GEOMETRIC ASPECTS OF THE ADS/CFT CORRESPONDENCE

MICHAEL T. ANDERSON

Abstract. We discuss classical gravitational aspects of the AdS/CFT correspondence, with the aim of obtaining a rigorous (mathematical) understanding of the semi-classical limit of the gravitational partition function. The paper surveys recent progress in the area, together with a selection of new results and open problems.

Contents

0. Introduction

In this paper we discuss certain geometrical aspects of the AdS/CFT or Maldacena correspondence [42], [31], [47]. From a physics point of view, only the purely classical gravitational aspects of the correspondence on the AdS side are considered; thus no scalar, p-form or gauge fields, and no supergravity or string corrections are considered. On the CFT side, we only take account of the conformal structure on the boundary, and its corresponding stress-energy tensor. The discussion is also confined, by and large, to the Euclidean (or Riemannian) version of the correspondence. On the other hand, within this modest and restricted framework, there is quite a bit that is now known on a rigorous mathematical basis.

Broadly speaking, the AdS/CFT correspondence states the existence of a duality equivalence between gravitational theories (such as string or M theory) on anti-de Sitter spaces M and conformal field theories on the boundary at conformal infinity ∂M. In the restricted semi-classical framework above, the correspondence as formulated by Witten [47] states that

$$Z_{CFT}[\gamma] = \sum e^{-I(g)}, \tag{0.1}$$

where Z_{CFT} is the partition function of a CFT attached to a conformal structure $[\gamma]$ on ∂M, and $I(g)$ is the renormalized Einstein-Hilbert action of an Einstein metric g on M with conformal infinity $[\gamma]$. The sum is over all manifolds and metrics (M, g) with the given boundary data $(\partial M, [\gamma])$.

The main focus of the paper is on developing a framework in which the right side of (0.1) can be given a rigorous understanding. We survey existing work on geometrical aspects of the correspondence related to this issue, and discuss several new results, mostly in the later sections.

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Section 1 discusses general background information on the structure of conformally compact Einstein metrics, while §2 and §3 discuss uniqueness and existence issues for the Dirichlet-Einstein problem respectively. Section 4 explains the role of positive scalar curvature boundary data in the existence theory. In §5 it is shown that the correspondence becomes much more explicit in the case of self-dual or anti-self-dual metrics on 4-manifolds. Finally in §6 we discuss the continuation to de Sitter-type metrics, and the construction of globally self-similar solutions to the vacuum Einstein equations. Throughout the text, a number of open questions and problems are presented.

1. Conformally compact Einstein metrics

Let M be the interior of a compact $n+1$ dimensional manifold $\bar{M}$ with non-empty boundary ∂M. A complete Riemannian metric g on M is $C^{m,\alpha}$ conformally compact if there is a defining function ρ on $\bar{M}$ such that the conformally equivalent metric

$$\widetilde{g} = \rho^2 g \tag{1.1}$$

extends to a $C^{m,\alpha}$ metric on the compactification $\bar{M}$. A defining function ρ is a smooth, non-negative function on $\bar{M}$ with $\rho^{-1}(0) = \partial M$ and $d\rho \neq 0$ on ∂M. The induced metric $\gamma = \widetilde{g}|_{\partial M}$ is called the boundary metric associated to the compactification $\widetilde{g}$. There are many possible defining functions, and hence many conformal compactifications of a given metric g, and so only the conformal class $[\gamma]$ of γ on ∂M, called conformal infinity, is uniquely determined by (M, g). Any manifold M carries many conformally compact metrics but we are mainly interested in Einstein metrics g, normalized so that

$$Ric_g = -ng. \tag{1.2}$$

It is well-known, and easily seen, that C^2 conformally compact Einstein metrics are asymptotically hyperbolic (AH), in that $|K_g + 1| = O(\rho^2)$, where K_g denotes sectional curvature of (M, g), and these two notions will be used interchangeably.

A compactification $\bar{g} = \rho^2 g$ as in (1.1) is called geodesic if $\rho(x) = dist_{\bar{g}}(x, \partial M)$. These compactifications are especially useful for computational purposes, and for the remainder of the paper we work only with geodesic compactifications. Each choice of boundary metric $\gamma \in [\gamma]$ determines a unique geodesic defining function ρ associated to (M, g).

The Gauss Lemma gives the splitting

$$\bar{g} = d\rho^2 + g_\rho, \quad g = \rho^{-2}(d\rho^2 + g_\rho), \tag{1.3}$$

where g_ρ is a curve of metrics on ∂M. The Fefferman-Graham expansion [27] is a formal Taylor-type series expansion for the curve g_ρ. The exact form of the expansion depends on whether n is odd or even. For n odd, one has

$$g_\rho \sim g_{(0)} + \rho^2 g_{(2)} + ... + \rho^{n-1} g_{(n-1)} + \rho^n g_{(n)} + \rho^{n+1} g_{(n+1)} + ... \tag{1.4}$$

This expansion is even in powers of ρ up to order $n-1$. The coefficients $g_{(2k)}$, $k \leq (n-1)/2$ are locally determined by the boundary metric $\gamma = g_{(0)}$; they are explicitly computable expressions in the curvature of γ and its covariant derivatives. The term $g_{(n)}$ is transverse-traceless, i.e.

$$tr_\gamma g_{(n)} = 0, \quad \delta_\gamma g_{(n)} = 0, \tag{1.5}$$

but is otherwise undetermined by γ; it depends on global aspects of the AH Einstein metric (M, g). If n is even, one has

$$g_\rho \sim g_{(0)} + \rho^2 g_{(2)} + ... + \rho^{n-2} g_{(n-2)} + \rho^n g_{(n)} + \rho^n \log\rho \, h + \rho^{n+1} g_{(n+1)} + ... \tag{1.6}$$

Again the terms $g_{(2k)}$ up to order $n-2$ are explicitly computable from the boundary metric γ, as is the transverse-traceless coefficient h of the first $\log\rho$ term. The term h is an important term relating to the CFT on ∂M; it is the metric variation of the conformal anomaly, cf. [22]. In odd dimensions, this always vanishes. The term $g_{(n)}$ satisfies

$$tr_{\gamma}g_{(n)} = \tau, \quad \delta_{\gamma}g_{(n)} = \delta, \tag{1.7}$$

where τ and δ are explicitly determined by the boundary metric γ and its derivatives, but $g_{(n)}$ is otherwise undetermined by γ and as above depends on the global geometry of (M, g). In addition $(\log\ \rho)^k$ terms appear at order $> n$ when $h \neq 0$. Note also that these expansions depend on the choice of boundary metric $\gamma \in [\gamma]$. Transformation properties of the coefficients $g_{(i)}$, $i \leq n$, and h as γ varies over $[\gamma]$ are also readily computable, cf. [22].

Mathematically, these expansions are formal, obtained by compactifiying the Einstein equations and taking iterated Lie derivatives of $\bar{g}$ at $\rho = 0$;

$$g_{(k)} = \frac{1}{k!}\mathcal{L}_T^{(k)}\bar{g}, \tag{1.8}$$

where $T = \bar{\nabla}\rho$. If $\bar{g} \in C^{m,\alpha}(\bar{M})$, then the expansions hold up to order $m + \alpha$. However, boundary regularity results are needed to ensure that if an AH Einstein metric g with boundary metric γ satisfies $\gamma \in C^{m,\alpha}(\partial M)$, then the compactification $\bar{g} \in C^{m,\alpha}(\bar{M})$.

In both cases n even or odd, the Einstein equations determine all higher order coefficients $g_{(k)}$, $k > n$, in terms of $g_{(0)}$ and $g_{(n)}$, so that an AH Einstein metric is formally determined by $g_{(0)}$ and $g_{(n)}$. The term $g_{(0)}$ corresponds to Dirichlet boundary data on ∂M, while $g_{(n)}$ corresponds to Neumann boundary data, (in analogy with the scalar Laplace operator). Thus, on AH Einstein metrics, the correspondence

$$g_{(0)} \to g_{(n)} \tag{1.9}$$

is analogous to the Dirichlet-to-Neumann map for harmonic functions. As discussed below, the term $g_{(n)}$ corresponds essentially to the stress-energy tensor of the dual CFT on ∂M. However, the map (1.9) is only well-defined per se if there is a unique AH Einstein metric with boundary data $\gamma = g_{(0)}$.

Understanding the correspondence (1.9) is one of the key issues in the AdS/CFT correspondence, restricted to the setting of the Introduction. Formally, knowing $g_{(0)}$ and $g_{(n)}$ allows one to construct the bulk gravitational field, that is the AH Einstein metric via the expansion (1.4) or (1.6). However, one needs to know that the expansions (1.4) or (1.6) actually converge to g_ρ for this to be of any use.

More significantly, if n is odd, given any real-analytic metric $g_{(0)}$ and symmetric bilinear form $g_{(n)}$ on ∂M, satisfying (1.5), there exists a unique C^ω conformally compact Einstein metric g defined in a thickening $\partial M \times [0, \varepsilon)$ of ∂M. In particular, the expansion (1.4) converges to g_ρ. A similar statement holds when n is even, cf. [6] for $n = 3$ and [39] or [45] for general n. Thus, the terms $g_{(0)}$ and $g_{(n)}$ may be specified arbitrarily and independently of each other, subject only to the constraint (1.5) or (1.7), to give "local" AH Einstein metrics. This illustrates the global nature of the correspondence (1.9).

Next, we turn to the structure of the moduli space of AH Einstein metrics on a given $(n+1)$-manifold M. Let $E = E^\infty$ be the space of AH Einstein metrics on M which admit a C^∞ compactification $\bar{g}$ as in (1.1). When n is even, we assume here that C^∞ means C^∞ polyhomogeneous, i.e. $g_\rho = \phi(\rho, \rho^n \log \rho)$, where ϕ is a C^∞ function of the two variables. The topology on E is the C^∞ (polyhomogeneous) topology on metrics on $\bar{M}$. Let $\mathcal{E} = E/\mathrm{Diff}_1(\bar{M})$, where $\mathrm{Diff}_1(\bar{M})$ is the group of C^∞ diffeomorphisms of $\bar{M}$ inducing the identity on ∂M, acting on E in the usual way by pullback. (The CFT on ∂M is a gauge-type theory, and so is diffeomorphism covariant, not diffeomorphism invariant; hence, it is natural to require diffeomorphisms fixing ∂M).

As boundary data, let $Met(\partial M) = Met^{\infty}(\partial M)$ be the space of C^{∞} metrics on ∂M and $\mathcal{C} = \mathcal{C}(\partial M)$ the corresponding space of pointwise conformal classes. Occasionally we will also work with the spaces of real-analytic metrics C^{ω}, or $C^{m,\alpha}$.

There is a natural boundary map,

$$\Pi : \mathcal{E} \to \mathcal{C}, \quad \Pi[g] = [\gamma], \tag{1.10}$$

which takes an AH Einstein metric g on M to its conformal infinity on ∂M.

One then has the following general result on the structure of $\mathcal{E}$ and the map Π, building on previous work of Graham-Lee [30] and Biquard [13].

