensor of class C^r. Let Δ denote the Laplace operator on M. Assume $V \in L^\infty(M)$, $V \geq 0$ on M, and $V > 0$ on a set of positive measure. Consider $L = \Delta - V$.

A weak solution to the Neumann problem

$$Lu = f \quad \text{on} \quad \Omega, \qquad Nu = g \quad \text{on} \quad \partial\Omega \tag{5.3.1}$$

is an element $u \in H^1(\Omega)$ satisfying

$$\int_\Omega \langle du, d\psi \rangle \, dV = -\int_\Omega (Vu + f)\psi \, dV - \int_{\partial\Omega} g\psi \, dS, \tag{5.3.2}$$

for all $\psi \in H^1(\Omega)$. Here the volume element dV on Ω and the area element dS on $\partial\Omega$ are determined by the Riemannian metric tensor on M, as is the inner product $\langle \xi, \eta \rangle$ of 1-forms. We aim to prove the following:

Theorem 5.3.1. *Given $s \in (0,r)$, $u \in H^1(\Omega)$ satisfying (5.3.2), $p \geq n/(1-s)$,*

$$f \in L^p(\Omega), \; g \in C^s(\partial\Omega) \Longrightarrow u \in C^{1+s}(\overline{\Omega}). \tag{5.3.3}$$

Our first reduction is to show that it suffices to take $f = 0$. Indeed, extending f by 0 on $M \setminus \Omega$ we can solve $Lv = f$ on M. By Proposition 2.3 of [MT], we have $v \in C^{1+s}(\overline{\Omega})$, provided $L^p(M) \subset C_*^{-1+s}(M)$, which holds as long as $H^{1-s,p}(M) \subset C_*^0(M)$, i.e., as long as $p(1-s) \geq n$. Then $v|_\Omega$ satisfies

$$Lv = f \quad \text{on} \quad \Omega, \qquad Nv = g_0 \in C^s(\partial\Omega), \tag{5.3.4}$$

and it suffices to show that $w = u - v$ belongs to $C^{1+s}(\overline{\Omega})$.

Our next step is to look at

$$Lw = 0 \quad \text{on} \quad \Omega, \qquad Nw = g_1 \in C^s(\partial\Omega), \tag{5.3.5}$$

where $g_1 = g - g_0$, and produce a solution $w \in C^{1+s}(\overline{\Omega})$. (If $V \equiv 0$ on Ω, assume $\int_{\partial\Omega} g_1 \, dS = 0$.) Producing such a solution to (5.3.5) will prove Theorm 5.3.1, since a solution $w \in H^1(\Omega)$ is unique (up to an additive constant if $V \equiv 0$ on Ω). To produce such a solution, we use the method of layer potentials.

Thus let $E(x,y)$ be the integral kernel of $L^{-1} : H^{-1}(M) \to H^1(M)$, and define the single layer potential

$$\mathcal{S}h(x) = \int_{\partial\Omega} E(x,y)h(y)\,dS(y). \tag{5.3.6}$$

To proceed we need an analysis of $E(x,y)$. It is elementary to show that $h \in L^1(\partial\Omega) \Rightarrow \mathcal{S}h \in C^{1+r}_{\mathrm{loc}}(M \setminus \partial\Omega)$. In fact, given compact $\Sigma \subset \partial\Omega$,

$$h \in L^1(\partial\Omega),\ \operatorname{supp} h \subset \Sigma \Longrightarrow \mathcal{S}h \in C^{1+r}_{\mathrm{loc}}(M \setminus \Sigma). \tag{5.3.7}$$

This permits us to use partition of unity arguments and localize the study of $\mathcal{S}h$ to coordinate patches. We can choose local coordinates such that $\partial\Omega$ is given by $\{x : x^n = 0\}$.

As in [MT], we can write

$$E(x,y)\sqrt{g(y)} = e_0(x-y,y) + e_1(x,y), \tag{5.3.8}$$

where (if $n \geq 3$)

$$e_0(x-y,y) = C_n\Big(\sum g_{jk}(y)(x_j - y_j)(x_k - y_k)\Big)^{-(n-2)/2}. \tag{5.3.9}$$

As shown in Theorem 2.6 of [MT], we have, for each $\varepsilon > 0$,

$$\begin{aligned} |e_1(x,y)| &\leq C_\varepsilon |x-y|^{-(n-2-r+\varepsilon)}, \\ |\nabla_x e_1(x,y)| &\leq C_\varepsilon |x-y|^{-(n-1-r+\varepsilon)}. \end{aligned} \tag{5.3.10}$$

Also (2.67) of [MT] implies (for $0 < s < r$)

$$|\nabla_x e_1(x_1,y) - \nabla_x e_1(x_2,y)| \leq C_\varepsilon |x_1 - x_2|^s\, |x_1 - y|^{-(n-1-r+s+\varepsilon)}, \tag{5.3.11}$$

provided $|x_1 - x_2| \leq (1/2)|x_1 - y|$.

