

THE NIRENBERG PROBLEM OF PRESCRIBED GAUSS CURVATURE ON S^2

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ABSTRACT. We introduce a new perspective on the classical Nirenberg problem of understanding the possible Gauss curvatures of metrics on S^2 conformal to the round metric. A key tool is to employ the smooth Cheeger-Gromov compactness theorem to obtain general and essentially sharp *a priori* estimates for Gauss curvatures K contained in naturally defined stable regions. We prove that in such stable regions, the map $u \rightarrow K_g$, $g = e^{2u}g_{+1}$ is a proper Fredholm map with well-defined degree on each component. This leads to a number of new existence and non-existence results. We also present a new proof and generalization of the Moser theorem on Gauss curvatures of even conformal metrics on S^2 .

In contrast to previous work, the work here does not use any of the Sobolev-type inequalities of Trudinger-Moser-Aubin-Onofri.

1. INTRODUCTION

In this work, we study the well-known Nirenberg problem: which smooth functions K on S^2 are realized as the Gauss curvature of a metric g on S^2 pointwise conformal to the standard round metric g_{+1} of radius 1 on $S^2 \subset \mathbb{R}^3$? For $g = e^{2u}g_{+1}$, the equation for the Gauss curvature K of g is

$$(1.1) \quad e^{2u}K = 1 - \Delta u.$$

so that the Nirenberg problem asks to characterize for which K is the nonlinear PDE (1.1) solvable. It is also of interest to understand the uniqueness or multiplicity of solutions.

Prior work on the Nirenberg problem is based on its variational formulation initiated by Moser [36]. For any given K consider the functional

$$(1.2) \quad J_K(u) = \int_{S^2} (|du|^2 + 2u)dv_0 - \log\left(\int_{S^2} K e^{2u} dv_0\right),$$

where dv_0 is the usual round area form on S^2 , normalized to unit area. Critical points u of J are given by solutions of (1.1), (up to constants), i.e. (1.1) is the Euler-Lagrange equation of J_K . Moser in [37] began the analysis of J_K based on fundamental Sobolev inequalities due to Trudinger [46] and Moser [36], leading then to a solution of the existence problem for even functions K on S^2 .

The functional J_K does not satisfy the Palais-Smale Condition C and except in case $K = \text{const}$, the infimum of J_K is never realized, [27]. Most all general existence results after [36] have been obtained by identifying conditions on K that prevent the *a priori* possible blow-up or bubble behavior of suitable approximate solutions of (1.1). These approximate solutions are sequences approaching a minimax (mountain pass) critical point of the functional J_K . Further Sobolev inequalities of Onofri [40] and Aubin [4] play an important role in this analysis.

Fundamental progress on the Nirenberg problem was made by Kazdan-Warner [32], [33], Aubin [4], Chang-Yang [9]- [11], Chang-Gursky-Yang [12], Chen-Ding [16] with further important progress made by Han [25], Chang-Liu [13] and many others. The full literature on this problem is vast and we mention here only further work in [27], [29], [45]; see also [30] for a recent survey and further references. The problem is still far from a complete resolution, even in the case of positive Gauss curvature.

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In this paper, we give new proofs of almost all of the results above from a different perspective, and derive as well new existence and non-existence results. In particular, the proofs of these results do not rely on any of the sharp Sobolev inequalities or (with one exception) any variational formulation of the problem.

To introduce the point of view taken here, we recast the problem (1.1) as follows. Let $C^{m,\alpha}$, $m \geq 4$, $\alpha \in (0, 1)$ be the Banach space of $C^{m,\alpha}$ functions $u : S^2 \rightarrow \mathbb{R}$. This is considered as the space of conformal factors for metrics $g = e^{2u}g_{+1}$. Consider the map

$$(1.3) \quad \begin{aligned} \pi : C^{m,\alpha} &\rightarrow C_+^{m-2,\alpha}, \\ \pi(u) &= K_g, \end{aligned}$$

where K_g is the Gauss curvature of g and $C_+^{m-2,\alpha}$ is the space of $C^{m-2,\alpha}$ functions K such that $K(x) > 0$ for some $x \in S^2$. By integrating (1.1), one easily sees that π maps into $C_+^{m-2,\alpha}$. The Nirenberg problem then asks to describe the image $Im \pi$ of π . It is well-known that (1.3) is not solvable for all $K \in C_+^{m-2,\alpha}$. There is a fundamental obstruction due to Kazdan-Warner [32]: if K is the Gauss curvature of the conformal metric g , then

$$(1.4) \quad \int_{S^2} X(K) dv_g = 0,$$

for all linear vector fields X on S^2 ; $X = \nabla \ell$ where ℓ is the restriction of a linear function on \mathbb{R}^3 to S^2 . For example, the functions $K = 1 + \ell$ are not realizable as the Gauss curvature of a conformal metric $g = e^{2u}g_{+1}$.

From (1.1) and standard elliptic theory, π is a smooth (nonlinear) Fredholm map, of Fredholm index 0. A broad understanding of the global properties of the map π , for instance its image, requires understanding when the Fredholm map π is proper; recall that a map is proper if the inverse image of any compact set in the target is compact in the domain. Loosely speaking, $\pi : \mathcal{U} \rightarrow \mathcal{V}$ proper on a domain \mathcal{V} is equivalent to the statement that control of $K = \pi(u) \in \mathcal{V}$ implies control of any solution u of (1.1); this equivalent to the existence of *a priori* estimates for solutions of (1.1) with $K \in \mathcal{V}$.

Now a general principle in Riemannian geometry is that control of the curvature K implies control of the metric g , at least given suitable bounds on global quantities such as volume and diameter. This principle, which has origins in early work in global differential geometry, is expressed concretely in two fundamental results in geometry, namely the Cheeger finiteness theorem [14] and the (smooth) Gromov convergence theorem [24], [22], [41], now generally referred to as the Cheeger-Gromov convergence theorem; this is discussed in detail in Section 2. The control on the metric given by the control on the curvature comes however only *modulo diffeomorphisms*; the action of diffeomorphisms is crucial and cannot be avoided.

Applied to the case of S^2 , it follows that control of K implies control of $g = e^{2u}g_{+1}$ modulo diffeomorphisms of S^2 , given suitable control on the area and diameter of (S^2, g) . Now the Nirenberg problem per se is of course *not* diffeomorphism invariant. However, on S^2 there is a unique conformal class, and so all metrics may be pulled back by diffeomorphisms to the fixed conformal class $[g_{+1}]$. This pull-back is unique modulo the conformal group $\text{Conf}(S^2)$.

This highlights the central role played by the non-compact conformal group $\text{Conf}(S^2)$ of S^2 . The importance of the action of $\text{Conf}(S^2)$ is certainly well-known from all previous analyses of the Nirenberg problem from the purely PDE point of view. However, the emphasis is quite different here, taking the Riemannian geometry of the metrics much more closely into consideration.

To describe the first main result, we consider first the behavior of K near its zero level set. Thus, suppose $q \in S^2$ satisfies $K(q) = 0$ and that in a small neighborhood of q , K has the form

$$(1.5) \quad K = \chi r^{2a},$$

where $r(x) = \text{dist}_{S^2(1)}(x, q)$, $a \geq 1$ and χ is at least continuous near q . Thus we are considering points on $\{K = 0\}$ which are either non-degenerate local maxima or minima of K (when $a = 1$ and $\chi(q) \neq 0$) or arbitrary degenerate critical points (when $a > 1$ or $\chi(q) = 0$). Let

$$(1.6) \quad \mathcal{C} = \{K \in C_+^{m-2, \alpha} : \text{if (1.5) holds at } q, \text{ then } \Delta K(q) > 0\}.$$

In particular, if $K \in \mathcal{C}$ and $K(q) = 0$, then q is not a local maximum point of K . We note that if q is a non-degenerate saddle point of K , then (1.5) does not hold near q ; such K are thus in \mathcal{C} . We also note that it is straightforward to see that \mathcal{C} is invariant under the action of the diffeomorphism group $\text{Diff}^{m-2, \alpha}$ on $C_+^{m-2, \alpha}$. Next let

$$(1.7) \quad \mathcal{N} = \{K \in C_+^{m, \alpha} : |\nabla K|(p) + |\Delta K|(p) > 0, \forall p \text{ s.t. } K(p) > 0\}.$$

Thus $K \in \mathcal{N}$ if at any critical point q of K with $K(q) > 0$, $\Delta K(q) \neq 0$. When $K > 0$ everywhere, \mathcal{N} is exactly the space of non-degenerate functions as defined by Chang-Yang in [11], [12].

In contrast to \mathcal{C} , the space \mathcal{N} is *not* invariant under the action of the diffeomorphism group $\text{Diff}^{m-2, \alpha}$. It is easy to see that both \mathcal{N} and \mathcal{C} are open and dense in $C_+^{m-2, \alpha}$ and are invariant under the action of $\text{Conf}(S^2)$. Let

$$\mathcal{U} = \pi^{-1}(\mathcal{C} \cap \mathcal{N}),$$

and, setting

$$\mathcal{K} = \mathcal{C} \cap \mathcal{N},$$

let

$$(1.8) \quad \pi_0 = \pi|_{\mathcal{U}} : \mathcal{U} \rightarrow \mathcal{K}, \quad \pi_0(u) = K_u,$$

be the restriction of π to \mathcal{U} . The first main result is the following:

Theorem 1.1. *On the domain \mathcal{U} , the curvature map*

$$\pi_0 : \mathcal{U} \rightarrow \mathcal{K}$$

in (1.8) is a proper Fredholm map of index 0.