Theorem 1.1. [5,6] *Let M be a compact, oriented $(n+1)$-manifold with boundary ∂M. If $\mathcal{E}$ is non-empty, then $\mathcal{E}$ is a smooth infinite dimensional manifold. Further, the boundary map*

$$\Pi : \mathcal{E} \to \mathcal{C}$$

is a C^{∞} smooth Fredholm map of index 0. Thus the derivative $D\Pi$ has finite dimensional kernel and cokernel, has closed range, and

$$dim Ker D\Pi = dim Coker D\Pi.$$

Implicit in Theorem 1.1 is the boundary regularity statement that an AH Einstein metric with C^{∞} conformal infinity has a C^{∞} (polyhomogeneous) geodesic compactification. When $n+1 = 4$, this boundary regularity has been proved in [4], [6], including the cases of C^{ω} and $C^{m,\alpha}$ regularity. In dimensions $n+1 > 4$, boundary regularity has recently been proved by Chruściel et al. [19] in the C^{∞} case. Moreover, when $\gamma \in C^{\omega}$, Kichenassamy [39] and Rendall [45] have proved that the expansions (1.4) and (1.6) converge to g_{ρ}.

In addition, the regular points of Π, that is the metrics in $\mathcal{E}$ where $D\Pi$ is an isomorphism, are open and dense in $\mathcal{E}$. Hence, if $\mathcal{E} \neq \emptyset$, then $\Pi(\mathcal{E})$ has non-empty interior in $\mathcal{C}$. Thus, if M carries some AH Einstein metric, then it also carries a large set of them, parametrized at least by an open set in $\mathcal{C}$. The results above all hold with E in place of $\mathcal{E}$, without essential changes.

A basic issue in this area is the Dirichlet problem for AH Einstein metrics: given the topological data $(M, \partial M)$, and a conformal class $[\gamma] \in \mathcal{C}$, does there exist a unique AH Einstein metric g on M, with conformal infinity $[\gamma]$? In terms of the boundary map Π, global existence is equivalent to the surjectivity of Π, while uniqueness is equivalent to the injectivity of Π.

For Riemannian metrics, the Einstein-Hilbert action is (usually) given by

$$I = -\frac{1}{16\pi G}\int_M (R - 2\Lambda) dv - \frac{1}{8\pi G}\int_{\partial M} H dA, \tag{1.11}$$

where R is the scalar curvature, Λ the cosmological constant and H is the mean curvature; (sometimes I is replaced by $-I$). In the following, units are chosen so that $16\pi G = 1$.

Critical points of I satisfy the Einstein equations

$$Ric - \frac{R}{2} g + \Lambda g = 0, \tag{1.12}$$

and in the normalization (1.2), $\Lambda = \frac{1}{2}\frac{n-1}{n+1}R = -\frac{1}{2}n(n-1)$. However, both terms in (1.11) are infinite on metrics in $\mathcal{E}$. A number of schemes have been proposed by physicists to obtain a finite expression for I on $\mathcal{E}$. Among these, the most natural is the holographic renormalization, c.f. [47], [34], [12], [22], described as follows. Given a fixed geodesic defining function ρ for g, let $B(\varepsilon) = \{x \in (M, g) : \rho(x) \geq \varepsilon\}$. If n is odd, from the expansion (1.4), one has an expansion of the volume of $B(\varepsilon)$ in the form

$$vol B(\varepsilon) = v_{(0)}\varepsilon^{-n} + ... + v_{(n-1)}\varepsilon^{-1} + V + o(1), \tag{1.13}$$

where the terms $v_{(k)}$ are explicitly computable from $(\partial M, \gamma)$. For an Einstein metric as in (1.2), $R - 2\Lambda = -2n$, so that

$$-\int_{B(\varepsilon)} (R - 2\Lambda)dv = 2n(v_{(0)}\varepsilon^{-n} + ... + v_{(n-1)}\varepsilon^{-1} + V) + o(1). \tag{1.14}$$

A similar expansion of the boundary integral in (1.11) over $S(\varepsilon)$ has a form similar to (1.13), but with no constant term V. In fact local and covariant counterterms $v_{(k)}(\varepsilon)$ for the integral in (1.14), and the corresponding boundary integral, can be constructed in terms of the metric γ_ε induced on the finite cut-off $S(\varepsilon) = \partial B(\varepsilon)$. These counterterms $v_{(k)}(\varepsilon)$, when suitably rescaled, converge to the counterterms $v_{(k)}$ at infinity; this is one important aspect of the AdS/CFT correspondence.

Thus, define the renormalized action I^{ren} by

$$I^{ren}(g) = 2nV. \tag{1.15}$$

Similarly, if n is even, the expansion (1.6) gives

$$volB(\varepsilon) = v_{(0)}\varepsilon^{-n} + ... + v_{(n-2)}\varepsilon^{-2} + L\log\varepsilon + V + o(1), \tag{1.16}$$

and again the terms $v_{(k)}$ as well as L are explicitly computable from $(\partial M, \gamma)$. The coefficient L, equal to the integral of $trg_{(n)}$ over ∂M, agrees with the conformal anomaly of the dual CFT on ∂M in all known cases, [34]. The renormalized action is again defined by (1.15).

When n is odd, I^{ren} is independent of the choice of boundary metric $\gamma \in [\gamma]$, and thus I^{ren} is a smooth functional on the moduli space $\mathcal{E}$. When n is even, this is not the case; I^{ren} does depend on the choice of boundary metric, and so only gives a smooth functional on the space E, (or E quotiented out by diffeomorphisms equal to the identity to order n at ∂M). On the other hand, L is independent of the choice of boundary metric.

Consider the variation of I^{ren} at a given $g \in E$, i.e.

$$dI^{ren}(\dot{g}) = \frac{d}{dt}I^{ren}(g + t\dot{g}), \tag{1.17}$$

where $\dot{g}$ is tangent to E. Thus, dI^{ren} may be considered as a 1-form on the manifold E (or $\mathcal{E}$ when n is odd). In general, the differential dI^{ren} is the stress-energy (or energy-momentum) tensor of the action. Since Einstein metrics are critical points of I or I^{ren}, it is clear that dI^{ren} must be supported on ∂M. In [22] it is proved that

$$dI^{ren} = -\frac{n}{2}g_{(n)} + r_{(n)}, \tag{1.18}$$

where $r_{(n)} = 0$ if n is odd, and is explicitly determined by $\gamma = g_{(0)}$ if n is even. Thus

$$dI^{ren}(\dot{g}) = -\frac{n}{2}\int_{\partial M} \langle g_{(n)} + r_{(n)}, \dot{g}_{(0)}\rangle dv_\gamma, \tag{1.19}$$

where $\dot{g}_{(0)}$ is the variation of the boundary metric γ induced by $\dot{g}$. On the gravitational side, the 1-form dI^{ren} is the (Brown-York) quasi-local stress-energy tensor T; via the AdS/CFT correspondence, this corresponds to the expectation value $\langle T\rangle$ of the stress-energy tensor of the dual CFT on ∂M.

In dimension 4, the renormalized action or volume can be given a quite different interpretation. Namely, by means of the Chern-Gauss-Bonnet theorem, one finds, on (M^4, g),

$$\int_M |W|^2 = 8\pi^2\chi(M) - I^{ren}, \tag{1.20}$$

where $\chi(M)$ is the Euler characteristic and W is the Weyl curvature of (M, g), cf. [3]. Thus the renormalized action, which involves only the scalar curvature and volume, in fact controls much more; it controls L^2 norm of the Weyl curvature W on-shell, i.e. on $\mathcal{E}$.

Since the left side of (1.20) is non-negative, an immediate consequence is that the renormalized action is uniformly bounded above on the full space $\mathcal{E}$, depending only on a lower bound for $\chi(M)$. Moreover, I^{ren} has an absolute maximum on hyperbolic metrics. Thus for example on the 4-ball B^4, the Poincaré metric has the largest action among all AH Einstein metrics on B^4. It is an interesting open question whether such a result also holds in higher dimensions.

2. Uniqueness Issue.

It is not unreasonable to believe that there should be some relation between the existence and uniqueness problems for the Einstein-Dirichlet problem. For example, the usual Fredholm alternative relates these two issues at the linearized level. In this section, we discuss the uniqueness question on the basis of a selection of examples.

Examples 2.1. The first example of non-uniqueness was that found by Hawking-Page [32] in their analysis of the AdS Schwarzschild, or AdS S^2 black hole metric. In general dimensions, this is a curve of AH Einstein metrics on $M = \mathbb{R}^2 \times S^{n-1}$ given by

$$g_m = V^{-1}dr^2 + Vd\theta^2 + r^2 g_{S^{n-1}(1)}, \tag{2.1}$$

where

$$V(r) = 1 + r^2 - \frac{2m}{r^{n-2}}. \tag{2.2}$$

Here $r \in [r_+, \infty)$, where r_+ is the largest root of V, and the circular parameter $\theta \in [0, \beta]$, where $\beta = 4\pi r_+/(nr_+^2 + (n-2))$. It is easy to see that the conformal infinity of g_m is given by the conformal class of the product metric on $S^1(\beta) \times S^{n-1}(1)$. The action of g_m is given by

$$I^{ren}(g_m) = -\beta\omega_{n-1}(r_+^n - r_+^{n-2} + c_n), \tag{2.3}$$

where $c_n = 0$ if n is odd, and $c_n = (-1)^{n/2}\frac{(n-1)!!^2}{n!}$ if n is even, with $\omega_{n-1} = volS^{n-1}(1)$, cf. [24]. The stress-energy tensor of g_m is

$$dI^{ren} = -(r_+^n + r_+^{n-2} + \frac{2c_n}{n-1})diag(1-n, 1, ...1), \tag{2.4}$$

cf. [25]. As a function of $m \in (0, \infty)$, observe that β has a maximum value of $\beta_0 = 2\pi/(n(n-2))^{1/2}$, and for every $m \neq m_0$, there are two values $m^\pm$ of m giving the same value of β. Thus two metrics have the same conformal infinity; the boundary map Π is a fold map, (of the form $x \to x^2$) along the curve g_m. Exactly the same formulas and behavior hold if $S^{n-1}(1)$ is replaced by any closed Einstein manifold (N, g_N) with $Ric_{g_N} = (n-2)g_N$, with ω_{n-1} replaced by $vol_{g_N}N$.

Note that if one allows the filling manifold M to change, a further metric has the same conformal infinity. Thus, choose $M = B^{n+1}/\mathbb{Z} = S^1 \times \mathbb{R}^n$, with the quotient of the hyperbolic metric g_{-1} on B^{n+1} by a translation isometry along a geodesic.

Examples 2.2. As discussed in [4], a more drastic example of non-uniqueness occurs in the family of AdS toral black hole metrics. These are metrics on $M = \mathbb{R}^2 \times T^{n-1}$, where T^{n-1} is the $(n-1)$-torus, and the standard form of the metrics g_m is the same as in (2.1), with $S^{n-1}(1)$ replaced by any flat metric on T^{n-1} and V in (2.2) replaced by $V(r) = r^2 - \frac{2m}{r^{n-2}}$, $\beta = 4\pi/nr_+$. The conformal infinity of these metrics is the flat metric on the product $S^1(\beta) \times T^{n-1}$. Here β is monotone in m, and so on this space of metrics, the boundary map Π is 1-1. The action and stress-energy tensor are given by, [24], [25]:

$$I^{ren}(g_m) = -\beta\omega_{n-1}r_+^n, \quad dI^{ren} = -r_+^n diag(1-n, 1, ..., 1), \tag{2.5}$$

where $\omega_{n-1} = volT^{n-1}$.