Given h supported in Σ, the intersection of $\partial\Omega$ with a coordinate patch, we analyze $\mathcal{S}h$ as a sum of two pieces, $\mathcal{S}h = \mathcal{S}_0 h + \mathcal{S}_1 h$, where

$$\begin{aligned} \mathcal{S}_0 h(x) &= \int_{\partial\Omega} e_0(x-y,y)g(y)^{-1/2}h(y)\,dS(y), \\ \mathcal{S}_1 h(x) &= \int_{\partial\Omega} e_1(x,y)g(y)^{-1/2}h(y)\,dS(y). \end{aligned} \tag{5.3.12}$$

Lemma 5.3.2. *Given $s \in (0,r)$, we have*

$$\mathcal{S}_1 : L^\infty(\Sigma) \longrightarrow C^{1+s}(M). \tag{5.3.13}$$

Proof. We need to show that, for $x_j \in M,\ h \in L^\infty(\partial\Omega)$, supported in Σ,

$$|\nabla \mathcal{S}_1 h(x_1) - \nabla \mathcal{S}_1 h(x_2)| \le C|x_1 - x_2|^s \,\|h\|_{L^\infty}. \tag{5.3.14}$$

There are two cases to consider.

Case I. $|x_1 - x_2| \le (1/2)\mathrm{dist}(x_1, \Sigma)$.
Then use (5.3.11) to get (5.3.14).

Case II. $|x_1 - x_2| \ge (1/2)\mathrm{dist}(x_1, \Sigma)$.
Set $\mathcal{O} = \{y \in \Sigma : |x_1 - y| \le 4|x_1 - x_2|\}$. Use (5.3.10) for $y \in \mathcal{O}$ to analyze separately $\nabla_x \int_{\mathcal{O}} e_0(x_j - y, y) g(y)^{-1/2} h(y)\, dS(y)$, and use (5.3.11) for $y \in \Sigma \setminus \mathcal{O}$ to complete the analysis of the left side of (5.3.14).

Lemma 5.3.3. *Given $s \in (0,r)$, we have*

$$\mathcal{S}_0 : C^s(\partial\Omega) \longrightarrow C^{1+s}(\overline{\Omega}). \tag{5.3.15}$$

Proof. This is a standard layer potential estimate. One has

$$\|\nabla \mathcal{S}_0 h\|_{L^\infty(\Omega)} \le C\|h\|_{C^s(\partial\Omega)}, \tag{5.3.16}$$

and

$$|\partial^2 \mathcal{S}_0 h(x', x_n)| \le C|x_n|^{s-1} \,\|h\|_{C^s(\partial\Omega)}, \tag{5.3.17}$$

which gives

$$|\nabla \mathcal{S}_0 h(x_1) - \nabla \mathcal{S}_0 h(x_2)| \le C|x_1 - x_2|^s \,\|h\|_{C^s(\partial\Omega)}, \quad x_j \in \overline{\Omega}; \tag{5.3.18}$$

compare Proposition 8.7 in Chapter 13 of [T1].

The next step in solving (5.3.5) in the form $w = \mathcal{S}h$ is to analyze $N\mathcal{S}h|_{\partial\Omega}$. We have the standard formula

$$N\mathcal{S}h\big|_{\partial\Omega} = \Big(-\frac{1}{2}I + K^*\Big)h, \tag{5.3.19}$$

with

$$K^*h(x) = \mathrm{PV}\int_{\partial\Omega} N_x E(x,y) h(y)\, dS(y), \quad x \in \partial\Omega. \tag{5.3.20}$$

Compare (2.81) of [MT] for such a formula in the more general context of a Lipschitz boundary $\partial\Omega$. We can break K^* into two pieces, $K^* = K_0^* + K_1^*$, using (5.3.8):

$$K_0^* h(x) = \mathrm{PV} \int_{\partial\Omega} N_x e_0(x-y, y) g(y)^{-1/2} h(y)\, dS(y), \tag{5.3.21}$$

and

$$K_1^* h(x) = \int_{\partial\Omega} N_x e_1(x, y) g(y)^{-1/2} h(y)\, dS(y). \tag{5.3.22}$$