In particular, for any $K \in \mathcal{K}$, the space of solutions $\pi^{-1}(K)$ of (1.1) is compact and is finite for generic $K \in \mathcal{K}$. Theorem 1.1 is also essentially sharp in that π does not extend to a proper map on any larger open domain \mathcal{V} containing \mathcal{U} , cf. Remark 5.11. Namely, we show that bubble solutions of unbounded area may form as $K \in \mathcal{K}$ approaches $\partial\mathcal{C}$. This follows from the basic work of Ding-Liu in [18] and later by Borer-Galimberti-Struwe in [8], cf. Remark 2.7. Similarly, we show that solutions in \mathcal{U} degenerate due to the non-compactness of the conformal group $\text{Conf}(S^2)$ as $K \in \mathcal{K}$ approaches $\partial\mathcal{N}$. This in turn follows from the basic work of Chang-Yang [9]- [11].

Theorem 1.1 gives *a priori* estimates for solutions u of (1.1) for all $K = \pi(u) \in \mathcal{K}$, cf. Remark 2.10. Such estimates have been previously obtained by Chang-Gursky-Yang [12] for $K \in \mathcal{K}$ in case $K > 0$ and extended by Chen-Li in [13] to a larger, but still strictly proper, subset of $K \in \mathcal{K}$ with K of variable sign. Theorem 1.1 gives the first such estimates in the full region \mathcal{K} . Moreover, the proof of Theorem 1.1 is quite different than previous work on this topic. As noted above, it does not rely on any delicate analysis related to sharp Sobolev inequalities of Trudinger-Moser-Aubin type or on any blow-up analysis or proof by contradiction. The proof is arguably much simpler than previous proofs.

We prove in Section 2 that the complement $\partial\mathcal{K} = C_+^{m-2, \alpha} \setminus \mathcal{K}$ is a closed, rectifiable set of codimension one in $C_+^{m-2, \alpha}$. Generically $\partial\mathcal{K}$ is a collection of smooth, codimension one hypersurfaces.

It is well-known that proper Fredholm maps $F : X \rightarrow Y$ of index zero between Banach spaces X, Y , (or orientable Banach manifolds with Y connected) have a well-defined \mathbb{Z} -valued degree given

by

$$\deg F = \sum_{x \in F^{-1}(y)} \text{sign}(DF_x) \in \mathbb{Z}$$

where y is any regular value of F and the sign of DF_x is ± 1 according to whether $DF(x)$ preserves or reverses orientation at x .

Thus let \mathcal{K}^i denote the collection of path components of $\mathcal{K} \subset C_+^{m-2,\alpha}$ and let $\mathcal{U}^i = (\pi_0)^{-1}(\mathcal{K}^i)$ be the inverse image in $C^{m,\alpha}$. Note that *a priori* \mathcal{U}^i may not be connected. The restriction of π_0 to \mathcal{U}^i gives a proper Fredholm map

$$(1.9) \quad \pi^i : \mathcal{U}^i \rightarrow \mathcal{K}^i,$$

with a well-defined degree. To calculate the degree, let K be any regular value of π^i in \mathcal{K}^i and without loss of generality, assume K is a Morse function on S^2 . Let

$$(1.10) \quad Z_i(K) = \{q : \nabla K(q) = 0, \Delta K(q) < 0 \text{ and } K(q) > 0\},$$

so that $Z_i(K)$ is a finite set of points in S^2 . Let $\text{ind } q$ be the index of the non-degenerate critical point q of K ; ($\text{ind } q$ is even for q a local minimum or maximum, while $\text{ind } q$ is odd if q is a saddle point). The degree of π^i on the domain of positive curvature functions $K > 0$ in \mathcal{K}^i was calculated by Chang-Gursky-Yang [12] and we prove that the same formula extends to the full domains \mathcal{K}^i .

Theorem 1.2. *One has*

$$(1.11) \quad \deg \pi^i = \sum_{q \in Z_i(K)} (-1)^{\text{ind } q} - 1,$$

for any $Z_i(K)$ as in (1.10). Equivalently, if M is the number of positive local maxima K and s^- is the number of saddle points of K where $K > 0$ and $\Delta K < 0$, then

$$\deg \pi^i = M - s^- - 1.$$

It is not difficult to see that $\deg \pi^i$ may assume any value in \mathbb{Z} , i.e. for each $n \in \mathbb{Z}$, there exists i such that $\deg \pi^i = n$, cf. Lemma 4.1. It is not clear (and is perhaps unlikely) whether the components \mathcal{K}^i of \mathcal{K} are uniquely determined by their degree, cf. Remark 4.2 for further discussion.

Combining the variational formulation (1.2) of equation (1.1) with Theorem 1.1 leads to a second formula for the degree.

Theorem 1.3. *The degree of π^i is also given (up to a fixed sign determined by orientation) by*

$$(1.12) \quad \deg \pi^i = \sum_{u_j \in (\pi^i)^{-1}(K)} (-1)^{\text{ind}(u_j)},$$

where K is any regular value of π^i .

Here the index $\text{ind}(u_j)$ is the dimension of the negative eigenspace of the Hessian $D^2 J_K$ at u_j , for J_K as in (1.2).

Theorems 1.2 or 1.3 prove existence of solutions of (1.1), together with a signed count on the number of generic solutions, for all $K \in C_+^{m-2,\alpha}$ except K in the degree zero components and K on the boundary $K \in \partial \mathcal{K}$. We prove in Remark 4.2 that the degree zero component is connected and so call this component \mathcal{K}^0 . Of course all K satisfying (1.4) are necessarily in \mathcal{K}^0 or $\partial \mathcal{K}$.

To obtain further information in these regions, we analyse the structure of the regular and singular points of the Fredholm map π . The Sard-Smale theorem [44] implies that the regular values of π are open and dense in the target space $C_+^{m-2,\alpha}$. Of course, *a priori*, on the degree zero component

$$(1.13) \quad \pi^0 : \mathcal{U}^0 \rightarrow \mathcal{K}^0,$$

there may be no corresponding regular points of π ; a generic value in the target space \mathcal{K}^0 may have empty inverse image. (For instance, the image may be of high codimension or even empty).

Theorem 1.4. *The set Σ of singular points of π in (1.3) is a stratified space with strata of codimension $s \geq 1$ in $C^{m,\alpha}$. In particular, the regular points and regular values of π are open and dense in the domain and range of π .*

We refer to Section 5 for the definition of a stratified space. This result gives a sharpening of the Perturbation Theorem in [33]. The image of π^0 in (1.13),

$$\mathcal{E} = \text{Im } \pi^0 \subset \mathcal{K}^0 \subset C_+^{m-2,\alpha}$$

is a non-empty *closed* subset of \mathcal{K}^0 . This is a consequence of the fact that π^0 is proper, cf. Proposition 5.7 below. The Kazdan-Warner obstruction (1.4) implies that π^0 is not surjective, so the complement of the image $\mathcal{D} = (\text{Im } \pi^0)^c \subset \mathcal{K}^0$ is a non-empty open set in \mathcal{K}^0 (and hence open in $C_+^{m-2,\alpha}$). We show in Proposition 4.4 that for any given K , if ℓ is any linear function with $|\ell|$ sufficiently large (depending on K), then (1.1) is not solvable for $K + \ell$, i.e. $K + \ell \notin \text{Im } \pi$.

In the region \mathcal{E} , solutions to the Nirenberg problem typically come in pairs, with J_K -index of opposite parity. We prove in Proposition 5.8 that the boundary $\mathcal{B} \subset \partial\mathcal{E}$ separating the existence region \mathcal{E} from the non-existence region \mathcal{D} in $C_+^{m-2,\alpha}$ is generically a smooth bifurcation locus for the creation or annihilation of pairs of solutions of the Nirenberg problem, cf. also Remark 5.9.

Next, the boundary $\partial\mathcal{K}$ decomposes as a union

$$\partial\mathcal{K} = \partial\mathcal{C} \cup \partial\mathcal{N}.$$

The first region corresponds to a wall-crossing of a local maximum of K passing through the value 0 while the second region corresponds to a wall-crossing where the value of ΔK changes sign at a critical point q of K with $K(q) > 0$. As noted above, degenerations given by bubbles of infinite area may form on approach to $\partial\mathcal{C}$ while the non-compactness of the conformal group $\text{Conf}(S^2)$ causes degeneration on approach to $\partial\mathcal{N}$.

Regarding $\partial\mathcal{C}$, we have the following result:

Theorem 1.5. *For any two distinct components $\mathcal{K}^m, \mathcal{K}^n$ of degree m and n respectively with $m \neq 0$ and $m < n$, all Morse functions $K \in \mathcal{M}$ in $\partial\mathcal{C}_m^n = \partial\mathcal{C} \cap \partial\mathcal{K}^m \cap \partial\mathcal{K}^n$ are realizable as Gauss curvatures of conformal metrics, i.e.*

$$\partial\mathcal{C}_m^n \cap \mathcal{M} \subset \text{Im } \pi.$$

This is proved in Proposition 4.7 below. A result of this generality is not known for $\partial\mathcal{N}$. We prove a partial result holds near the point $K = 1$, cf. the discussion preceding Remark 5.11. Also large families of explicit solutions in $\partial\mathcal{N}$ are constructed in Proposition 4.5 and Remark 4.6; we show that if u is any eigenfunction of the Laplacian on $S^2(1)$ then $\pi(u) = K_u \in \partial\mathcal{N}$.