However, the actual situation is a little more subtle. The metrics g_m are all locally isometric, and so are isometric in the universal cover $\mathbb{R}^2 \times \mathbb{R}^{n-1}$,

$$\widetilde{g}_m = V^{-1}dr^2 + Vd\theta^2 + r^2 g_{\mathbb{R}^{n-1}}. \tag{2.6}$$

Let (T^n, g_0) be any flat metric on the n-torus, and let σ be any simple closed geodesic in (T^n, σ). Topologically, one may glue on a disc $D^2 = \mathbb{R}^2$ onto σ to obtain a solid torus $\mathbb{R}^2 \times T^{n-1}$. Metrically, this is carried out as follows. Given σ, for any R sufficiently large, there exists S, and a covering map $\pi = \pi_R : S^1 \times \mathbb{R}^{n-1} \to T^n$, such that

$$\pi(S^1) = \sigma, \quad \text{and} \quad \pi^*(S^2 g_0) = V(R)d\theta^2 + R^2 g_{\mathbb{R}^{n-1}}.$$

Here S is determined by the relation that $V(R)^{1/2}\beta = SL(\sigma)$, where $L(\sigma)$ is the length of σ in (T^n, g_0). Thus π takes the circle factor S^1 to σ and maps the flat metric on $S^1 \times \mathbb{R}^{n-1}$ to $S^2 g_0$ on T^n. The map π is given by dividing by a unique (twisted) isometric $\mathbb{Z}^{n-1}$-action on $S^1 \times T^{n-1}$ and this action clearly extends to an isometric $\mathbb{Z}^{n-1}$-action on $\widetilde{g}_m$. Letting $R \to \infty$ and taking the corresponding limiting map π and $\mathbb{Z}^{n-1}$-action gives a "twisted" toral black hole metric

$$\hat{g}_m = V^{-1}dr^2 + [Vd\theta^2 + r^2 g_{\mathbb{R}^{n-1}}]/\mathbb{Z}^{n-1} \tag{2.7}$$

on $\mathbb{R}^2 \times T^{n-1}$ with conformal infinity (T^n, g_0).

By varying the choices of σ, this gives infinitely many isometrically distinct AH Einstein metrics on $\mathbb{R}^2 \times T^{n-1}$ with the same conformal infinity (T^n, σ), so that Π is ∞-to-1; (note however that these metrics are all locally isometric). These metrics all lie in distinct components of the moduli space $\mathcal{E}$, so $\mathcal{E}$ has infinitely many components on $\mathbb{R}^2 \times T^{n-1}$; these components are permuted by the action of "large" diffeomorphisms on the boundary T^n, not isotopic to the identity, corresponding to the choices of simple closed geodesic σ.

One may take limits of any infinite sequence of these metrics, with fixed conformal infinity, by taking $L(\sigma) \to \infty$. All sequences have a unique limit given by the hyperbolic cusp metric

$$g_C = ds^2 + e^{2s} g_0, \tag{2.8}$$

on $\mathbb{R} \times T^n$. It is not difficult to compute exactly the action I^{ren} of $\hat{g}_m$, and it is easy to see that as $L(\sigma) \to \infty$,

$$I^{ren}(\hat{g}_m) \to I^{ren}(g_C) = 0,$$

(corresponding to $\beta \to \infty$).

Remark 2.3. Let (N, g_N) be any closed $(n-1)$-dimensional Einstein manifold, with $Ric_{g_N} = -(n-1)g_N$. Such metrics generate AdS black hole metrics just as in (2.1); thus

$$g_m = V^{-1}dr^2 + Vd\theta^2 + r^2 g_N, \tag{2.9}$$

is an AH Einstein metric on $\mathbb{R}^2 \times N$, with conformal infinity $S^1(\beta) \times (N, g_N)$. Here $V = -1 + r^2 - 2m/r^{n-2}$, $r_+ > 0$ is the largest root of V and $\beta = 4\pi r_+/(nr_+^2 - (n-2))$. Again β is a monotone function of m. However, g_m is well-defined for negative values of m; in fact, g_m is well-defined for $m \in [m_-, \infty)$, where

$$m_- = -\frac{1}{n-2}\left(\frac{n-2}{n}\right)^{n/2}, \quad \text{with} \quad r_+ = \left(\frac{n-2}{n}\right)^{1/2}. \tag{2.10}$$

For the extremal value m_- of m, $V(r_+) = V'(r_+) = 0$, and a simple calculation shows that the horizon $\{r = r_+\}$ occurs at infinite (geodesic) distance to any given point in (M, g_{m_-}); the horizon in this case is called degenerate, (with zero surface gravity). Note that $\beta(m_-) = \infty$, so that the θ-circles are in fact lines $\mathbb{R}$. As m decreases to m_-, the horizon diverges to infinity, (in the opposite direction from the conformal infinity), while the length of the θ-circles expands to ∞. Thus, the metric g_{m_-} is a complete metric on the manifold $\mathbb{R} \times \mathbb{R} \times N = \mathbb{R}^2 \times N$, but is no longer conformally compact; the conformal infinity is $\mathbb{R} \times (N, g_N)$. The action also diverges to $-\infty$ as $m \to m_-$.

However, one may divide the infinite θ-factor of the extremal metric g_{m_-} by $\mathbb{Z}$ to obtain a complete metric $\hat{g}$ on $\mathbb{R} \times S^1 \times N$. The metric $\hat{g}$ is an AH Einstein metric with a single cusp-like end, and with conformal infinity $S^1 \times (N, g_N)$; the length of the S^1 factor may be arbitrary. This is a non-standard example of an AH Einstein metric, with a cusp-like end, as opposed to the standard hyperbolic cusp of (2.8).

Note however that in contrast to the situation in Examples 2.2, $\hat{g}$ does not arise as a limit of the curve g_m as $m \to m_-$; as $m \to m_-$, the conformal infinity of g_m also degenerates.

Examples 2.4. As a final example of non-uniqueness, let (N, g_0) be any complete, geometrically finite hyperbolic $(n+1)$-manifold. We assume that N has a conformal infinity $(\partial N, \gamma_0)$, as well as a finite number of parabolic or cusp ends, of the form (2.8). There are numerous examples of such manifolds in any dimension. If n is odd, the renormalized action and stress-energy tensor are given by

$$I^{ren}(N, g_0) = (-1)^m 2n \frac{2^{2m}\pi^m m!}{(2m)!}\chi(N), \quad dI^{ren}(N, g_0) = 0, \tag{2.11}$$

where $n = 2m - 1$, cf. [26]. If n is even, an explicit general formula for the renormalized action is not known, although of course it is finite, while $dI^{ren}(N, g_0)$ is explicitly computable from the $g_{(2)}$ term in (1.6), cf. [22] for example.

Now one may truncate and cap off the cusp ends of (N, g) by glueing in solid tori $\mathbb{R}^2 \times T^{n-1}$ with boundary $\partial(\mathbb{R}^2 \times T^{n-1}) = T^n$. In 3 dimensions, this is the process of hyperbolic Dehn filling, due to Thurston. Essentially exactly as in Examples 2.2, a disc $\mathbb{R}^2 = D^2$ can be attached to any simple closed geodesic σ in T^n.

Using the Dehn filling results in [7], G. Craig has recently shown [21] that all the cusp ends of N may be capped off in this way to produce infinitely many distinct manifolds M_i, with AH Einstein metrics g_i, all with conformal infinity given by the original $(\partial N, \gamma_0)$. The construction implies that as the lengths of all geodesics σ_i diverge to infinity, (M_i, g_i) converges to the original manifold (N, g_0) in the Gromov-Hausdorff topology. Moreover, for all i large, $I^{ren}(M_i, g_i) < I^{ren}(N, g_0)$, and

$$I^{ren}(M_i, g_i) \to I^{ren}(N, g_0), \quad dI^{ren}(M_i, g_i) \to dI^{ren}(N, g_0), \tag{2.12}$$

in any dimension.

Taken together, the results above suggest that in general, there may be some difficulties in obtaining a well-defined (purely gravitational) semi-classical partition function Z_{AdS}. Analogous difficulties in defining the partition function for Euclidean quantum gravity have been discussed briefly in [9]. Thus, given $(\partial M, \gamma)$, the correspondence (0.1) requires summing over the moduli space of all AH Einstein manifolds (M, g) with the given boundary data $(\partial M, \gamma)$. In the infinite sets of AH Einstein metrics with fixed boundary data constructed above, the action I^{ren} is uniformly bounded above and converges to a limit. Moreover, all metrics are strictly stable, in the sense that the 2^{nd} variation of the action among transverse-traceless perturbations vanishing at infinity is positive definite - there are no negative or zero eigenmodes present. Hence, all solutions contribute a definite positive amount to the partition function, and so the partition function is likely to be badly divergent. In slightly more detail, the zero-loop approximation to the partition function is badly divergent, while the one-loop appoximation is also likely to be, unless there happen to be infinitely many other AH Einstein metrics giving rise to cancellations.

As will be seen below, this phenomenon does not occur when the boundary metric γ has positive scalar curvature, at least in dimension 4. This is perhaps then another reason for restricting the correspondence to boundary data of positive scalar curvature, as suggested by Witten [47], [48] for reasons related to the stability of the CFT.

However, a simple sum as in (0.1), even with the addition of higher loop corrections, may be ignoring certain important geometric information. Thus, suppose one had a finite dimensional connected moduli space Λ of solutions with fixed boundary data, (so that there is a finite dimensional space of zero modes). In this case, one would not simply sum over the distinct solutions g_λ, but integrate the function e^{-I} over the (presumably finite volume) moduli space Λ with respect to the volume form induced by the L^2 metric on the space of metrics. It seems reasonable and natural that a similar prescription should be used when one has an infinite sequence of isolated points $\{g_i\}$ converging to a limit set X_∞. The metrics g_i satisfy

$$dist_{L^2}(x_i, X_\infty) \to 0, \quad \text{as} \quad i \to \infty,$$

and it is natural to include weight factors, depending on $dist_{L^2}(x_i, X_\infty)$, in the sum (0.1). What is not clear is exactly what weight factors one should choose.

Such infinite behavior in the gravitational partition function does appear in the remarkable paper of Dijkgraff et al. [23], where the authors deal with the infinite family of BTZ black hole metrics, parametrized by relatively prime integers (c, d) corresponding to simple closed geodesics on T^2; in the terminology above, these are just the different hyperbolic Dehn fillings of a 2-torus. The partition function found in [23] does have suitable weight factors, leading to a convergent sum.