By Lemma 5.3.2 we have $K_1^* : L^\infty(\partial\Omega) \to C^s(\partial\Omega)$. As for K_0^*, it is a pseudodifferential operator with double symbol, of the sort studied in Chapter I, §9 of [T2], with symbol

$$a(x, y, \xi) \in C^r S^0_{1,0}. \tag{5.3.23}$$

Furthermore, the appearance of N_x in (5.3.21) produces the following important cancellation effect:

$$a(x, x, \xi) = 0. \tag{5.3.24}$$

(This is part of what makes analysis on domains with C^{1+r} boundary easier than analysis on Lipschitz domains.) Hence, by Proposition 9.15 of [T2], Chapter I, we have $K_0^* : L^\infty(\partial\Omega) \to C^s(\partial\Omega)$, for all $s < r$. In summary,

$$K^* : L^\infty(\partial\Omega) \longrightarrow C^s(\partial\Omega), \quad \forall\, s < r, \tag{5.3.25}$$

and hence

$$K^* : C^s(\partial\Omega) \longrightarrow C^s(\partial\Omega) \ \text{ is compact, for all } \ s \in (0, r). \tag{5.3.26}$$

Thus

$$-\frac{1}{2} I + K^* : C^s(\partial\Omega) \longrightarrow C^s(\partial\Omega) \ \text{ is Fredholm, of index } \ 0. \tag{5.3.27}$$

Using this we can prove the following.

Proposition 5.3.4. *Let $g_1 \in C^s(\partial\Omega)$, $0 < s < r$. If $V > 0$ somewhere (i.e., on a set of positive measure) on Ω, then (5.3.5) has a unique solution $w = \mathcal{S}h \in C^{1+s}(\overline{\Omega})$, where*

$$\Big(-\frac{1}{2} I + K^*\Big) h = g_1, \quad h \in C^s(\partial\Omega). \tag{5.3.28}$$

If $V \equiv 0$ on Ω, then (5.3.5) has a solution $w \in C^{1+s}(\overline{\Omega})$ if and only if $\int_{\partial\Omega} g_1 \, dS = 0$, and such w is unique up to an additive constant.

Proof. This is a standard argument in layer potential theory. One shows that, if $V > 0$ somewhere on Ω, then $-(1/2)I + K^*$ is injective on $C^s(\partial\Omega)$, hence, by (5.3.27), bijective. If $V \equiv 0$ on Ω, then $-(1/2)I+K^*$ has a one-dimensional kernel in $C^s(\partial\Omega)$, and the constant function 1 annihilates its range. Compare the treatment of Theorem 3.4 in [MT], carried out in the more general context of a Lipschitz boundary. Once one solves (5.3.28) for $h \in C^s(\partial\Omega)$, the fact that $w = \mathcal{S}h$ solves (5.3.5) and belongs to $C^{1+s}(\overline{\Omega})$ follows from the previous analysis.

With this result, the proof of Theorem 5.3.1 is complete.

§5.4: Local regularity for the Neumann problem

It is useful to strengthen the global regularity result of §5.3 to a local regularity result for a weak solution to the Neumann problem. With M, Ω, and L as in §5.3, let $\mathcal{O}$ be an open subset of $\partial\Omega$ and suppose u is a weak solution to

$$Lu = f \quad \text{on} \quad \Omega, \qquad Nu\big|_{\mathcal{O}} = g. \tag{5.4.1}$$

That is to say, we assume $u \in H^1(\Omega)$ and that (5.3.2) holds for all $\psi \in H^1(\Omega)$ such that $\psi|_{\partial\Omega}$ vanishes on a neighborhood of $\partial\Omega \setminus \mathcal{O}$ in $\partial\Omega$. We prove the following.

Theorem 5.4.1. *Let u be a weak solution to (5.4.1). Assume $u \in H^{1,q}(\Omega)$ with $q \geq 2$ and either $q > 1/r$ or $q \geq p$. (Recall $g_{jk} \in C^r$.) As in Theorem 5.3.1, assume $0 < s < r < 1$ and $p \geq n/(1-s)$. Then*

$$f \in L^p(\Omega), \ g \in C^s(\mathcal{O}) \Longrightarrow u \in C^{1+s}(\Omega \cup \mathcal{O}). \tag{5.4.2}$$