We conclude the paper with an existence result for functions K which have a symmetry which breaks the action of the conformal group $\text{Conf}(S^2)$ on $C_+^{m-2,\alpha}$, as in Moser's theorem [37] on the existence of solutions of (1.1) with K even. Thus, let Γ be a finite subgroup of $O(3) = \text{Isom}(S^2(1))$. The action of Γ is said to break the (non-compact) action of the conformal group if the Γ -action and $\text{Conf}(S^2)/O(3)$ -action on S^2 do not commute. Thus for any conformal dilation $\varphi \in \text{Conf}(S^2)/O(3) \simeq \mathbb{R}^3$, with $\varphi \neq \text{Id}$, there exists $\gamma \in \Gamma$ such that

$$\gamma \circ \varphi \neq \varphi \circ \gamma.$$

For instance, it is easy to see that the \mathbb{Z}_2 -action generated by the antipodal map breaks the non-compact action of $\text{Conf}(S^2)$.

Theorem 1.6. *Let $\Gamma \subset O(3)$ be a finite subgroup breaking the action of $\text{Conf}(S^2)$. Then any Γ -invariant $C_+^{m-2,\alpha}$ function $K : S^2 \rightarrow \mathbb{R}$ is the Gauss curvature of a Γ -invariant conformal metric g on S^2 .*

The contents of the paper are briefly as follows. In Section 2, we prove Theorem 1.1 using the geometric convergence theory of Cheeger-Gromov, combined with a basic estimate of Astala [3] on regularity of quasi-conformal mappings and the basic work of Chang-Yang [11]. We also relate the decomposition (1.9) into path components of \mathcal{K} to the more well-known decomposition of the space of Morse functions in $C_+^{m-2,\alpha}$ into Morse chambers. Theorems 1.2 and 1.3 are proved in Section 3, building on the basic degree formula of Chang-Gursky-Yang [12] for Theorem 1.2 and on general degree theoretic properties for proper Fredholm maps for Theorem 1.3. Section 4 is a bridge between the earlier and later sections and discusses several existence and non-existence results, including Theorem 1.5, in the regions \mathcal{K}^0 , $\partial\mathcal{C}$ and $\partial\mathcal{N}$. In Section 5, we analyse the structure of singular and regular points of π and prove Theorem 1.4. Finally, Section 6 is devoted to the proof of Theorem 1.6. Further results on the topics above are given in the individual sections.

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2. PROPERNESS OF π .

The basic issue in studying the global properties of the Fredholm map π in (1.3) is whether π is proper. Thus, given any sequence $g_i = e^{2u_i}g_{+1}$ of metrics in the standard round conformal class with Gauss curvature $K_i = K_{g_i}$ such that

$$(2.1) \quad K_i \rightarrow K \quad \text{in } C_+^{m-2,\alpha},$$

the issue is when does a subsequence of $\{u_i\}$ converge to a limit in $C^{m,\alpha}$. It is well-known that this is not true in general, due to the non-compactness of the conformal group $\text{Conf}(S^2) \simeq PGL(2, \mathbb{C})$ of fractional linear transformations of $(S^2(1), g_{+1})$. The maximal compact subgroup of $PGL(2, \mathbb{C})$ is $O(3)$, the isometry group of $S^2(1)$, and the quotient satisfies

$$(2.2) \quad PGL(2, \mathbb{C})/O(3) \simeq \mathbb{H}^3 \simeq \mathbb{R}^3.$$

Elements in the quotient \mathbb{R}^3 are identified with the conformal dilations as follows. Given a point $p \in S^2$, in the chart given by stereographic projection from the pole p ,

$$\varphi_{p,t}(x) = tx,$$

represents the conformal dilation by t with source $-p$ and sink p . One has $\varphi_{p,1} = Id$, which serves as the origin $0 \in \mathbb{R}^3$. Also $(\varphi_{p,t})^{-1} = \varphi_{p,t^{-1}} = \varphi_{-p,t}$. Similarly the map $\varphi_{p,q,t} = t(x - x_0)$ represents the conformal dilation by t with source q and sink p , where q maps to x_0 under stereographic projection. (This corresponds to a conjugation of $\varphi_{p,t}$ by a rotation).

The action (or skew-action) of $\text{Conf}(S^2)$ on the target space $C_+^{m-2,\alpha}$ of curvature functions is by pre-composition,

$$(K, \varphi) \rightarrow K \circ \varphi.$$

This action is clearly not proper, since the constant functions $K = \text{const}$ are fixed points of the action. On the other hand, it is easy to see that the action of $\text{Conf}(S^2)$ is proper on $C_+^{m-2,\alpha}$ away from the constant functions.

To describe the action of $\text{Conf}(S^2)$ on the domain space $C^{m,\alpha}$ of conformal factors, let $\varphi \in \text{Conf}(S^2)$ be an orientation preserving conformal diffeomorphism, so that $\varphi^*g_{+1} = \chi_\varphi^2 g_{+1}$, where

$\chi_\varphi^2 = \det D\varphi > 0$. For $u \in C^{m,\alpha}$ with $g = e^{2u}g_{+1}$, one has $\varphi^*g = e^{2u\circ\varphi}\varphi^*g_{+1} = e^{2u\circ\varphi}\chi_\varphi^2g_{+1}$. Thus $\text{Conf}(S^2)$ acts on $C^{m,\alpha}$ as

$$(2.3) \quad (\varphi, u) \rightarrow \varphi^*u = u_\varphi = (u \circ \varphi) + \log \chi_\varphi.$$

For conformal dilations $\varphi_{p,t}$, the volume distortion $\chi_{p,t}^2$ tends to the Dirac measure supported at p as $t \rightarrow \infty$. Note that since $\varphi_{-p,s} \circ \varphi_{p,t} = \varphi_{p,\frac{t}{s}}$, $\varphi_{-p,s}^*(\varphi_{p,t}^*(0)) = \log \chi_{p,\frac{t}{s}}$ so that

$$(2.4) \quad \varphi_{-p,s}^*(\log \chi_{p,t}) = \log \chi_{p,t} \circ \varphi_{-p,s} + \log \chi_{-p,s} = \log \chi_{p,\frac{t}{s}}.$$

For later purposes (e.g. Lemma 2.5), observe that the level set $\{\log \chi_{p,t} = 0\}$ converges to the point p as $t \rightarrow \infty$ while the level set $\{\varphi_{-p,s}^* \log \chi_{p,t} = 0\}$ remains bounded away from p and $-p$ as $t, s \rightarrow \infty$ but $\frac{t}{s}$ remains bounded.

In contrast to the action of $\text{Conf}(S^2)$ on the target space $C_+^{m-2,\alpha}$ of curvature functions, the action of $\text{Conf}(S^2)$ on the domain $C^{m,\alpha}$ is smooth and proper; all orbits are properly embedded. The map π is equivariant with respect to these actions of $\text{Conf}(S^2)$, i.e.

$$(2.5) \quad \pi(\varphi^*u) = \varphi^*\pi(u).$$

In particular, π maps $\text{Conf}(S^2)$ orbits in the domain $C^{m,\alpha}$ to $\text{Conf}(S^2)$ orbits in the target $C_+^{m-2,\alpha}$ and so π is not proper onto a neighborhood of the constant functions $K = \text{const}$.

Following [4], let \mathcal{S} be the space of conformal factors e^{2u} with zero center of mass,

$$(2.6) \quad \mathcal{S} = \{u \in C^{m,\alpha} : \forall i, \int_{S^2} e^{2u} x_i dv_{+1} = 0\},$$

where x_i are the restrictions of the standard linear coordinate functions on \mathbb{R}^3 to $S^2(1)$. The space \mathcal{S} is a smooth codimension 3 hypersurface in $C^{m,\alpha}$. It is proved in [13] that \mathcal{S} is a smooth global slice for the non-compact action of $\text{Conf}(S^2)$ on $C^{m,\alpha}$, in that there is a smooth diffeomorphism

$$(2.7) \quad C^{m,\alpha} \simeq \mathcal{S} \times (\text{Conf}(S^2)/O(3)) \simeq \mathcal{S} \times \mathbb{R}^3.$$

In particular, for any $u \in C^{m,\alpha}$ there is a unique conformal dilation $\varphi_{p,t}$, $(p,t) \in \mathbb{R}^3$, such that $(\varphi_{p,t})^*u \in \mathcal{S}$. The dilation $\varphi_{p,t}$ depends smoothly on u .

Given this background, we study the properness issue in general by first recalling the Cheeger-Gromov convergence theorem, expressed here in the very special case of S^2 . Let $W^{k,p}$ denote the Sobolev space of functions on S^2 with k weak derivatives in L^p ; convergence in the weak topology in $W^{k,p}$ is denoted by $W_*^{k,p}$.