3. Existence Issue.

Next we turn to the global existence question, i.e. the surjectivity of the boundary map Π. Locally, the map Π is quite simple; its domain and target are smooth manifolds and the linearization $D\Pi$ has finite equi-dimensional kernel and cokernel. However, globally the domain $\mathcal{E}$ of Π is highly non-compact. To obtain a good global theory relating the domain and image of Π, one needs the map Π to be proper, i.e. for any compact set $K \subset \mathcal{C}$, $\Pi^{-1}(K)$ is compact in $\mathcal{E}$. In particular, for any $[\gamma] \in \mathcal{C}$, $\Pi^{-1}([\gamma])$ should be a compact set in $\mathcal{E}$. If this fails, for instance if $\Pi^{-1}([\gamma])$ is not compact, one needs to understand the possible limit structures of metrics in $\Pi^{-1}([\gamma])$.

The lack of uniqueness or even finiteness discussed in §2 shows that in general Π is not proper. Any general results on the compactness of a space of Einstein metrics having a compact set of boundary metrics must rely on a simpler theory of compactness of Einstein metrics on closed manifolds, i.e. the study of moduli spaces of Einstein metrics on closed manifolds. In dimension 2, the moduli space of Einstein metrics is described by Teichmüller theory. Unfortunately, in general dimensions, such a theory does not exist, and seems out of current reach. However, there is quite a well-developed theory of moduli of Einstein metrics on closed manifolds in 4-dimensions, and this allows one to develop an analogous theory in the case of AH Einstein metrics.

The results described below are thus restricted to 4-manifolds M, with ∂M a 3-manifold. It seems reasonable that these results can be generalized to higher dimensions in the presence of extra symmetry via Kaluza-Klein type symmetry reductions, and progress in this direction would be very interesting.

Let $\mathcal{C}^0$ be the space of conformal classes on a 3-manifold ∂M which have a *non-flat* representative metric of non-negative scalar curvature. (Of course not all 3-manifolds admit such a metric). Let $\mathcal{E}^0 = \Pi^{-1}(\mathcal{C}^0)$, and consider the restricted map

$$\Pi^0 : \mathcal{E}^0 \to \mathcal{C}^0. \tag{3.1}$$

Theorem 3.1. [5] *Let M be a 4-manifold satisfying*

$$H_2(\partial M) \to H_2(M) \to 0. \tag{3.2}$$

Then the map Π^0 in (3.1) is proper. Further, Π has a well-defined degree, given by

$$deg\Pi = \sum_{g_i \in \Pi^{-1}[\gamma]} (-1)^{ind g_i}. \tag{3.3}$$

Here $[\gamma]$ is any regular value of Π in $\mathcal{C}^0$, (recall the regular values are dense in $\mathcal{C}^0$). Since Π^0 is proper, the sum above is finite, and ind_{g_i} is the L^2 index of g_i, that is the dimension of the space of transverse-traceless L^2 forms on which the 2^{nd} variation of the action is negative definite, (the number of negative eigenmodes).

It is obvious from the definition that

$$deg\Pi \neq 0 \Rightarrow \Pi^0 \ \text{ is surjective.} \tag{3.4}$$

On the other hand, Π^0 may or may not be surjective when $deg\Pi^0 = 0$. Similarly, $deg\Pi^0 = \pm 1$ does not imply uniqueness of an AH Einstein metric with a given conformal infinity; it implies that generic boundary metrics have an odd number of AH Einstein filling metrics.

Remark 3.2. The condition (3.2) is used only to rule out degeneration of Einstein metrics to orbifolds. If one enlarges the space $\mathcal{E}$ to include orbifold Einstein metrics $\mathcal{E}_s$, then Π^0 is proper on the enlarged space $\hat{\mathcal{E}} = \mathcal{E} \cup \mathcal{E}_s$. However, it is not currently known if $\hat{\mathcal{E}}$ has the structure of a smooth manifold, or of a manifold off a set of codimension k, for some $k \geq 2$, although one certainly expects this to be the case.

Theorem 3.1 implies in particular that there are only finitely many components C_λ of the moduli space $\mathcal{E}^0$ for which $\cap_\lambda \Pi(C_\lambda) \neq \emptyset$, i.e. only finitely many components have a given boundary metric in common. Of course the degree $deg\Pi^0$ may depend on the choice of the component. This result also holds when one allows the manifold M to vary. Thus, given any $K < \infty$, there are only finitely many diffeomorphism types of 4-manifolds M, with a given boundary ∂M, with $\chi(M) \leq K$, and for which $\Pi(\mathcal{E})$ contains any given element $[\gamma] \in \mathcal{C}^0$.

Thus, the infinities discussed in §2 cannot arise when $R_\gamma > 0$, and so the sum in (0.1) is essentially well-defined. The sum could be infinite only if there exist (M_i, g_i) with $\Pi(g_i) = \gamma$, with $\chi(M_i) \to +\infty$, (slightly analogous to the divergence of the string partition function). The role of the hypothesis $R_\gamma > 0$ will be explained in more detail in §4, but we outline the general idea of the proof of Theorem 3.1 to explain how this condition arises.

By way of background, consider first the structure of the moduli space of unit volume Einstein metrics on a fixed closed 4-manifold. Modulo the possibility of orbifold degenerations, the overall structure of the moduli space is quite similar to that of the Teichmüller theory for the moduli space of constant curvature metrics on a closed surface. Recall that sequences $\{g_i\}$ of unit area constant curvature metrics on a surface S have subsequences that either:

- Converge to a limit (S, g), (for example on S^2 where the moduli space is a point),
- Collapse in the sense that the injectivity radius converges to 0 everywhere, (as in the case of a divergent sequence of flat metrics on T^2),
- Form cusps (N, g), $N \subset S$, as in the case of hyperbolic metrics on S. Thus S is a finite union of hyperbolic surfaces with cusp ends (N_k, g_k) together with a finite number of annuli $\mathbb{R} \times S^1$ which are collapsed by the sequence $\{g_i\}$.

A similar basic trichotomy holds in dimension 4, cf. [2]. Thus, analogous to (1.20), the Chern-Gauss-Bonnet theorem implies a uniform upper bound on the L^2 norm of the Weyl curvature of an Einstein metric on a 4-manifold M. Using this, sequences of such metrics have subsequences that either converge, collapse, or form cusps as above, although one must allow also for the formation of orbifold singularities.

Now suppose instead that $\{g_i\}$ is a sequence of AH Einstein metrics on a 4-manifold M for which the corresponding conformal infinities γ_i are contained in a compact set, so that

$$\gamma_i \to \gamma, \tag{3.5}$$

(in a subsequence). Then using (3.5) and (1.20), one can again prove $I^{ren}(g_i)$ remains uniformly bounded, (so that, roughly speaking, I^{ren} is proper on $\mathcal{C}$) and the trichotomy above still holds.

The possibility of collapse can also be ruled out by the control on the boundary metrics. However, in general, the formation of cusps cannot be ruled out, (as seen from the examples in §2).

More precisely, define an AH Einstein metric with cusps (N, g) to be a complete Einstein metric g on an $(n+1)$-manifold N which has two types of ends, namely AH ends and cusp ends. A cusp end of (N, g) is an end E such that $vol_g E < \infty$, and hence is not conformally compact. The bound on the volume follows from the bound on the renormalized action, via (1.15). On any divergent sequence of points $x_k \in E$, the injectivity radius $inj_g(x_k) \to 0$ as $k \to \infty$, so the metric g is collapsing at infinity in E. For instance, as discussed in Examples 2.2, infinite sequences of twisted toral black hole metrics limit on a complete hyperbolic cusp metric (2.8). Similarly, infinite sequences in Examples 2.4 limit on a complete hyperbolic manifold with cusp ends.

It will be seen in §4 that the hypothesis $R_\gamma > 0$, (or $R_\gamma \geq 0$ and γ not Ricci-flat), rules out the possible formation of cusps, (in any dimension). This shows that $\{g_i\}$ above has a subsequence converging to a limit AH Einstein metric g on M, so that Π^0 is proper.

The examples of cusp formation discussed in §2 all take place on sequences of metrics g_i lying either in distinct components of $\mathcal{E}$, or on different smooth manifolds M_i. Another interesting open question is whether cusps can actually form within a given or fixed component of $\mathcal{E}$, on a fixed manifold M. On closed manifolds, it is clear that cusps can form at the endpoints of curves of Einstein metrics. For example, let $M = \Sigma_{g_1} \times \Sigma_{g_2}$ be a product of surfaces of genus $g_i \geq 2$. Products of hyperbolic metrics on each surface are Einstein metrics on M and so as in Teichmüller theory there are smooth curves of Einstein metrics limiting on cusps (N, g) associated to M. However, it is not so easy to see, and in any case is unknown, if analogues of such constructions hold in the AH setting; compare also with Remark 2.3.

It would also be very interesting to know if the possible formation of cusps is restricted by the topology of the ambient manifold M. For example, the topological condition (3.2) rules out the formation of orbifold singularities. In the example above on $M = \Sigma_{g_1} \times \Sigma_{g_2}$, this is clearly the case; the fundamental group of the collapsed region is non-trivial and injects in the fundamental group of M. One might conjecture for instance that on the 4-ball B^4, or $(n+1)$-ball B^{n+1}, cusp formation is not possible.

If one knows that no cusp formation is possible on limits of sequences in $\mathcal{E} = \mathcal{E}(M)$ within a compact set of boundary metrics, then Theorem 3.1 holds in general, without the restriction to $\mathcal{E}^0$.

Consider briefly the situation in general where cusps may form. Given a fixed 4-manifold M, let $\bar{\mathcal{E}}$ be the completion of $\mathcal{E}$ in the Gromov-Hausdorff topology. Thus $(X, g) \in \bar{\mathcal{E}}$ iff there is a sequence $\{g_i\} \in \mathcal{E}$ such that $(M, g_i) \to (X, g)$ in the (pointed) Gromov-Hausdorff topology. The analysis above implies that, in general,

$$\bar{\mathcal{E}} = \mathcal{E} \cup \mathcal{E}_s \cup \mathcal{E}_c, \tag{3.6}$$

where $\mathcal{E}_s$ consists of orbifold AH Einstein metrics and $\mathcal{E}_c$ consists of AH Einstein metrics with cusps, associated to M. The boundary map Π extends to a continuous map

$$\bar{\Pi} : \bar{\mathcal{E}} \to \mathcal{C}, \tag{3.7}$$

and $\bar{\Pi}$ is proper. However, the structure of $\bar{\mathcal{E}}$ is not well understood. If for example $\bar{\mathcal{E}}$ is a manifold off a singular set of codimension at least 2, then $deg\bar{\Pi}$ is well defined, and is given by (3.3). In particular, (3.4) then holds. Moreover, one clearly has $deg\bar{\Pi} = deg\Pi^0$, when $\mathcal{C}^0 \neq \emptyset$. However, if $\bar{\mathcal{E}}$ is something like a manifold with boundary, then $deg\bar{\Pi}$ is not well-defined and the global behavior of $\bar{\Pi}$ is less clear.

We now return to applications of Theorem 3.1 itself. The degree $deg\Pi^0$ is a smooth invariant of $(M, \partial M)$. In many specific cases, the degree can be calculated by means of the following isometry extension result; this result is very natural from the perspective of the AdS/CFT correspondence, and holds in all dimensions.