Proof. Take $p_0 \in \mathcal{O}$ and pick $\varphi \in C_0^\infty(M)$, equal to 1 on a neighborhood of p_0, such that $\varphi = 0$ on a neighborhood of $\partial\Omega \setminus \mathcal{O}$ in M. Let $v = \varphi u|_\Omega$, so $v \in H^{1,q}(\Omega)$. We seek to establish extra regularity of v. It is readily verified that v is a (global) weak solution of

$$Lv = \tilde{f} \quad \text{on} \quad \Omega, \qquad Nv = \tilde{g} \quad \text{on} \quad \partial\Omega, \tag{5.4.3}$$

with

$$\begin{aligned} \tilde{f} &= \varphi f + 2\langle d\varphi, du \rangle + u\Delta\varphi, \\ \tilde{g} &= \varphi g + u(N\varphi)\big|_{\partial\Omega}. \end{aligned} \tag{5.4.4}$$

We have

$$\tilde{f} \in L^{p\wedge q}(\Omega), \quad \tilde{g} \in C^s(\partial\Omega) + C^r(\partial\Omega) \cdot B^{1-1/q}_{q,q}(\partial\Omega). \tag{5.4.5}$$

Extend $\tilde{f}$ by 0 on $M \setminus \Omega$ and set $v_1 = L^{-1}\tilde{f}$. By (2.16) of [MT], we have

$$L^{p\wedge q}(M) \subset H^{\rho-1,\tilde{p}}(M) \Longrightarrow v_1 \in H^{\rho+1,\tilde{p}}(M), \quad \rho = r - \varepsilon. \tag{5.4.6}$$

In fact, if $q \geq p$, we can use Proposition 2.3 of [MT] as in the beginning of the proof of Theorem 5.3.1, and say $v_1 \in C^{1+s}(M)$. If $q < p$, then we can take $\tilde{p} > q$ in (5.4.6).

Now $v_2 = v - v_1|_\Omega$ is a weak solution to

$$Lv_2 = 0 \text{ on } \Omega, \quad Nv_2 = g_2 = \tilde{g} - Nv_1, \tag{5.4.7}$$

and

$$Nv_1 \in C^r(\partial\Omega) \cdot B^{\rho-1/\tilde{p}}_{\tilde{p},\tilde{p}}(\partial\Omega), \tag{5.4.8}$$

if $q < p$, and in $C^s(\partial\Omega)$ if $q \geq p$. Note that $\tilde{p}\rho > q(r-\varepsilon)$ can be assumed to be > 1 by the hypothesis $rq > 1$, if we take $\varepsilon > 0$ small enough. At this point we have $g_2 \in L^{\tilde{q}_2}(\partial\Omega)$, with $\tilde{q}_2 > q$. From this we can deduce

$$v_2 \in H^{1,\tilde{q}_2}(\Omega), \quad \tilde{q}_2 > q. \tag{5.4.9}$$

The result (5.4.9) is a special case of much stronger known results; let us sketch the proof. First, parallel to (5.3.26), we have

$$K^* : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \text{ is compact, for all } p \in (1,\infty). \tag{5.4.10}$$

Indeed, the compactness of K_0^* follows from Proposition 9.5 in Chapter I of [T2], together with (5.3.23)–(5.3.24), and the compactness of K_1^* follows from the estimates in (5.3.10). Having (5.4.10), we see that

$$-\frac{1}{2}I + K^* : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega) \text{ is Fredholm, of index } 0, \tag{5.4.11}$$

for each $p \in (1,\infty)$. Then an argument as in the proof of Proposition 5.3.4 yields

$$v_2 = \mathcal{S}h_2, \quad h_2 = \Big(-\frac{1}{2}I + K^*\Big)^{-1} g_2 \in L^{\tilde{q}_2}(\partial\Omega), \tag{5.4.12}$$

up to an additive constant if $V \equiv 0$ on Ω. Now we have the non-tangential maximal function estimate

$$\|(\nabla\mathcal{S}h_2)^*\|_{L^{\tilde{q}_2}(\partial\Omega)} \leq C_{\tilde{q}_2}\|h_2\|_{L^{\tilde{q}_2}(\partial\Omega)}, \quad 1 < \tilde{q}_2 < \infty, \tag{5.4.13}$$

which is stronger than (5.4.9). The estimate (5.4.13), in the setting of (5.4.6) for a C^r-metric tensor, is proven in (2.77) of [MT], in the more general context of a Lipschitz domain Ω.

Now (5.4.6) and (5.4.9) together give

$$v = v_1 + v_2 \in H^{1,q_2}(\Omega), \quad q_2 > q. \tag{5.4.14}$$

Now we go back to u. After shrinking Ω to a smaller neighborhood of p_0, we can replace our hypothesis $u \in H^{1,q}(\Omega)$ by $u \in H^{1,q_2}(\Omega)$. Iterating this argument yields $u \in H^{1,q_\nu}(\Omega)$ with $q < q_2 < q_3 < \cdots$. After a finite number of iterations we reach a point where Theorem 5.3.1 is applicable to $v = \varphi u$, and Theorem 5.4.1 is proven.