Proposition 2.1. (Cheeger-Gromov). *Let g_i be a sequence of C^2 metrics on S^2 with uniformly bounded Gauss curvature $K_i = K_{g_i}$,*

$$(2.8) \quad |K_i| \leq K_0 < \infty.$$

Suppose there are constants $v_0 > 0$ and $D_0 < \infty$ such that

$$(2.9) \quad \text{area}_{g_i} S^2 \geq v_0 \quad \text{and} \quad \text{diam}_{g_i} S^2 \leq D_0.$$

Then there is a sequence of diffeomorphisms $\psi_i \in \text{Diff}^{2,\alpha} \cap W^{3,p}$ such that

$$(2.10) \quad \tilde{g}_i = \psi_i^* g_i$$

is uniformly bounded in $C^{1,\alpha} \cap W^{2,p}$, for any $\alpha < 1$, $p < \infty$. In particular, a subsequence converges

$$(2.11) \quad \tilde{g}_i \rightarrow \tilde{g} \quad \text{in} \quad C^{1,\alpha},$$

and weakly in $W^{2,p}$ (i.e. in $W_^{2,p}$) for any $p < \infty$ to a limit $C^{1,\alpha} \cap W^{2,p}$ metric \tilde{g} on S^2 .*

Proof: This is a standard result that holds in all dimensions and detailed proofs are given in [22], [41], [42] for instance. For later purposes, we sketch some of the main ideas of the proof. A key idea is to locally represent any metric g by well-controlled harmonic coordinate charts, i.e. local diffeomorphisms

$$F : B(r_0) \subset \mathbb{R}^2 \rightarrow S^2,$$

with $F^{-1} = (x_1, x_2)$ given by a pair of harmonic functions x_i with respect to g (or equivalently with respect to g_{+1} when g is conformal to g_{+1}). The local charts F are constructed so that the metric $\bar{g} = F^*g$ is bounded in the $C^{1,\alpha} \cap W^{2,p}$ norm by the L^∞ norm of $\bar{K} = K \circ F$, given the global bounds (2.9). The key to this control is the well-known expression for the components $\bar{g}_{\alpha\beta}$ of a metric in local harmonic coordinates given by

$$(2.12) \quad \Delta_{\bar{g}} \bar{g}_{\alpha\beta} + Q_{\alpha\beta}(\bar{g}, \partial \bar{g}) = -2(Ric_{\bar{g}})_{\alpha\beta} = -2\bar{K} \bar{g}_{\alpha\beta},$$

where Q is a lower order term, quadratic in \bar{g} and its first derivatives. Here $\Delta_{\bar{g}}$ is the Laplacian of \bar{g} , given in the harmonic coordinate chart as

$$\Delta_{\bar{g}} = \bar{g}^{ab} \partial_a \partial_b.$$

The system (2.12) is an elliptic system for $\bar{g}_{\alpha\beta}$ and the $C^{1,\alpha} \cap W^{2,p}$ bounds on $\bar{g}_{\alpha\beta}$ follow from standard elliptic estimates, given L^∞ control on \bar{K} .

The diffeomorphisms ψ_i in (2.10) are constructed by assembling the overlap maps of such local charts by a geometric gluing, such as a center of mass averaging. Elliptic regularity again shows that each ψ_i is $C^{2,\alpha} \cap W^{3,p}$ smooth. The uniform $C^{1,\alpha} \cap W^{2,p}$ local control on the metrics $F_i^*g_i$ then easily leads to the convergence (2.11) by the Arzela-Ascoli theorem. ■

We may apply Proposition 2.1 to a sequence $g_i = e^{2u_i}g_{+1}$ of conformal metrics on S^2 satisfying (2.8)-(2.9). The uniformization theorem on S^2 implies that any $C^{1,\alpha} \cap W^{2,p}$ metric \tilde{g} is of the form $\tilde{g} = \psi^*(e^{2u}g_{+1})$ for some $\psi \in \text{Diff}^{2,\alpha} \cap W^{3,p}$ and $u \in C^{1,\alpha} \cap W^{2,p}$. Thus, in a subsequence,

$$(2.13) \quad \tilde{g}_i = \psi_i^*(g_i) \rightarrow \tilde{g} = \psi^*(e^{2u}g_{+1}),$$

in $C^{1,\alpha} \cap W_*^{2,p}$. Replacing ψ_i by $\tilde{\psi}_i = \psi_i \circ \psi^{-1}$ and dropping the tilde from the notation, (2.13) then becomes

$$(2.14) \quad \psi_i^*(e^{2u_i}g_{+1}) \rightarrow e^{2u}g_{+1},$$

in $C^{1,\alpha} \cap W_*^{2,p}$. The equation (2.14) means that the diffeomorphisms ψ_i are $(1 + \varepsilon_i)$ -quasi-conformal with $\varepsilon_i \rightarrow 0$ as $i \rightarrow \infty$. By a standard result (cf. [35]) any sequence of $(1 + \varepsilon_i)$ -quasi-conformal diffeomorphisms has a subsequence converging to a limit, modulo the action of the conformal group. (This is the normal family principle for quasi-conformal mappings). Any such limit is clearly a conformal map of S^2 . Thus, there exist conformal maps $\varphi_i \in \text{Conf}(S^2)$ such that

$$(2.15) \quad \eta_i := \varphi_i^{-1} \circ \psi_i \rightarrow Id.$$

The diffeomorphisms η_i are $(1 + \varepsilon_i)$ quasi-conformal with $\varepsilon_i \rightarrow 0$ as $i \rightarrow \infty$. The equations (2.14)-(2.15) will be a main focus of attention after Remark 2.7 below. First however we must address the situation when one of the hypotheses in (2.9) fails.

Remark 2.2. We first note that Proposition 2.1 is essentially local; suitable versions of the result hold for bounded domains or complete, non-compact manifolds, when one works with respect to given base points, cf. [42] for further discussion of the general situation.

In the case at hand, consider a pointed sequence (S^2, g_i, y_i) , $g_i = e^{2u_i}g_{+1}$ satisfying (2.8), where y_i is any sequence of base points $y_i \in S^2$. Suppose the following local non-collapse assumption (analogous to (2.9)) holds:

$$(2.16) \quad \text{area}_{g_i} B_{y_i}(1) \geq v_0 > 0.$$

Here v_0 is an arbitrary positive constant and $B_{y_i}(1)$ denotes the g_i -geodesic ball of radius 1 about y_i . Then the Cheeger-Gromov theory implies that a subsequence converges in the pointed $C^{1,\alpha} \cap W_*^{2,p}$ topology modulo diffeomorphisms to a complete Riemannian surface $(\Omega, \tilde{g}, \tilde{y})$. In general Ω is an open surface; if Ω is closed (and hence compact), then necessarily $\Omega = S^2$. If Ω is open, then $\text{diam}_{g_i} S^2 \rightarrow \infty$. The convergence modulo diffeomorphisms means that there is an exhaustion Ω_k of Ω , with $\Omega_k \subset \Omega_{k+1}$, $\cup \Omega_k = \Omega$, $\tilde{y} \in \Omega_k$ for all k , and embeddings $\psi_k : \Omega_k \rightarrow S^2$, $\psi_k(\tilde{y}) = y_k$, such that $\tilde{g}_i = \psi_k^*(e^{2u_i} g_{+1}) \rightarrow \tilde{g}|_{\Omega_k}$. In addition, by the uniformization theorem, there is a conformal embedding $F : \Omega \rightarrow S^2$ of the abstract space (Ω, \tilde{g}) so that $\tilde{g} = F^*(e^{2u} g_{+1})$ for some function u defined on $\text{Im}(F)$. The mappings ψ_k and F are not unique; one may compose them with diffeomorphisms for instance.

Passing to a diagonal subsequence of $\{i, k\}$ and relabeling, it follows from the above that

$$(2.17) \quad (F^{-1})^* \psi_i^*(e^{2u_i} g_{+1}) \rightarrow e^{2u} g_{+1},$$

in $C^{1,\alpha} \cap W_*^{2,p}$ uniformly on compact subsets of $F(\Omega)$, analogous to (2.14). As above, the maps $\psi_i \circ F^{-1}$ form a sequence of quasi-conformal diffeomorphisms with dilatation $1 + \varepsilon_i \rightarrow 1$ and as in (2.15), there exists a sequence of conformal transformations φ_i such that

$$(2.18) \quad \varphi_i \circ \psi_i \circ F^{-1} \rightarrow Id,$$

on $\text{Im}(F)$.

This shows that an analog of the discussion following Proposition 2.1 holds when the hypothesis (2.9) is replaced by the weaker non-collapse assumption (2.16).

Finally, we note that if one assumes $g_i \in C^{m+2}$ and a stronger bound on the covariant derivatives of the Gauss curvatures,

$$(2.19) \quad |\nabla^j K_i|_{g_i} \leq K_m,$$

$1 \leq j \leq m$, then the convergence in (2.11) or (2.17) above can be improved to convergence in $C^{m+1,\alpha} \cap W_*^{m+2,p}$. Note the bound (2.19) is invariant under diffeomorphism. On the other hand, the convergence $K_i \rightarrow K$ in $C^{m-2,\alpha}$ as in (2.1) is not invariant under diffeomorphism. Such convergence does not imply convergence of $K_i \circ \psi_i$ to a limit \tilde{K} in, say, C^α , if the diffeomorphisms ψ_i tend to infinity.

The Gauss-Bonnet theorem gives

$$4\pi = \int_{S^2} K_g dv_g \leq \max |K_g| \text{area}_g S^2,$$

so that the lower area bound in (2.9) (but not necessarily (2.16)) holds automatically under uniform curvature bounds. If one has a uniform positive lower bound

$$(2.20) \quad K_g \geq K_0 > 0$$

on the Gauss curvature, then the diameter bound in (2.9) is also automatic by the well-known Bonnet-Myers theorem. In general, when (2.20) does not hold for some $K_0 > 0$, there may be sequences for which $\text{diam}_{g_i} S^2 \rightarrow \infty$.