Theorem 3.3. [5] *Let g be a C^2 conformally compact Einstein metric on an $(n+1)$ manifold with conformal infinity $[\gamma]$ on ∂M. Then any 1-parameter group of conformal isometries of $(\partial M, \gamma)$ extends to a 1-parameter group of isometries of (M, g).*

In particular, any AH Einstein metric whose conformal infinity is highly symmetric is itself highly symmetric. Einstein metrics which have a transitive or cohomogeneity 1 isometric group action have essentially been classified. Using this, the degree $deg\Pi^0$ can be computed in a number of interesting cases; the currently known results are summarized in the table below.

Table

M	∂M	Seed Metric	$deg\Pi^0$	Π^0 onto $\mathcal{C}^0$
B^4	S^3	Poincaré	1	Yes
$\mathbb{CP}^2 \setminus B^4$	S^3	AdS Taub-Bolt	0	No
$S^2 \times \mathbb{R}^2$	$S^2 \times S^1$	AdS Schwarzschild	0	No
$\mathbb{R}^3 \times S^1$	$S^2 \times S^1$	Poincaré/$\mathbb{Z}$	1	Yes
$E_k \to S^2$, $k \geq 2$	$S^3/\mathbb{Z}_k$	AdS Taub-Bolt	1	Yes
X_k, $k \geq 2$	$S^3/\mathbb{Z}_k$	Self-dual CS metrics	0	No

Here $(M, \partial M)$ is the given manifold with boundary and the existence of a seed metric implies $\mathcal{E} \neq \emptyset$. The degree is given for the component of $\mathcal{E}$ containing the seed metric.

The manifold E_k is the $\mathbb{R}^2$ bundle over S^2 with Chern class k, while X_k is a resolution of the orbifold $\mathbb{C}^2/\mathbb{Z}_k$ with $c_1(X_k) < 0$. The seed metric on X_k is an element of the family of self-dual AH Einstein metrics on X_k constructed by Calderbank-Singer, [17].

Mazzeo-Pacard [43] have shown that if M_1 and M_2 admit an AH Einstein metric, then so does the boundary connected sum $\hat{M} = M_1 \#_b M_2$; for $\hat{M}$ one has $\partial \hat{M} = \partial M_1 \# \partial M_2$. Further, if $\mathcal{E}^0(M_i) \neq \emptyset$, then $\mathcal{E}^0(\hat{M}) \neq \emptyset$. It would be interesting to determine the degree of $\hat{M}$ in terms of the degree of each M_i.

Remark 3.4. Let M be any $(n+1)$ manifold with $\partial M = S^n$. If $M \neq B^{n+1}$, then Π cannot be surjective onto $\mathcal{C}^0$; in particular when $n+1 = 4$, $deg\Pi^0 = 0$. In fact the round metric g_{+1} on S^n cannot be in Im Π, for Theorem 3.3 implies that any such AH Einstein metric must be the hyperbolic metric on the ball. The same argument shows that the conformal class of the round product metric $S^1(\beta) \times S^n(1)$, for $\beta > \beta_0 = 2\pi/(n(n-2))^{1/2}$, is not in Im Π on any manifold M with $\partial M = S^1 \times S^n$ except $\mathbb{R}^n \times S^1$, where it is uniquely realized by the hyperbolic metric.

In sum, one has the following examples of uniqueness results from Theorem 3.3. The Poincaré metric is the unique AH Einstein metric with boundary metric the round metric on S^n, while the AdS Schwarzschild and (quotient) Poincaré metric are unique among AH Einstein metrics with boundary the round product on $S^{n-1} \times S^1$. Similarly, the only manifold carrying an AH Einstein metric with boundary metric the round metric on $S^3/\mathbb{Z}_k$ is E_k when $k \geq 2$, and it is uniquely realized by the AdS Taub-Bolt metric, cf. [32] for example.

(ii). By the same reasoning as above, any AH Einstein metric on M with conformal infinity a flat metric on the n-torus T^n is necessarily a twisted AdS toral black hole metric, as in Examples 2.2. It follows from a simple Wick rotation argument that the Lorentzian AdS soliton metric of Horowitz-Myers [37], is the unique static AdS metric, when the conformal infinity is compactified to a flat torus; see also [29], [10] for previous work on the uniqueness of the AdS solition.

4. Role of $R \geq 0$.

In this section, we discuss the role of the hypothesis that the boundary metric γ on ∂M has non-negative scalar curvature. This condition was first suggested by Witten in [47], who pointed out that the corresponding CFT is unstable when $R_\gamma < 0$, suggesting that the AdS/CFT correspondence may break down in this region. This result led shortly thereafter to the work of Witten-Yau [48], proving that ∂M is necessarily connected when ∂M carries a metric of positive scalar curvature, see also the work of Cai-Galloway [16] for a different proof which handles in addition the case of non-negative scalar curvature..

In this section we give a very elementary proof of the Witten-Yau result. In fact the result is stronger in that it gives a definite bound on the distance of any point in M to its boundary, in the geodesic compactification. This shows that the condition $R_\gamma > 0$ not only proves that ∂M must be connected, but also prevents the formation of new, not necessarily conformally compact, boundary components in families of AH metrics.

Let g be an AH metric (not necessarily Einstein), on an $(n+1)$-manifold M, possibly with several boundary components; thus g is merely assumed to be C^3 conformally compact. Given a fixed boundary component $\partial_0 M$, with associated boundary metric γ, let ρ be the geodesic defining function defined by $(\partial_0 M, \gamma)$, so that if $\bar{g} = \rho^2 g$ is the associated (partial) C^2 compactification of (M, g), then $\rho(x) = dist_{\bar{g}}(x, \partial_0 M)$. Observe that if M has other boundary components, then these lie at infinite distance with respect to $\bar{g}$ to any point in M. Note also that since g is C^2 conformally compact, $|Ric_g + ng| = O(\rho^2)$.

Theorem 4.1. *Let $\bar{g} = \rho^2 g$ be a partial geodesic compactification of an AH metric g on M, satisfying*

$$Ric_g + ng \geq 0 \quad \text{and} \quad |Ric_g + ng| = o(\rho^2). \tag{4.1}$$

If $R_\gamma = const > 0$, then for all $x \in M$,

$$\rho^2(x) \leq 4n(n-1)/R_\gamma. \tag{4.2}$$

Proof: Along the $\bar{g}$-geodesics normal to $\partial_0 M$, one has the Riccati equation

$$\bar{H}' + |\bar{K}|^2 + \bar{R}ic(T,T) = 0, \tag{4.3}$$

where $T = \bar{\nabla}\rho$, $\bar{K}$ is the 2nd fundamental form of the level sets $S(\rho)$ of ρ, and $\bar{H} = tr\bar{K}$ is the mean curvature, with $\bar{H}' = \partial \bar{H}/\partial\rho$. Thus $\bar{K} = \bar{D}^2\rho$, $\bar{H} = \bar{\Delta}\rho$. Here and in the following, the computations are with respect to $\bar{g}$. Standard formulas for conformal changes of metric give

$$\bar{R}ic = -(n-1)\frac{\bar{D}^2\rho}{\rho} - \frac{\bar{\Delta}\rho}{\rho}\bar{g} + (Ric_g + ng) \geq -(n-1)\frac{\bar{D}^2\rho}{\rho} - \frac{\bar{\Delta}\rho}{\rho}\bar{g}. \tag{4.4}$$

Hence

$$\bar{R} = -2n\frac{\bar{\Delta}\rho}{\rho} + (R + n(n+1))/\rho^2 \geq -2n\frac{\bar{\Delta}\rho}{\rho}. \tag{4.5}$$

In particular, $\bar{R}ic(T,T) = -\frac{\bar{\Delta}\rho}{\rho} + (Ric_g + ng)(T,T)/\rho^2$.

Dividing (4.3) by ρ then gives

$$\frac{(\bar{\Delta}\rho)'}{\rho} - \frac{\bar{\Delta}\rho}{\rho^2} + (Ric_g + ng)(T,T)/\rho^3 + \frac{|\bar{D}^2\rho|^2}{\rho} = 0,$$

so that

$$(\frac{\bar{\Delta}\rho}{\rho})' = -\frac{|\bar{D}^2\rho|^2}{\rho} - (Ric_g + ng)(T,T)/\rho^3 \leq -\frac{|\bar{D}^2\rho|^2}{\rho}. \tag{4.6}$$

By Cauchy-Schwartz, $|\bar{D}^2\rho|^2 \geq \frac{1}{n}(\bar{\Delta}\rho)^2$ and so, setting $\phi = -\bar{\Delta}\rho/\rho$ one has

$$\phi' \geq \frac{1}{n}\rho\phi^2. \tag{4.7}$$

A simple computation using the Gauss equations at ∂M together with the fact that $A = 0$ at ∂M implies that

$$(n-1)\phi(0) = \tfrac{1}{2}R_\gamma + \lim_{\rho\to 0}\tfrac{1}{\rho^2}[(Ric_g + ng)(T,T) - \tfrac{1}{2}(R + n(n+1))]. \tag{4.8}$$

The hypothesis (4.1) implies that the limit is 0, and hence $\phi(0) > 0$. A simple integration then gives

$$\rho^2 \leq 2n/\phi(0), \tag{4.9}$$

which gives (4.2). ■

Note that if $\gamma \in [\gamma]$ has non-negative scalar curvature, then there exists a representative $\bar{\gamma} \in [\gamma]$ with constant non-negative scalar curvature, by the solution of the Yamabe problem.

As noted above, the estimate (4.2) immediately implies that ∂M is connected, since any other boundary component would have to lie at infinite ρ-distance to $\partial_0 M$. Similarly, simple topological arguments then imply

$$\pi_1(\partial M) \to \pi_1(M) \to 0.$$

More importantly for our purposes, the estimate (4.2) also immediately implies that AH Einstein metrics with cusps, as described in §3, cannot form as the limit of sequences of AH Einstein metrics with boundary metrics of uniformly positive scalar curvature. This explains then the role of $\mathcal{C}^0$ in Theorem 3.1. A straightforward extension of the method above (based on the Cheeger-Gromoll splitting theorem) shows also that if $R_\gamma = 0$ and ρ is unbounded on (M,g), then (M,g) is isometric to

$$g = ds^2 + e^{2s}g_{N^n},$$

where g_{N^n} is Ricci-flat, cf. [16]. Thus when $n = 3$, N must be flat so a finite cover of (M,g) is isometric to a hyperbolic cusp metric (2.8). In particular, this can only happen if $(\partial M, \gamma)$ is flat.

Remark 4.2. Theorem 4.1 was proved in [4]. The proof is included here partly for completeness, and partly because the Lorentzian version of this result will be used in §6.

An elementary consequence of Theorem 4.1 is that a geometrically finite hyperbolic manifold with conformal infinity satisfying $R_\gamma > 0$ has no parabolic ends.

5. Self-Duality.

The analysis in the previous sections describes the beginnings of a well-defined existence theory for the Einstein-Dirichlet problem, at least in 4-dimensions. From the point of view of the AdS/CFT correspondence, one would like however much more detailed information about the correspondence (1.9) of the Dirichlet and Neumann boundary data.