§5.5: Neumann data in L^s

We produce more regularity results on weak solutions to

$$Lu = f \quad \text{on} \quad \Omega, \qquad Nu = g \quad \text{on} \quad \partial\Omega, \tag{5.5.1}$$

starting with the following. As before, assume $g_{jk} \in C^r$ for some $r > 0$.

Proposition 5.5.1. *Let $u \in H^{1,2}(\Omega)$ be a weak solution to (5.5.1). Assume*

$$f \in L^p(\Omega), \quad g \in L^s(\partial\Omega), \quad p > n, \ 1 < s < \infty. \tag{5.5.2}$$

Then we have

$$(\nabla u)^* \in L^s(\partial\Omega), \quad u \in H^{1,ns/(n-1)}(\Omega). \tag{5.5.3}$$

Proof. Since $p > n$, we can solve $Lv = f$ (with f extended by 0) on a neighborhood $\mathcal{O}$ of $\overline{\Omega}$, with $v \in C^{1+\sigma}(\mathcal{O})$, $\sigma > 0$. This reduces our consideration to the case $f = 0$ in (5.5.2). Then u is given by

$$u = \mathcal{S}h, \tag{5.5.4}$$

with

$$\Big(-\frac{1}{2}I + K^*\Big)h = g, \quad h \in L^s(\partial\Omega). \tag{5.5.5}$$

The fact that

$$h \in L^s(\partial\Omega) \Longrightarrow (\nabla\mathcal{S}h)^* \in L^s(\partial\Omega) \tag{5.5.6}$$

is established in the context of a Hölder continuous metric tensor (and in the more general context of a Lipschitz domain) in [MT].

The last part of (5.5.3) follows from the mapping property

$$\mathcal{S} : L^s(\partial\Omega) \longrightarrow H^{1,ns/(n-1)}(\Omega). \tag{5.5.7}$$

This is demonstrated, in the more general context of a Lipschitz domain, in [MT2]. We describe here the basic structure of the argument. We write $\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_1$, as in (5.3.12). Harmonic analysis techniques are brought to bear to establish

$$\mathcal{S}_0 : L^s(\partial\Omega) \longrightarrow H^{1,ns/(n-1)}(\Omega). \tag{5.5.8}$$

(Such a result is easier for a smooth domain than for a Lipschitz domain.) As for $\mathcal{S}_1$, we already have in (5.3.13) that

$$\mathcal{S}_1 : L^\infty(\partial\Omega) \longrightarrow C^{1+\sigma}(\overline{\Omega}), \tag{5.5.9}$$

for some $\sigma > 0$. Meanwhile the estimate (5.3.10) on $\nabla_x e_1(x, y)$ is more than adequate to give

$$\mathcal{S}_1 : L^1(\partial\Omega) \longrightarrow H^{1,n/(n-1)}(\Omega), \tag{5.5.10}$$

and then interpolation gives more than

$$\mathcal{S}_1 : L^s(\partial\Omega) \longrightarrow H^{1,ns/(n-1)}(\Omega). \tag{5.5.11}$$

We now establish a useful local regularity result.

Proposition 5.5.2. *Let $\mathcal{O}$ be an open subset of $\partial\Omega$ and assume u is a weak solution to*

$$Lu = f \quad on \quad \Omega, \quad Nu\big|_{\mathcal{O}} = g. \tag{5.5.12}$$

Take $p > n$, $s \in (1, \infty)$, and assume

$$u \in H^{1,p}(\Omega), \quad f \in L^p(\Omega), \quad g \in L^s(\mathcal{O}). \tag{5.5.13}$$

Then each $p \in \mathcal{O}$ has a neighborhood $\overline{U}$ in $\overline{\Omega}$ such that

$$u \in H^{1,ns/(n-1)}(U). \tag{5.5.14}$$

Proof. As in the proof of Theorem 5.4.1, we consider $v = \varphi u$, which is a global weak solution of (5.4.3), with $\tilde{f}, \tilde{g}$ given by (5.4.4). Under our current hypotheses we have

$$\tilde{f} \in L^p(\Omega), \quad \tilde{g} \in L^s(\partial\Omega), \tag{5.5.15}$$

so Proposition 5.5.1 gives $v \in H^{1,ns/(n-1)}(\Omega)$, and Proposition 5.5.2 is proven.

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