We show next that the diameter must remain uniformly bounded when the metrics g are in a fixed conformal class with curvature K in a compact subset of \mathcal{C} , for \mathcal{C} defined as in (1.6). Let $[g]$ denote the pointwise conformal class of a metric g .

Theorem 2.3. *Let g_i be a sequence of $C^{m,\alpha}$ metrics on S^2 with $m \geq 4$. If $[g_i] = [g_{+1}]$ and $K_i \in \mathcal{C}$ with*

$$(2.21) \quad K_i \rightarrow K \in \mathcal{C} \subset C_+^{m-2,\alpha},$$

then there is a constant D_0 such that

$$(2.22) \quad \text{diam}_{g_i} S^2 \leq D_0 < \infty,$$

and hence $\text{area}_{g_i} S^2 \leq A_0$, for some $A_0 < \infty$.

The proof of this result is rather long and broken into the following lemmas and propositions. The first proposition does not require the limit curvature $K \in \mathcal{C}$.

Proposition 2.4. *Suppose $[g_i] = [g_{+1}]$ with $K_i \rightarrow K$ in $C^{m-2,\alpha}$ and suppose in addition there is a constant $A_0 < \infty$ such that*

$$(2.23) \quad \text{area}_{g_i} S^2 \leq A_0.$$

Then (2.22) holds.

Proof: The proof is by contradiction and so we assume there is a sequence g_i as above satisfying (2.23) but for which

$$(2.24) \quad \text{diam}_{g_i} S^2 \rightarrow \infty.$$

For any $\varepsilon > 0$ and any metric g on S^2 one has a thick-thin decomposition of (S^2, g) ,

$$(2.25) \quad S^2 = U^\varepsilon \cup U_\varepsilon,$$

where U^ε is the set of points x where the injectivity radius $\text{inj}_g(x) \geq \varepsilon$ while U_ε is the complement. We first claim that if (2.23) and (2.24) hold, then both sets U^ε and U_ε are non-empty, for any given $\varepsilon > 0$ sufficiently small, provided i is sufficiently large.

To see this, we recall the collapse theory of Cheeger-Fukaya-Gromov [15] which states that if a manifold (M, g) satisfies $|Rm| \leq \Lambda$ and $\text{inj}_g M \leq \varepsilon_0$, then M has an \mathcal{F} -structure determined by the collection of short geodesic loops based at any point $p \in M$. Here ε_0 is a fixed constant, depending only on Λ and the dimension n of M while Rm is the Riemann curvature of (M, g) . In two dimensions, \mathcal{F} -structures are particularly simple; for any $p \in M$, there is a unique short geodesic loop based at p , of length equal to $2\text{inj}_g(p)$, (assuming $\text{inj}_g(p)$ is sufficiently small compared with Λ^{-1}). Such loops may be smoothed at the base points to give a foliation of M by circles. In particular any surface with an \mathcal{F} -structure has Euler characteristic $\chi(M) = 0$; (the same holds in all dimensions by [15]). Since $\chi(S^2) \neq 0$, it follows that

$$U^\varepsilon \neq \emptyset,$$

for all $\varepsilon \leq \varepsilon_0$, where ε_0 depends only on $\max |K|$. On the other hand, if $U_\varepsilon = \emptyset$, then every point of (S^2, g_i) has injectivity radius at least ε . Since the curvature K is uniformly bounded, standard comparison geometry, cf. [42] for example, shows that $\text{area}_{g_i}(B_p(1)) \geq a_0$, where a_0 depends only on ε and $\max |K|$. The area bound (2.23) then implies a uniform upper bound on the number of disjoint geodesic 1-balls $B_{p_j}(1)$ which can be contained in (S^2, g_i) . Taking a maximal collection of such balls, the balls $B_{p_j}(2)$ of radius 2 then cover the manifold (S^2, g_i) which implies that $\text{diam}_{g_i} S^2$ is uniformly bounded above. This contradicts (2.24). This proves the claim above.

Let \tilde{U}^ε be the unit tubular neighborhood of U^ε , $\tilde{U}^\varepsilon = \{x \in (S^2, g) : \text{dist}(x, U^\varepsilon) \leq 1\}$. Again, standard comparison geometry, cf. [24] or [42], implies that $\tilde{U}^\varepsilon \subset U^{\lambda\varepsilon}$, for a (small) constant λ depending only on a bound for $\max |K|$. As above, since each connected component of \tilde{U}^ε has a uniform lower bound on its area, it follows that \tilde{U}^ε has a bounded number of components (independent of i). By passing to a subsequence, we may assume \tilde{U}^ε has a fixed number of components (for any given choice of $\varepsilon \leq \varepsilon_0$). Further, the same covering argument as above shows there is a uniform upper bound on the diameter of each component of \tilde{U}^ε ;

$$\text{diam}_{g_i} \tilde{U}^\varepsilon \leq D,$$

where D depends only on ε and $\max |K_i|$.

Let $\tilde{U}_\varepsilon = S^2 \setminus \tilde{U}^\varepsilon$, so that $\tilde{U}_\varepsilon \subset U_\varepsilon$. In the following, we assume ε is chosen so that $\varepsilon \leq \varepsilon_0$, so that each component of \tilde{U}_ε has an \mathcal{F} -structure; more precisely each component of \tilde{U}_ε has a small thickening $\tilde{U}'_\varepsilon \supset \tilde{U}_\varepsilon$ which has an \mathcal{F} -structure so that \tilde{U}'_ε has a foliation by disjoint short circles. We relabel \tilde{U}'_ε to \tilde{U}_ε and similarly relabel \tilde{U}^ε to $S^2 \setminus \tilde{U}_\varepsilon$. Thus each component of \tilde{U}_ε is topologically an annulus $I \times S^1$ whose boundary components (diffeomorphic to circles) are also boundary components of $\partial\tilde{U}^\varepsilon$. In particular, since \tilde{U}^ε has a bounded number of components, so does \tilde{U}_ε . By (2.24), there must be some components A^j of \tilde{U}_ε with $\text{diam}_{g_i} A^j \rightarrow \infty$ as $i \rightarrow \infty$. On those components A^j with uniformly bounded diameter, the injectivity radius is uniformly bounded below (again by standard comparison geometry, cf. [42]). Thus, by choosing ε smaller, we may assume the decomposition (2.25) satisfies: \tilde{U}^ε consists of a fixed number of components of uniformly bounded diameter, \tilde{U}_ε consists of a fixed number of annuli A^j with

$$(2.26) \quad \text{diam}_{g_i} A^j \rightarrow \infty \text{ as } i \rightarrow \infty,$$

and so with $\text{inj}_{g_i}(y_i) \rightarrow 0$ at some points $y_i \in A^j$ between the boundary components. These thin regions, also frequently called cusp regions, join the thick components together to form S^2 .

Next we claim that for any j , $\inf_{A^j} K_i \rightarrow 0$ as $i \rightarrow \infty$. In fact let $A_L \subset A^j$ be any subannulus for which the g_i -distance to its two boundary components ∂A_L is at least L ; then

$$(2.27) \quad \inf_{A_L} K_i \leq \pi L^{-1}.$$

The estimate (2.27) is an immediate consequence of the standard Myers' theorem in comparison geometry (cf. [42]).

The extremal length $\ell_{\text{ext}}(A)$ of an annulus A , cf. [1], [31] for instance, is defined to be the supremum of the ratio $\ell^2(\sigma)/\text{area}(A)$, where $\ell(\sigma)$ is the minimal length of curves σ joining distinct boundary components of A and the supremum is over all conformally equivalent metrics on A . By construction, the extremal length is a conformal invariant of A and is increasing under inclusion. By the uniformization theorem, any annulus A is conformally equivalent to a standard annulus $A(r_1, r_2)$ of inner and outer radii r_1, r_2 in the Euclidean plane \mathbb{R}^2 . One has

$$(2.28) \quad \ell_{\text{ext}}(A(r_1, r_2)) = \frac{1}{2\pi} \log\left(\frac{r_2}{r_1}\right),$$

cf. again [1], [31]. A similar formula holds for annuli in spherical geometry. By construction, the thin annuli (A^j, g_i) above satisfy

$$\ell_{\text{ext}}(A^j, g_i) \rightarrow \infty,$$

as $i \rightarrow \infty$. Since $\text{area}_{g_i} A^j$ is bounded above by (2.23), it follows that there are curves $\gamma^j \subset A^j$, homologous to a boundary component of A^j , such that $\ell_{g_i}(\gamma^j) \rightarrow 0$ as $i \rightarrow \infty$. Since then $\text{inj}_{g_i}(q_j) \rightarrow 0$ for any point $q_j \in \gamma^j$, one has $\text{dist}_{g_i}(\gamma^j, \partial A^j) \rightarrow \infty$ as $i \rightarrow \infty$. For convenience, we choose γ^j to be a shortest closed geodesic homotopic to S^1 in $A^j \simeq I \times S^1$.