Again, restricting to dimension 4, a good deal more can be said, using the splitting of the curvature tensor into self-dual and anti-self-dual parts. Thus, let $M = M^4$ be an oriented 4-manifold with boundary. As first noticed by Hitchin [35], given any C^2 conformally compact metric g on M, (not necessarily Einstein), a simple calculation using the Atiyah-Patodi-Singer index theorem gives

$$\frac{1}{12\pi^2}\int_M |W^+|^2 - |W^-|^2 = \sigma(M) - \eta(\gamma), \tag{5.1}$$

where $W^\pm$ are the self-dual and anti-self-dual parts of the Weyl curvature, $\sigma(M)$ is the signature of M, and $\eta(\gamma)$ is the eta invariant of the conformal infinity $(\partial M, \gamma)$; note that (5.1) is conformally invariant. Combining this with the formula (1.20) for the renormalized action of an AH Einstein metric gives the formula

$$I^{ren} \leq 8\pi^2\chi(M) - 12\pi^2|\sigma(M) - \eta(\gamma)|, \tag{5.2}$$

with equality if and only if g is self-dual, (or anti-self-dual).

Hence if (M, g) is self-dual and Einstein,

$$I^{ren} = 12\pi^2\eta(\gamma) + (8\pi^2\chi(M) - 12\pi^2\sigma(M)), \tag{5.3}$$

while $I^{ren} = -12\pi^2\eta(\gamma) + (8\pi^2\chi(M) + 12\pi^2\sigma(M))$, if (M, g) is anti-self-dual Einstein.

This leads to the following result:

Theorem 5.1. *Let $\mathcal{E}_{sd}$ be the moduli space of self-dual AH Einstein metrics on a 4-manifold M. Then $I^{ren} : \mathcal{E}_{sd} \to \mathbb{R}$ is given by*

$$I^{ren} = 12\pi^2\eta + c_M, \tag{5.4}$$

where $c_M = 8\pi^2\chi(M)$ - $12\pi^2\sigma(M)$ is topological.

The stress-energy tensor $g_{(3)}$ at a self-dual Einstein metric is given by

$$dI^{ren} = -\tfrac{3}{2}g_{(3)} = 12\pi^2 d\eta = -\tfrac{1}{2} * dRic. \tag{5.5}$$

Proof: Both (5.4) and (5.5) follow immediately from (5.3). In (5.5), Ric is viewed as a 1-form with values in $T(\partial M)$ and $*$ is the Hodge $*$-operator. The formula for $d\eta$ comes from the original work of Chern-Simons [18], cf. also [3] for more details. ■

Similarly on the moduli space of anti-self-dual AH Einstein metrics $\mathcal{E}_{ads}$, $I^{ren} = -12\pi^2\eta + c'_M$, and $dI^{ren} = -12\pi^2 d\eta = \frac{1}{2} * dRic$.

Thus, on the moduli spaces $\mathcal{E}_{sd}$ or $\mathcal{E}_{asd}$, the renormalized action is explicitly computable from the *global* geometry of the boundary metric $(\partial M, \gamma)$, while the stress-energy is *locally computable* on $(\partial M, \gamma)$.

Remark 5.2. This exact identification of the renormalized gravitational action and its stress-energy tensor on $\mathcal{E}_{sd}$ or $\mathcal{E}_{asd}$ appears to closely resemble the identification of the renormalized action with the Liouville action on the boundary in dimensions 2 + 1, cf. [38], [20]. It would be very interesting to pursue this analogy further.

The discussion above suggests that a self-dual (or anti-self-dual) AH Einstein metric should be uniquely determined by its boundary metric γ. This is true at least for real-analytic boundary data, $\gamma \in C^\omega$, as proved by LeBrun [40], using twistor methods. A more elementary proof, using just the Cauchy-Kovalewsky theorem, was given recently in [6]. This uniqueness implies that the topology of the bulk manifold M is determined by the boundary data $(\partial M, \gamma)$, up to covering spaces, i.e. any two self-dual AH Einstein metrics $(M_1, g_1), (M_2, g_2)$ with the same boundary data are locally isometric; in particular they are isometric in some covering space.

Similarly, in analogy to the discussion following (1.9), [40] or [6] imply that any metric $\gamma \in C^\omega(\partial M)$ is the boundary metric of a self-dual or anti-self-dual AH Einstein metric g, defined on a thickening $\partial M \times [0, \varepsilon)$ of ∂M. Of course, for general γ, this metric will not extend to a smooth Einstein metric on a compact manifold M with boundary ∂M.

For example, consider the boundary $S^2 \times S^1$, with boundary metric the round conformally flat product metric. This bounds a self-dual AH Einstein metric in a neighborhood of $S^2 \times S^1$, and uniqueness implies that this metric must be the hyperbolic metric on $B^3 \times S^1$, given by $H^4(-1)/\mathbb{Z}$. The AdS Schwarzschild metric $S^2 \times \mathbb{R}^2$ is thus of course not self-dual.

It would interesting to know if the moduli space $\mathcal{E}_{sd}$ has the structure of an infinite dimensional manifold, as is the case with $\mathcal{E}$ itself. Noteworthy in this respect is a result of Biquard [14] that in a neighborhood of the hyperbolic metric on B^4, the spaces $\mathcal{E}_{sd}$ and $\mathcal{E}_{asd}$ are smooth infinite dimensional manifolds, which intersect transversally at the hyperbolic metric g_{-1}. In particular, any metric $g \in \mathcal{E}$ near g_{-1} can be uniquely written as a sum $g = g^+ + g_{-1} + g^-$, where $g^+ + g_{-1}$ is self-dual and $g_{-1} + g^-$ is anti-self-dual. It seems that this result should be useful in understanding the Dirichlet-Neumann correspondence (1.9) near g_{-1}.

There are a number of interesting and explicit or semi-explicit examples of self-dual AH Einstein metrics. Thus, the AdS Taub-NUT [32] or Pedersen metrics [44] are self-dual on B^4, with boundary metric a Berger sphere, (the 3-sphere squashed along the S^1 fibers of the Hopf fibration). More generally, Hitchin [36] has constructed self-dual AH Einstein metrics on B^4 with conformal infinity any left-invariant metric on $SU(2) = S^3$. In addition, LeBrun [41] has proved the existence of an infinite dimensional family of self-dual AH Einstein metrics on B^4. More recently, Calderbank and Singer [17] have constructed families of self-dual AH Einstein metrics on resolutions of orbifolds $\mathbb{C}^2/\mathbb{Z}_k$ having negative Chern class. This gives 4-manifolds with arbitrarily large Betti number b_2 for which $\mathcal{E}_{sd} \neq \emptyset$.

6. Continuation to de Sitter and self-similar vacuum space-times.

In this section we discuss the continuation of AH Einstein metrics to de Sitter-type Lorentz metrics, and the possibility of constructing global self-similar vacuum solutions of the Einstein equations in higher dimensions. The basic model for this picture is the decomposition of Minkowski space-time $(\mathbb{R}^4, \eta)$ into foliations by hyperbolic metrics in the interior of the past and future light cones of a point $\{0\}$, and foliations by deSitter metrics in the exterior of the light cone. These foliations are of course given as the level sets of the distance function to $\{0\}$. Some aspects of this work have previously appeared in [8], see also [46] for a formal treatment of some of these issues.

Let M be any compact $(n+1)$ manifold with boundary ∂M and let g be any AH Einstein metric on M, with boundary metric $(\partial M, \gamma)$, with respect to a geodesic defining function ρ. As is well-known, and observed by Fefferman-Graham [27] in their original work on the subject, (M, g) then generates a vacuum solution to the Einstein equations on $\mathcal{M} = \mathbb{R}^+ \times M$ given by

$$\mathfrak{g} = -d\tau^2 + \tau^2 g, \tag{6.1}$$

where $\tau \in (0, \infty)$. This is a Lorentzian cone metric on the Riemannian metric g, and satisfies the vacuum equations

$$Ric_{\mathfrak{g}} = 0. \tag{6.2}$$

At least in a neighborhood of $\mathbb{R}^+ \times \partial M$, by (1.3), the metric (6.1) may be rewritten in the form

$$\mathfrak{g} = -d\tau^2 + (\frac{\tau}{\rho})^2(d\rho^2 + g_\rho). \tag{6.3}$$

The space-time $(\mathcal{M}, \mathfrak{g})$ is globally hyperbolic, with Cauchy surface given by M and Cauchy data $(g, K) = (g, g)$. The time evolution with respect to the time parameter τ is given by trivial rescalings of the time 1 spatial metric (M, g). When $(M, g) = (\mathbb{R}^3, g_{-1})$ is the Poincaré metric on the 3-ball, $(\mathcal{M}, \mathfrak{g})$ is the interior of the past (or future) light cone of a point $\{0\}$ in Minkowski space-time $\mathbb{R}^4$, (also called the Milne universe). Similarly, for any (M, g) with $g \in \mathcal{E}$, the vacuum solution $(\mathcal{M}, \mathfrak{g})$ is the interior of the past (or future) light cone (H, γ_0), where $H = \mathbb{R}^+ \times \partial M$ and the degenerate metric γ_0 on H is given by

$$\gamma_0 = v^2 \gamma,$$

with $v \in \mathbb{R}^+$ given by $v = \tau/\rho$. Thus, (H, γ_0) is the smooth Cauchy horizon for the space-time $(\mathcal{M}, \mathfrak{g})$. To be definite, we choose $(\mathcal{M}, \mathfrak{g})$ to be the interior of the past light cone of the vertex $\{0\} = \{v = 0\}$, and will later set $(\mathcal{M}, \mathfrak{g}) = (\mathcal{M}^-, \mathfrak{g}^-)$ and $H = H^-$.

In general, the metric $\mathfrak{g}$ is not C^∞ up to the Cauchy horizon H. Thus, under the change of variables $(\tau,\rho)\to(v,x)$, with $\rho=\sqrt{x}$, (6.3) becomes

$$\mathfrak{g}=-xdv^2-vdvdx+v^2g_{\sqrt{x}}, \tag{6.4}$$

and for $g_{\sqrt{x}}$ one has the Fefferman-Graham expansion

$$g_{\sqrt{x}}=\gamma+xg_{(2)}+...+x^{n/2}g_{(n)}+\tfrac{1}{2}x^{n/2}\log x\ h+.... \tag{6.5}$$

Hence, if n is odd, the metric $\mathfrak{g}$ is $C^{n/2}$ up to the horizon H, while if n is even, $\mathfrak{g}$ is $C^{n/2-\varepsilon}$ up to H. This degree of smoothness cannot be improved by passing to other coordinate systems; only in very rare instances where $g_{(n)}=0$ when n is odd, or $h=0$ when n is even, will g be C^∞ up to H.

Suppose first that $n=3$ and let γ be a real-analytic metric on ∂M. Then by [6], the compactification $\bar{g}=\rho^2 g$ of (M,g) is also real-analytic on $\bar{M}$, and so the Fefferman-Graham expansion (1.4) converges to g_ρ. Thus, the curve g_ρ can be extended past $\rho=0$ to purely imaginary values of ρ. This corresponds to replacing $\sqrt{x}$, $x>0$, by $-\sqrt{|x|}$, $x<0$, (i.e. $\rho\to i\rho$), and so gives the curve

$$g_\rho^{ext}=\gamma-\rho^2g_{(2)}-\rho^3g_{(3)}+\rho^4g_{(4)}+...., \tag{6.6}$$

obtained from the expansion for g_ρ by replacing ρ by $i\rho$ and dropping the i coefficients. Of course one could also continue the Fefferman-Graham expansion into the region $\rho<0$; this would give an AH Riemannian Einstein metric on "the other side" of ∂M, defined at least in some neighborhood of ∂M. However, this extension will not be of concern here.

Thus, although the metric $(\mathcal{M},\mathfrak{g})$ is only $C^{3/2}$ at H, it extends via (6.6) and (6.4) across the horizon H into the exterior of the light cone. Returning to the original variables (τ,ρ) in (6.3) then gives the metric

$$\mathfrak{g}^{ext}=d\tau^2+(\frac{\tau}{\rho})^2(-d\rho^2+g_\rho^{ext}). \tag{6.7}$$

Formally, this is obtained from $g=g^-$ by the replacement $\tau\to i\tau$, $\rho\to i\rho$, interchanging a spatial and time direction. This gives an extension of the metric $\mathfrak{g}$ into a region $\mathcal{M}^{ext}$ exterior to the light cone H, defined for all $\tau\in(0,\infty)$, $\rho\in[0,\varepsilon)$, for some $\varepsilon>0$. The metric $\mathfrak{g}^{ext}$ is C^ω where $\rho\neq 0$, but is only $C^{3/2}$ up to H where $\rho=0$.

The metric $\mathfrak{g}^{ext}$ is also a Lorentzian cone metric, now however with a space-like self-similarity in τ in place of the previous time-like self-similarity. The slices $\tau=const$ are all homothetic, and are Lorentzian metrics of the form

$$\mathfrak{g}^{dS}=(\frac{1}{\rho})^2(-d\rho^2+g_\rho^{ext}). \tag{6.8}$$

The metric $\mathfrak{g}^{dS}$ is a solution to the vacuum Einstein equations with a positive cosmological constant $\Lambda=\frac{1}{2}n(n-1)$, i.e. when $n=3$,

$$Ric_{\mathfrak{g}^{dS}}=3\mathfrak{g}^{dS}, \tag{6.9}$$

and so $\mathfrak{g}^{dS}$ is a deSitter-type (dS) space-time, (just as the initial metric g is of anti-deSitter type).

Exactly the same discussion holds in all dimensions. Thus, suppose again that $\gamma\in C^\omega(\partial M)$. As noted in §1, it follows from the recent regularity result of Chruściel et al. [19], that the compactification $\bar{g}=\rho^2g$ is C^∞ polyhomogeneous. Moreover, recent work of Kichenassamy [39] or Rendall [45] implies the Fefferman-Graham expansion (1.4) or (1.6) converges to g_ρ, in both cases n even or n odd. Hence, exactly the same arguments as above hold for any n, and give a dS-type Einstein metric of the form (2.8) satisfying

$$Ric_{\mathfrak{g}^{dS}}=n\mathfrak{g}^{dS}, \tag{6.10}$$

with g^{ext}_ρ given by

$$g^{ext}_\rho = \gamma - \rho^2 g_{(2)} - \rho^3 g_{(3)} + ... \pm \rho^n g_{(n)} \pm \rho^n \log \rho \ h + ... \tag{6.11}$$

The terms $g_{(k)}$ are defined as in (1.8), where $T = \bar{\nabla}\rho$ is the future-directed unit vector.

We summarize some of this discussion in the following result.

Corollary 6.1. *Let γ be a real-analytic metric on an n-manifold ∂M, and $g_{(n)}$ a real-analytic symmetric bilinear form on ∂M satisfying the constraint conditions (1.5) or (1.7). Then there is a 1-1 correspondence between Riemannian AH Einstein metrics g with boundary metric γ, and deSitter-type Lorentzian Einstein metrics $\mathfrak{g}^{dS}$ with past (or future) boundary metric γ, given by (6.3)-(6.8).*

∎

This correspondence thus gives a rigorous form of "Wick rotation" between these types of Einstein metrics. The Fefferman-Graham expansion holds for Einstein metrics of any signature, (again as observed in [27]). In the correspondence between AH and dS Einstein metrics, one has

$$g^{AH}_{(k)} = \pm g^{dS}_{(k)}, \tag{6.12}$$

where $+$ occurs if $k \equiv 0, 1 \pmod 4$, while $-$ occurs if $k \equiv 2, 3 \pmod 4$.

In dimension 4, the formulas (1.20) and (1.19) for the renormalized action and its variation also have analogues for dS space-times. Thus, let $(\mathcal{S}, \mathfrak{g})$ be a solution of (6.10) which is asymptotically simple, in that $(\mathcal{S}, \mathfrak{g})$ has a smooth past and future conformal infinity $(\mathcal{I}^-, \gamma^-)$ and $(\mathcal{I}^+, \gamma^+)$. In particular, $\mathcal{S}$ is geodesically complete and globally hyperbolic with compact Cauchy surface Σ, a 3-manifold diffomorphic to $\mathcal{I}^-$ and $\mathcal{I}^+$. In the following, we will forgo the exact determination of signs, which are best computed on an example; note that the Einstein-Hilbert action (1.11) is usually replaced by its negative for Lorentzian metrics.

Proposition 6.2. *Let $(\mathcal{S}, \mathfrak{g})$ be a 4-dimensional asymptotically simple vacuum dS space-time. Then*

$$\pm I^{ren} = \int_{\mathcal{S}} |W|^2 dV, \tag{6.13}$$

where $|W|^2 = W_{ijkl}W^{ijkl}$ and

$$\pm \, dI^{ren} = g^+_{(3)} - g^-_{(3)}, \tag{6.14}$$

where the terms are taken with respect to the past unit normal.

Since the metric g is Lorentzian, note that $|W|^2$ is not apriori non-negative.

Proof: The proof of (6.13) is exactly the same as the proof of (1.20) in [3], using the Lorentz version of the Chern-Gauss-Bonnet theorem, cf. [1] for example, in place of the Riemannian version. Since $\mathcal{S} = \mathbb{R} \times \Sigma$ topologically, where Σ is a closed 3-manifold, $\chi(\mathcal{S}) = 0$. Similarly, the proof of (1.19) in [3] holds equally well for Lorentzian metrics, and gives (6.14). ∎

Returning to the discussion preceding Corollary 6.1, the extended metric $\mathfrak{g}$ on the enlarged space $\mathcal{M}^- \cup \mathcal{M}^{ext}$ is still a solution to the vacuum Einstein equations, (with $\Lambda = 0$). This is clear if $n > 4$, since the metric is everywhere at least $C^{5/2}$. For $n = 3, 4$, the metric is not C^2, but is easily verified to still be a weak solution of the vacuum Einstein equations, i.e. it satisfies the equations (6.2) distributionally.

Now the initial AH Einstein metric (M, g) is global. It is natural to ask if the dS metric $\mathfrak{g}^{dS}$ is also global; the formula (6.7) is only defined for $\rho \in [0, \varepsilon)$, for some $\varepsilon > 0$. In general, the answer is no. In fact, the work in §4 carries over to this setting almost identically, and gives the following result, proved independently by the author (unpublished) and Andersson-Galloway [11].

Proposition 6.3. *Let $(\mathcal{S}, \mathfrak{g})$ be an $(n+1)$ dimensional globally hyperbolic space-time, with compact Cauchy surface Σ, which is C^3 conformally compact to the past, so that past conformal infinity $(\mathcal{I}^-, \gamma)$ is C^3. Suppose $(\mathcal{S}, \mathfrak{g})$ satisfies the strong energy and decay conditions*

$$(Ric_{\mathfrak{g}} - n\mathfrak{g})(T,T) \geq 0 \ \text{ and } \ |(Ric_{\mathfrak{g}} - n\mathfrak{g})(T,T)| = o(\rho^2), \tag{6.15}$$

for T time-like. Let γ be a representative for $[\gamma]$ with constant scalar curvature R_γ. If $R_\gamma < 0$, then

$$\rho^2(x) \leq 4n(n-1)/|R_\gamma|, \tag{6.16}$$

where ρ is the geodesic defining function associated to $(\mathcal{I}^-, \gamma)$.

In particular, any time-like geodesic in $\mathcal{S}$ is future incomplete, and no Cauchy surface Σ_ρ exists, even partially, for $\rho^2 > 4n(n-1)/|R_\gamma|$, so that $\mathcal{I}^+ = \emptyset$.

Proof: The proof is essentially identical to that of Theorem 4.1. Let $\bar{\mathfrak{g}} = \rho^2 \mathfrak{g}$ be the C^2 geodesic compactification determined by the data $(\mathcal{I}^-, \gamma)$; as before the computations below are with respect to $\bar{\mathfrak{g}}$. The equation (4.3) holds for Lorentzian metrics, (where it is known as the Raychaudhuri equation). The vector field $T = \bar{\nabla}\rho$ is now a unit time-like vector field, so $\mathfrak{g}(T,T) = -1$. This has the implication that $H = -\bar{\Delta}\rho$ while

$$\bar{R}ic = -(n-1)\frac{\bar{D}^2\rho}{\rho} - \frac{\bar{\Delta}\rho}{\rho}\bar{\mathfrak{g}} + (Ric_{\mathfrak{g}} - n\mathfrak{g}) \geq -(n-1)\frac{\bar{D}^2\rho}{\rho} - \frac{\bar{\Delta}\rho}{\rho}\bar{\mathfrak{g}},$$

and

$$\bar{R} = -2n\frac{\bar{\Delta}\rho}{\rho} + (R - n(n+1))/\rho^2 \geq -2n\frac{\bar{\Delta}\rho}{\rho}.$$

In particular, $\bar{R}ic(T,T) = \frac{\bar{\Delta}\rho}{\rho} + (Ric_g - ng)(T,T)/\rho^2$. The same arguments as in (4.4)-(4.7) then give

$$\phi' \leq -\frac{1}{n}\rho\phi^2, \tag{6.17}$$

where again $\phi = -\bar{\Delta}\rho/\rho$. The formula (4.8) also holds, and hence $R_\gamma < 0$ implies $\phi(0) < 0$. Integrating (6.17) as before gives (6.16). ∎

It is straightforward to extend Proposition 6.3 to the situation where $R_\gamma = 0$. As in the case of Theorem 4.1, $(\mathcal{S}, \mathfrak{g})$ has $\mathcal{I}^+ = \emptyset$ and no time-like geodesic is future-complete unless $(\mathcal{S}, \mathfrak{g})$ is isometric to

$$\mathfrak{g} = -dt^2 + e^{2t} g_{N^n},$$

where g_{N^n} is Ricci-flat, cf. [11] for further details.

Proposition 6.3 implies that dS space-times $(\mathcal{S}, \mathfrak{g})$ satisfying the strong energy and decay conditions (6.15) cannot be geodesically complete if $R_\gamma < 0$, and have at most one component of conformal infinity. This exhibits the role of the hypothesis $R_\gamma > 0$ in a more drastic way than the AH case.

We are interested in understanding when the vacuum dS metric $\mathfrak{g}^{dS}$ constructed in (6.8) is also complete to the future, and has a smooth future conformal infinity $\mathcal{I}^+$. In dimensions $n+1 > 4$, it is an interesting open problem to find sufficient conditions guaranteeing the existence of complete asymptotically simple vacuum dS space-times. However, in dimension 4, a basic result of H. Friedrich does give global existence of dS vacuum solutions, for small perturbations of the exact deSitter metric.