Let B^j be one component of $A^j \setminus \gamma^j$, so B^j is a subannulus of A^j , still with $\ell_{\text{ext}}(B^j) \rightarrow \infty$. The thick boundary component of B^j is in \tilde{U}^ε while the inner or thin boundary γ^j has $\ell_{g_i}(\gamma^j) \rightarrow 0$. In addition, we choose an annulus $A_L \subset B^j$ as in (2.27) with γ^j equal to the inner boundary component of ∂A_L .

Up to this point, the annuli (A^j, g_i) and half-annuli (B^j, g_i) have been considered as abstract Riemannian manifolds. However, since g_i is pointwise conformal to g_{+1} , the annuli (B^j, g_i) are domains in the standard round sphere $S^2(1)$. In spherical geometry, as in (2.28) the extremal length of an annulus can become unbounded only if at least one boundary component converges to a point, so that we may assume $\ell_{g_{+1}}(\gamma^j) \rightarrow 0$. In particular one may choose a point $p^j \in B^j$ such that γ^j converges to p^j , in that $\text{dist}_{+1}(\gamma^j, p^j) \rightarrow 0$.

Next, the short curve γ^j bounds a disc (on each side) in S^2 and at least one such disc D_j^2 is small, i.e. contained in a spherical disc of (arbitrarily) small radius. Choose the component B^j so that $B^j \subset D_j^2$ with $\partial D_j^2 = \gamma^j$. Applying the Gauss-Bonnet theorem to such discs we obtain

$$(2.29) \quad \int_{D_j^2} K_i dv_{g_i} = 2\pi - \int_{\partial D_j^2} \kappa_i,$$

where κ_i is the geodesic curvature of the boundary $(\partial D_j^2, g_i)$. Since the boundary of D_j^2 is the geodesic γ^j , this gives

$$\int_{D_j^2} K_i dv_{g_i} = 2\pi.$$

The area bound (2.23) then implies there exists κ_0 , depending only on A_0 in (2.23) (and ε) such that

$$(2.30) \quad K(x) \geq \kappa_0,$$

for some $x \in D_j^2$. (Simple examples, e.g. a sequence of round spheres with radius diverging to infinity, show that upper area bound is necessary for (2.30)).

We now obtain a contradiction as follows. The discs D_j^2 (which depend on i) are contained in g_{+1} -geodesic discs $D_{p^j}(\delta_i)$ with $\delta_i \rightarrow 0$ as $i \rightarrow \infty$. Such discs thus contain points where both (2.30) and (2.27) hold with L^{-1} arbitrarily small. Since the sequence $\{K_i\}$ is uniformly bounded in C^α , this gives a contradiction. ■

Next we turn to the existence of an apriori area bound. Of course if $\min K_i \geq \kappa_0$ for some $\kappa_0 > 0$, then the area bound is automatic; the arguments below through and including Proposition 2.6 are only needed when $\liminf K_i \leq 0$. We first collect several general facts about the behavior of $\{u_i\}$.

Lemma 2.5. *Let $g_i = e^{2u_i} g_{+1}$ and suppose $K_i \rightarrow K \in C^{m-2, \alpha}$. Then the following results hold.*

(i). *For any $\varepsilon > 0$ there is a constant $A_1 = A_1(\varepsilon) < \infty$ such that*

$$(2.31) \quad \text{area}_{g_i} \{K_i < -\varepsilon\} \leq A_1.$$

(ii). *There is a constant $M = M(\varepsilon) < \infty$ such that*

$$(2.32) \quad u_i(x) \leq M,$$

for all $x \in \{K_i \leq -\varepsilon\}$.

(iii). *If $\text{area}_{g_i} S^2 \rightarrow \infty$, then*

$$(2.33) \quad \text{area}_{g_i} \{-\varepsilon < K_i < \varepsilon\} \rightarrow \infty.$$

(iv). *If $\liminf K_i \leq 0$ and $K \in C_+^{m-2, \alpha}$, then there exists a constant $m_0 < \infty$ such that on S^2 ,*

$$(2.34) \quad u_i \geq -m_0.$$

Proof: The computations and estimates to follow in the proof apply to each u_i , but we will generally drop the index i from the notation; thus $u = u_i$ and $K = K_i$ in the following. The estimates obtained will all be uniform, i.e. independent of i large, using in particular the convergence $K_i \rightarrow K \in C^{m-2, \alpha}$.

To begin, one has

$$\text{area}_g \{K < -\varepsilon\} = \int_{\{K < -\varepsilon\}} e^{2u} dv_{g_{+1}}.$$

The estimate (2.31) is obvious in the region where $u \leq 0$, so we concentrate in the region where $u \geq 0$. Without loss of generality, choose ε such that all points in the interval $[-\varepsilon, -\varepsilon/2]$ are regular values of K . Let $\eta = \eta(K)$ be a smooth cutoff function of K with $\eta \in [0, 1]$, $\eta = 1$ on $\{K \leq -\varepsilon\}$

and $\eta = 0$ on $\{K \geq 0\}$. Let $U^0 = \{u \geq 0\}$ and $L_0 = \partial U^0 = \{u = 0\}$. (In the following, one should replace 0 by a regular value δ of u arbitrarily close to 0, but this will make no difference in the argument below). From (1.1), one has

$$-\varepsilon \int_{U_0} \eta^4 e^{2u} \geq \int_{U_0} \eta^4 K e^{2u} \geq - \int_{U_0} \eta^4 \Delta u,$$

where the integration is with respect to $dv_{g_{+1}}$. Applying the divergence theorem to the last term gives $-\int \eta^4 \Delta u = -\int u \Delta \eta^4 + \int_{L_0} u \partial_N \eta^4 - \int_{L_0} \eta^4 \partial_N u$. The first boundary integral vanishes since $u = 0$ on L_0 while the second is non-negative since N points out of U_0 . Hence

$$-\varepsilon \int_{U_0} \eta^4 e^{2u} \geq - \int_{U_0} u \Delta \eta^4.$$

Since $\Delta \eta^4 = 4\eta^3 \Delta \eta + 12\eta^2 |d\eta|^2$, it follows that

$$\int_{U_0} \eta^4 e^{2u} \leq C \int_{U_0} \eta^2 u,$$

where C depends only on ε and the C^2 norm of K (and the fixed choice of $\eta(K)$). By the Cauchy-Schwarz inequality, one has $\int_{U_0} \eta^2 u \leq \sqrt{4\pi} (\int_{U_0} \eta^4 u^2)^{1/2}$. Moreover, $e^{2u} \geq u^2$ on U^0 and $e^{2u} \gg u$ for u large, so that the integrand on the left side of the inequality above dominates the right side integrand where u is large. It follows that $\int_{U_0} \eta e^{2u} \leq C'$ so that

$$\text{area}_g(\{K \leq -\varepsilon\} \cap U^0) \leq \int_{U^0} \eta^4 e^{2u} \leq C'.$$

This proves (2.31).

To prove (ii), the area bound (2.31) gives a uniform L^1 bound on e^{2u} , $u = u_i$, and hence a uniform L^2 bound on u on $\{K \leq -\varepsilon\}$. Since on this set $\Delta u \geq 1$ by (1.1), the upper bound (2.32) is an immediate consequence of the well-known DeGiorgi-Nash-Moser interior estimate for subsolutions of elliptic equations, cf. [21]. Of course here we use the estimate on the domains $V' = \{K \leq -\varepsilon\} \subset V = \{K \leq -\varepsilon/2\}$ for regular values $\varepsilon/2, \varepsilon$ as well as the convergence $K_i \rightarrow K \in C^{m-2, \alpha}$.

For the third claim, by the Gauss-Bonnet theorem, one has

$$(2.35) \quad \int_{\{K \leq -\varepsilon\}} K e^{2u} + \int_{\{K \in [-\varepsilon, \varepsilon]\}} K e^{2u} + \int_{\{K \geq \varepsilon\}} K e^{2u} = 4\pi,$$

Now (2.31) implies the first integral in (2.35) is bounded below. If $\text{area}_{g_i} \{-\varepsilon < K_i < \varepsilon\}$ is bounded, then the second integral in (2.35) is bounded above and hence so is the third integral. That in turn implies $\text{area}_{g_i} \{K_i \geq \varepsilon\}$ is uniformly bounded, which implies $\text{area}_{g_i} S^2$ is uniformly bounded. Hence if $\text{area}_{g_i} S^2 \rightarrow \infty$, then (2.33) must hold.

To prove (iv), a simple computation using (1.1) gives

$$(2.36) \quad (\Delta + 2)e^{-2u} = 2K + 4e^{-2u}|du|^2.$$

Now the first eigenvalue λ_1 of the Laplacian on (S^2, g_{+1}) is $\lambda_1 = 2$, with eigenfunctions given by restrictions of the linear functions ℓ on \mathbb{R}^3 to $S^2(1)$. Any proper smooth domain $U \subset\subset S^2$ thus has lowest eigenvalue λ_1 for the Laplacian with Dirichlet boundary values on ∂U satisfying $\lambda_1 > 2$. The operator $\Delta + 2$ thus has no kernel on smooth functions on U vanishing at ∂U . Observe that (2.36) implies $(\Delta + 2)e^{-2u} \geq -C$, where C is a lower bound for $2K$. Let U be any smooth domain for which there is a spherical disc $D_p(\varepsilon) \subset S^2 \setminus U$. It then follows from the well-known maximum principle for subsolutions of $\Delta + 2$, cf. [21], that

$$(2.37) \quad \sup_U e^{-2u} \leq C'(\varepsilon) + \sup_{\partial U} e^{-2u},$$

where $C'(\varepsilon)$ depends only on ε . Thus if e^{-2u} (i.e. the sequence e^{-2u_i}) is uniformly bounded above on ∂U , then e^{-2u} is bounded, i.e. u is uniformly bounded below, on U , provided the complement U^c contains a small disc $D_p(\varepsilon)$ of fixed radius. This implies that if $u_i \rightarrow -\infty$ anywhere on S^2 , then $u_i \rightarrow -\infty$ at least on a dense set in S^2 . Note that exactly this behavior occurs for the family of conformal dilations in (2.3), $u_i = \log \chi_{p,t_i}$ with $t_i \rightarrow \infty$.