Theorem 6.4. [28] *Let γ be a smooth metric on S^3 and σ be a smooth transverse-traceless symmetric bilinear form on S^3. Suppose that γ is sufficiently close to the round metric g_{+1} on S^3 and $|\sigma|$ is sufficiently small, (measured with respect to γ). Then there exists a unique asymptotically simple vacuum dS space-time $(\mathcal{S}, \mathfrak{g})$, $\mathcal{S} = S^3 \times \mathbb{R}$ with smooth conformal compactification $\bar{\mathcal{S}} = \mathcal{S} \cup \mathcal{I}^- \cup \mathcal{I}^+$*

for which the Fefferman-Graham expansion satisfies $g_{(0)} = \gamma$ *and* $g_{(3)} = \sigma$ *on* $\mathcal{I}^-$. *If* γ *and* σ *are* C^ω, *then* $(\mathcal{S}, \mathfrak{g})$ *is* C^ω *conformally compact.*

Note that, in contrast to the AH or AdS situation, $g_{(0)}$ and $g_{(3)}$ are freely specifiable on $\mathcal{I}^-$, subject to smallness conditions. An alternate version of the result should allow one to freely specify $g_{(0)}$ on $\mathcal{I}^+$ and $\mathcal{I}^-$, provided they are both close to the round metric on S^3; this remains to be proved however.

Remark 6.5. This result is an exact analogue of the result of Graham-Lee [30] on AH Einstein perturbations of the Poincaré metric on the ball B^{n+1}; (since Friedrich's result predates that of Graham-Lee, the opposite statement is more accurate). It would be very interesting if a higher dimensional analogue of Friedrich's result could be proved, as in the Graham-Lee theorem.

We may now apply this result to the "initial" AH Einstein metric (M, g), $g = g^-$. Thus, on $M = B^4$, let g^- be an AH Einstein metric with C^ω boundary metric γ^- close to the round metric γ_{+1} on S^3, (so g^- is close to the Poincare metric on B^4). The metric g^- determines the terms $\gamma = g_{(0)} = g_{(0)}^{AH}$ and $g_{(3)}^{AH}$ in the Fefferman-Graham expansion. Let $\mathfrak{g}^{dS}$ be the unique vacuum dS solution given by Friedrich's theorem satisfying

$$(g_{(0)}^{dS})_{\mathcal{I}^-} = g_{(0)}^{AH}, \quad \text{and} \quad (g_{(3)}^{dS})_{\mathcal{I}^-} = -g_{(3)}^{AH}, \tag{6.18}$$

where $g_{(3)}^{dS}$ is defined as in (1.8) with respect to the future normal $T = \bar{\nabla}\rho$. Thus, the stress-energy tensors of g^- and $\mathfrak{g}^{dS}$ cancel at $\mathcal{I}^-$.

The vacuum dS solution $\mathfrak{g}^{dS}$ is globally defined, and has a C^ω compactification to $\mathcal{I}^-$ and $\mathcal{I}^+$. Let $(g_{(0)}^{dS})_{\mathcal{I}^+}$ and $(g_{(3)}^{dS})_{\mathcal{I}^+}$ be the boundary metric and stress-energy tensor of $\mathfrak{g}^{dS}$ at future conformal infinity $\mathcal{I}^+$. Then $(g_{(0)}^{dS})_{\mathcal{I}^+}$ is close to the round metric γ_{+1} on S^3, while $(g_{(3)}^{dS})_{\mathcal{I}^+}$ is close to 0, and both are real-analytic. By the Graham-Lee theorem [30], there is an AH Einstein metric g^+ on B^4 with boundary metric $\gamma_+ = (g_{(0)}^{dS})_{\mathcal{I}^+}$, and by boundary regularity [6], g^+ has a real-analytic compactification.

Thus, we have constructed a global $4+1$ dimensional space-time

$$(\mathcal{M}, \mathfrak{g}) = (\mathcal{M}^-, g^-) \cup (\mathcal{M}^{ext}, \mathfrak{g}^{ext}) \cup (\mathcal{M}^+, g^+). \tag{6.19}$$

This space-time is globally self-similar, with

$$\mathcal{L}_{\nabla\tau}\mathfrak{g} = 2\mathfrak{g}, \tag{6.20}$$

with a singularity at the vertex $\{0\}$. The metric $\mathfrak{g}$ is C^ω off the null cone $H = H^- \cup H^+$ and is $C^{3/2}$ across the null-cone away from $\{0\}$.

In general however, it is not clear if $(\mathcal{M}, \mathfrak{g})$ is a vacuum space-time. The stress-energy tensor $g_{(3)}^+$ of g^+ is globally determined by the boundary metric γ^+, and there is no apriori reason that one should have

$$g_{(3)}^+ = -(g_{(3)}^{dS})_{\mathcal{I}^+}, \tag{6.21}$$

as given by construction at $\mathcal{I}^-$. Thus, there may be an effective stress-energy of the gravitational field along the future light cone H^+ of $\{0\}$. Of course the vacuum equations (6.9) are satisfied everywhere off H^+.

It is of interest to understand if there exist non-trivial situations where (6.21) does hold, or to prove that it cannot hold. If (6.21) holds, then $(\mathcal{M}, \mathfrak{g})$ is a globally defined self-similar vacuum solution, with an isolated (naked) singularity at $\{0\}$. Of course in $3+1$ dimensions the only such space-time is empty Minkowski space $(\mathbb{R}^4, \eta)$.

We examine this issue on a particular family of examples.

Example 6.6. Let g^- be the AdS Taub-NUT metric on B^4, cf. [32] for example, (also called the Pedersen metric [44]), given by

$$g^- = \frac{E(r^2-1)}{F(r)}dr^2 + \frac{EF(r)}{(r^2-1)}\theta_1^2 + \frac{E(r^2-1)}{4}g_{S^2(1)}, \tag{6.22}$$

where $E \in (0,\infty)$ is any constant, $r \geq 1$, and

$$F(r) = Er^4 + (4-6E)r^2 + (8E-8)r + 4 - 3E. \tag{6.23}$$

The length of the S^1 parametrized by θ_1 is 2π. This metric is self-dual Einstein and has conformal infinity γ^- given by the Berger (or squashed) sphere with S^1 fibers of length $\beta = 2\pi E^{1/2}$ over $S^2(1)$. Clearly γ^- is C^ω, as is the geodesic compactification with boundary metric γ^-. Since g^- is self-dual, the stress-energy tensor $g_{(3)}$ is given by (5.5). When $E=1$, g^- is the Poincaré metric.

The deSitter continuation of g^- is the dS Taub-NUT metric on $\mathbb{R} \times S^3$, cf. [15] for instance, given by

$$\mathfrak{g}^{dS} = -\frac{E(\tau^2+1)}{A(\tau)}d\tau^2 + \frac{EA(\tau)}{(\tau^2+1)}\theta_1^2 + \frac{E(\tau^2+1)}{4}g_{S^2(1)}, \tag{6.24}$$

where $\tau \in (-\infty,\infty)$ and

$$A(\tau) = E\tau^4 - (4-6E)\tau^2 - (8E-8)\tau + 4 - 3E. \tag{6.25}$$

Again when $E=1$, $\mathfrak{g}^{dS}$ is the (exact) deSitter metric. For $\mathfrak{g}^{dS}$ to be complete and globally hyperbolic, without singularities, one needs $A(\tau) > 0$, for all τ. A lengthy but straightforward calculation shows this is the case exactly when

$$E \in (\frac{2}{3}, \frac{1}{3}(2+\sqrt{3})). \tag{6.26}$$

Suppose then E satisfies (6.26). By construction, the metrics g^- and $\mathfrak{g}^{dS}$ satisfy (6.18). Observe from the explicit form of (6.24) that

$$\gamma^- = \gamma^+. \tag{6.27}$$

Thus, even though $\mathfrak{g}^{dS}$ is not time-symmetric when $E \neq 1$, there is no gravitational scattering from past to future conformal infinity, in the sense that $\mathcal{I}^-$ is isometric to $\mathcal{I}^+$. However, further computation shows that (6.21) does not hold; instead one has $g^+_{(3)} = (g^{dS}_{(3)})_{\mathcal{I}^+}$, so that the full metric $\mathfrak{g}$ has an effective stress-energy tensor on the future null cone of $\{0\}$. We note however that there is an AH Taub-NUT metric satisfying (6.21) which has an isolated conical (nut) singularity at the origin of B^4; in this case, the formula (6.23) is replaced by a more general formula allowing two independent parameters, the mass and nut charge, in place of the one parameter E, cf. [24], [25]. If one fills in H^+ with such a metric, then the effective stress-energy tensor of $(\mathcal{M}, \mathfrak{g})$ is located on the future world line of the singularity $\{0\}$.

Returning to the discussion of the de Sitter metrics (6.24), at the extremal values $E_- = \frac{2}{3}$ and $E_+ = \frac{1}{3}(2+\sqrt{3})$, the metric $\mathfrak{g}^{dS}$ is still complete and globally hyperbolic. However, at these values, g^{dS} is not in the space of metrics with smooth $\mathcal{I}^+$ and $\mathcal{I}^-$; instead, it is in the boundary of this space. To explain this, let $\tau_- = -1$, and $\tau_+ = 2-\sqrt{3}$. Then at $E_\pm$, $A(\tau) \geq 0$, with $A(\tau) = 0$ exactly at $\tau_\pm$. Each metric g^{dS} breaks up into a pair of complete, globally hyperbolic metrics, g^{dS}_p and g^{dS}_f parametrized on $(-\infty, \tau_\pm)$ and $(\tau_\pm, \infty)$ respectively. The metric g^{dS}_p has a smooth past conformal infinity $\mathcal{I}^-$, but $\mathcal{I}^+ = \emptyset$, while g^{dS}_f has a smooth future conformal infinity $\mathcal{I}^+$ but $\mathcal{I}^- = \emptyset$. These metrics correspond to degenerate black hole metrics, and are analogous, in a dual sense, to the situation in Remark 2.3.

For E outside the range (6.26), the function $A(\tau)$ changes sign, and the metrics g^{dS} develop closed time-like curves, as with the behavior of the $\Lambda = 0$ Taub-NUT metrics

It is also straightforward to compute that $R_{\gamma^-} > 0$ exactly for E in the range $E \in (0, 4)$. This shows that the converse of Proposition 6.3 does not hold, i.e. the condition $R_\gamma > 0$ is not sufficient to imply that a vacuum dS solution with smooth $\mathcal{I}^-$ is complete to the future.

It would be very interesting to generalize this example. For instance, can the same construction of globally self-similar almost-vacuum solutions be carried out for general AdS and dS Bianchi IX space-times, which have conformal infinity a general $SU(2)$ invariant metric on S^3? The AdS Bianchi IX metrics are self-dual, and have been described in detail by Hitchin [36]. Is the relation (6.27), related to self-duality?

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Department of Mathematics
S.U.N.Y. at Stony Brook
Stony Brook, NY 11794-3651
anderson@math.sunysb.edu