Choose a point p_i realizing $\max u_i$ so that $u_i(p_i) \rightarrow \infty$. Next choose a point q_i with $K_i(q_i) = 0$. Without loss of generality, we may assume $u_i(q_i) \rightarrow -\infty$; (otherwise, one may replace q_i by q_j with $K_i(q_j) \rightarrow 0$ as $i, j \rightarrow \infty$ and $u_i(q_j) \rightarrow -\infty$ as $j \rightarrow \infty$ and take a diagonal subsequence). As in the process described in (2.4), apply a sequence of conformal dilations $\varphi_i = \varphi_{q_i, p_i, t_i}$ to u_i with source $q_i \rightarrow q$ and sink $p_i \rightarrow p$, so that $\tilde{u}_i = \varphi_i^* u_i$ satisfies

$$(2.38) \quad \int_{D_q(\frac{1}{2})} e^{2\tilde{u}_i} dv_{+1} = 1.$$

Note here that for any fixed u , the integral $\int_{D_q(\frac{1}{2})} e^{2\varphi^* u} dv_{+1}$ is a continuous function of $\varphi \in \text{Conf}(S^2)$ and varies between $(a \min e^{2u}, a \max e^{2u})$, $a = \text{area} D_q(\frac{1}{2})$. Since $u_i(q_i) \rightarrow -\infty$ and $u_i(p_i) \rightarrow +\infty$, the conformal dilations φ_i above exist and $t_i \rightarrow +\infty$. (The choice of the constants $\frac{1}{2}$ and 1 here can be replaced by other fixed constants). If $\inf_{D_q(\frac{1}{10})} \tilde{u}_i \rightarrow -\infty$, then one may dilate or rescale further so that $\hat{u}_i = \varphi_{p_i, q_i, s_i}^* \tilde{u}_i$ is bounded below in $D_q(\frac{1}{2})$ and the area with respect to \hat{u}_i in (2.38) is at most 1. Relabeling then \hat{u}_i to \tilde{u}_i if necessary, the estimate (2.37) then implies that \tilde{u}_i is bounded below globally on S^2 .

Now apply Remark 2.2 to the sequence (S^2, \tilde{g}_i, q_i) , $\tilde{g}_i = e^{2\tilde{u}_i} g_{+1}$. Since \tilde{u}_i is uniformly bounded below, $\tilde{g}_i \geq c g_{+1}$ for some uniform constant $c > 0$ and hence the geodesic ball $B_{\tilde{g}_i}^{\tilde{g}_i}(\delta_0) \subset D_q(\frac{1}{2})$ for a uniform $\delta_0 > 0$. We first note that the sequence \tilde{g}_i cannot collapse at q_i , i.e. (2.16) holds. For if collapse did occur, then very short \tilde{g}_i -geodesic loops form near q . Such loops bound a spherical disc $D_i \subset D_q(\frac{1}{2})$ near q , whose g_i -area is uniformly bounded by (2.38). This gives the same contradiction as at end of the proof of Proposition 2.4.

It then follows by Remark 2.2 that the pointed sequence (S^2, \tilde{g}_i, q_i) , converges in a subsequence in the based $C^{1,\alpha}$ topology modulo diffeomorphisms to a limit (Ω, \tilde{g}, q) , i.e. (2.17) holds for some limit function \tilde{u} . The dilations φ_i in (2.18) are bounded, since otherwise one would necessarily have $\tilde{u}_i \rightarrow -\infty$ almost everywhere, i.e. the conformal rescaling above removes the possible divergence in the conformal group. It follows from (2.17)-(2.18) that $\{\tilde{u}_i\}$ subconverges to a solution \tilde{u} of

$$\Delta \tilde{u} = 1,$$

on a domain $\Omega \subset S^2$. Now the metric $g = e^{2\tilde{u}} g_{+1}$ is complete and flat and hence is either the Euclidean metric g_{Eucl} on $\mathbb{R}^2 = S^2 \setminus \{p\}$ or a complete flat metric on a cylinder $\mathbb{R} \times S^1 \simeq S^2 \setminus \{p \cup p'\}$. However, the metric g_{Eucl} is not pointwise conformal to g_{+1} , giving a contradiction in that case. (Note that the pullback of g_{Eucl} to $S^2 \setminus \{p\}$ by stereographic projection is not pointwise conformal to g_{+1} , although it is of course conformally equivalent to g_{+1}). If g is a complete flat metric on a cylinder, then there are annuli $A_i \subset (S^2, g_i)$ which are almost flat, i.e. $K_i \rightarrow 0$ on A_i and with length $\ell(A_i) \rightarrow \infty$. Since $K_i \rightarrow K \in C_+^{m-2,\alpha}$, this gives the same contradiction as at the end of the proof of Proposition 2.4, again using the area bound (2.38). ■

Proposition 2.6. *Under the assumptions of Theorem 2.3, there is a constant $A_0 < \infty$ such that (2.23) holds, i.e.*

$$(2.39) \quad \text{area}_{g_i} S^2 \leq A_0.$$

Proof: The proof is again by contradiction. Thus assume

$$(2.40) \quad \text{area}_{g_i} S^2 = \int_{S^2} e^{2u_i} dv_{+1} \rightarrow \infty.$$

Consider the behavior of the (signed) measures

$$d\mu_i = -\Delta u_i dv_{+1} = (K_i e^{2u_i} - 1) dv_{+1}.$$

We first analyze the situation where these measures have uniformly bounded total variation (or mass), so that there is a constant $M < \infty$ such that

$$\int_{S^2} |\Delta u_i| dv_{+1} \leq M.$$

This bound is equivalent to

$$(2.41) \quad \int_{S^2} |K_i| dv_{g_i} = \int_{S^2} |K_i| e^{2u_i} dv_{+1} \leq M,$$

so there is a uniform L^1 bound on the total absolute curvature of (S^2, g_i) .

Let $y_i \in S^2$ be a sequence of base points; in the following it is convenient (but not necessary) to choose y_i to be points realizing $\max K_i$ with $y_i \rightarrow y \in S^2$. Recall that since $K_i \rightarrow K \in \mathcal{C}$, there is a constant $\kappa_0 > 0$ such that $K_i(y_i) \geq \kappa_0$ for all i . Hence there is an $\varepsilon_0 > 0$ and spherical disc $D_{y_i}(\varepsilon_0)$ about y_i of g_{+1} -radius ε_0 such that $K_i(x_i) \geq \kappa_0/2$, for $x_i \in D_{y_i}(\varepsilon_0)$. By (2.34), there is a g_i -geodesic disc $B_i = B_{y_i}(\delta_0)$ of g_i -radius δ_0 about y_i such that $B_{y_i}(\delta_0) \subset D_{y_i}(\varepsilon_0)$. Now it is well-known that geodesic discs of fixed radius cannot collapse, so $\text{inj}_{g_i}(y_i) \geq i_0 > 0$. This follows for instance from the Gauss-Bonnet theorem:

$$\int_{B_i} K_i = 2\pi\chi(B_i) - \int_{\partial B_i} \kappa_i.$$

If B_i collapses, then topologically B_i is an annulus so $\chi(B_i) = 0$. Standard comparison geometry for the exponential map in local spaces with $0 \leq K \leq K_0$ shows that the geodesic curvature $\kappa_i > 0$, which gives a contradiction.

Remark 2.2 then implies that the sequence (S^2, g_i, y_i) converges (in a subsequence) in the pointed $C^{1,\alpha}$ topology modulo diffeomorphisms, to a complete Riemannian surface $(\Omega, \tilde{g}, \tilde{y})$. The lower bound (2.34) and the fact that $K_i \rightarrow K$ in $C^{m-2,\alpha}$ implies that $|\nabla K_i|_{g_i}$ is uniformly bounded, so the convergence is actually in the $C^{2,\alpha}$ topology, again by Remark 2.2. The surface (Ω, g) is necessarily open; if Ω is closed, then $\Omega = S^2$ and $\text{area}_{g_i} S^2$ is uniformly bounded, contradicting (2.40).

By (2.41), (Ω, \tilde{g}) has finite total absolute curvature. A well-known theorem of Huber [28] then implies that (Ω, \tilde{g}) is conformally equivalent to S^2 punctured at a finite number of points q_j , i.e. each end E_j of Ω is parabolic in the sense of potential theory, cf. [1]. We note also that for any end E of Ω , there exists a divergent sequence of points $x_i \rightarrow \infty$ such that

$$(2.42) \quad \lim_{i \rightarrow \infty} K(x_i) \leq 0.$$

(For if (2.42) were not the case, then $K \geq \kappa > 0$ on E , for some constant $\kappa > 0$, which implies that E is of bounded diameter by the Myers theorem in comparison geometry, [41]; this is of course a contradiction).

Referring again to Remark 2.2, without loss of generality, we may assume $\psi_i^{-1}(y_i) = \tilde{y}_i \rightarrow \tilde{y}$ with $F(\tilde{y}) = y$. Since $K_i(y_i) > 0$ is bounded away from 0, it follows from (2.42) and the C^α convergence $K_i \rightarrow K$, that the image of the embeddings ψ_i must contain a small spherical disc of fixed size,

i.e. $D_{y_i}(\delta) \subset \psi_i(\Omega_i)$ for some $\delta > 0$. The embedding $F : \Omega \rightarrow S^2$ has image $F(\Omega) = S^2$ with a finite number of points $\{q_j\}$ removed. As in (2.17) one has

$$(2.43) \quad (F^{-1})^* \psi_i^*(e^{2u_i} g_{+1}) \rightarrow e^{2u} g_{+1},$$

on $F(\Omega)$ and there exists a sequence of conformal transformations φ_i such that

$$\varphi_i \circ \psi_i \circ F^{-1} \rightarrow Id,$$

on S^2 . Now we claim that the sequence φ_i cannot diverge to ∞ in $\text{Conf}(S^2)$. Namely, since the images $\psi_i(\Omega_i)$ contain a spherical disc of fixed size about y , φ_i can only diverge if it is within a bounded distance in $\text{Conf}(S^2)$ to a sequence of conformal dilations $\bar{\varphi}_i = \varphi_{y,t_i}$ sourced at y with $t_i \rightarrow \infty$. Setting $\tilde{u}_i = \bar{\varphi}_i^* u_i$, it follows from (2.43) that \tilde{u}_i subconverges in $C^\alpha \cap W_*^{1,2}$ to a solution \tilde{u} of $\Delta \tilde{u} = 1 - K(y)e^{2\tilde{u}}$. By elliptic regularity, \tilde{u} is smooth. This is the equation for a metric of constant curvature $K(y)$ which implies that Ω is compact, i.e. $\Omega = S^2$, giving a contradiction since Ω is open.

Thus $F^{-1} \circ \psi_i$ converges to a limit quasi-conformal diffeomorphism with dilatation 1, i.e. to a conformal diffeomorphism. Altering the embedding F by this fixed conformal map, we may assume F is the inclusion of a subset $\Omega \subset S^2$ and

$$\Omega = S^2 \setminus \cup\{q_j\}.$$

By (2.17), the functions u_i converge to the limit function u on Ω . Elliptic regularity associated with the equation (1.1) implies the convergence is in $C^{m,\alpha}$. Since $K = \lim K_i$, by (2.42) one must have $K(q_j) \leq 0$. In fact, one must have

$$K(q_j) = 0.$$

Namely if $K(q_j) < 0$, then the end E_j of Ω is a hyperbolic cusp and so of finite area. The end E_j must be capped off by a small spherical disc D_j about q_j in S^2 . Applying the Gauss-Bonnet theorem to (D_j, g_i) for i large shows that there exist points $q'_j \in D_j$ such that $K_i(q'_j) \geq 0$. For i sufficiently large, this gives the same contradiction as before, namely to the C^α convergence $K_i \rightarrow K$.

Now the measures $d\mu_i$ converge (in a subsequence) to a limit signed measure $d\mu$ on S^2 . Let $d\mu_r$ and $d\mu_s$ denote the absolutely continuous and singular measures of $d\mu$ with respect to Lebesgue measure:

$$(2.44) \quad d\mu = d\mu_r + d\mu_s.$$

As above, the functions u_i converge (in a subsequence) uniformly on compact subsets of Ω in $C^{m,\alpha}$, to a limit function u . The regular part of $d\mu$ is thus given by

$$d\mu_r = (Ke^{2u} - 1)dv_{+1} \text{ on } \Omega,$$

where $K = \lim K_i$. Since $\text{area}_{g_i} S^2 \rightarrow \infty$, u_i blows up, $u_i \rightarrow \infty$, at each q_j . Similarly, since g is complete, $u \rightarrow \infty$ at each q_j . As noted above, the support of the singular measure $d\mu_s$ is given by

$$Z = \cup q_j \subset \{K = 0\}.$$

Any measure supported at a point is a multiple of the Dirac delta measure, so that there exist $a_j \in \mathbb{R}$ such that

$$(2.45) \quad d\mu_s = 2\pi \sum a_j \delta_{q_j}.$$

One has

$$\int_{\Omega} d\mu_r = \int_{\Omega} K dv_g - 4\pi \leq 2\pi\chi(\Omega) - 4\pi,$$

where the last inequality follows from another well-known result of Huber [28]. Since the total mass of $d\mu$ is 0 (by the weak convergence of the measures $d\mu_i \rightarrow d\mu$), it follows that

$$\int d\mu_s = 2\pi \sum a_j \geq 4\pi - 2\pi\chi(\Omega),$$

so that $\sum a_j \geq 2 - \chi(\Omega)$. Note that the Euler characteristic $\chi(\Omega)$ is given by $\chi(\Omega) = 2 - \#\{\cup q_j\}$.

Let $G = G_q$ be the Green's function for the operator $1 - \Delta$ on $S^2(1)$ with pole at q (and $-q$), so that

$$\Delta G = 1 - 2\pi(\delta_q + \delta_{-q}).$$

Thus G_q corresponds to the Dirac measure δ_q with $a = 1$ (and measure δ_{-q}). The behavior of the Green's function at the antipodal point $-q$ will play no role in the discussion below. A simple computation shows that G is given explicitly by

$$(2.46) \quad G(x, q) = \log \frac{1}{\sin r(x, q)}$$

where $r(x, q) = \text{dist}_{+1}(x, q)$. In particular for x near q , G_q has the well-known logarithmic behavior $G(x, q) = -\log r(x, q) + \alpha$, where α is smooth.

Now we first note that (2.46) implies that in (2.45)

$$a_j \geq 1.$$

This follows from the completeness of (Ω, g) . Namely, the g -length of a curve γ ending at some q_j is given by

$$(2.47) \quad \ell_g(\gamma) = \int_{\gamma} e^u dr,$$

where r is the g_{+1} -arclength parameter for γ . Since g is complete, the integral in (2.47) must diverge; hence on approach to q_j , $u \geq \log r^{-1+\delta}$, for any fixed $\delta > 0$ (on some sequence of points p_k with $r(p_k) \rightarrow 0$). By (2.46), this implies $a_j \geq 1$.

The leading order behavior of the limit function u near the point $q \in \{q_j\}$ is governed by the behavior of the Green's function $G = G_q$. By Green's representation formula, the decomposition (2.44) gives the expression

$$(2.48) \quad u = aG + \beta,$$

near q where β is (at least) continuous in a neighborhood of $q \in \{q_j\}$. This gives

$$g = e^{2u} g_{+1} = e^{2(aG+\beta)} g_{+1} = e^{2\beta} g_a,$$

where $g_a = e^{2aG} g_{+1}$. One has

$$K_a e^{2aG} = 1 - \Delta aG = (1 - a) + 2\pi a(\delta_q + \delta_{-q}),$$

so that away from the pole q (and $-q$),

$$(2.49) \quad K_a = (1 - a)e^{-2aG} = (1 - a)(\sin r)^{2a} \leq 0.$$

For K_u we obtain

$$(2.50) \quad K_u = (1 - a - \Delta\beta)(\sin r)^{2a} e^{2\beta} = \chi r^{2a},$$

on Ω near q ; the last equality defines χ . Now $K_u = \lim K_i = K$, so that (2.50) holds also at q . Clearly then $K(q) = 0$ and q is a critical point of $K \in \mathcal{C}$. Since (1.5) thus holds at q , one must have $\Delta K(q) > 0$. This implies that

$$(2.51) \quad a = 1,$$

and also $\chi(q) > 0$.

Now the geometry of g_a for $a = 1$ is that of a complete flat metric on the cylinder $\mathbb{R} \times S^1$; $K_a = 0$ when $a = 1$. The pole q occurs at $+\infty$ while the pole $-q$ occurs at $-\infty$. As above, we ignore the behavior at $-q$. It follows that the geometry of (Ω, g) on approach to any $q \in \{q_j\}$ is that of a flat cylindrical end.

On the sequence (S^2, g_i) , these cylindrical ends are filled in or capped by very small discs $D_j = D_j(\delta_i)$ in (S^2, g_{+1}) about q_j , $\delta_i \rightarrow 0$. Thus topologically, $S^2 = \Omega \cup \{\cup D_j\}$. The metrics g_i are blowing up such very small discs either to discs of approximately unit area or to very large discs. The former case corresponds to capping of the cylindrical end $\mathbb{R}^+ \times S^1$ by a hemisphere. In this case, the Gauss curvature is necessarily bounded away from 0 at some point in the disc, (by the Gauss-Bonnet theorem for instance), which contradicts the C^α convergence $K_i \rightarrow K$. Hence the capping discs must become very large and of very small curvature $K_i \sim 0$ corresponding to a spherical bubble with area tending to infinity. Intuitively, this requires a band of negative curvature to open up the cylinder, capped off by a region of positive curvature to cap off the end.

To analyse this latter case in detail, fix any i sufficiently large; for r very small, by (2.50) the curvature K_i of g_i on the sphere $S_q(r)$ is small but uniformly positive, $K_i \geq \kappa_0 > 0$. Consider the exponential map of g_i -geodesics normal to $S_q(r)$ into the disc $D_q(r)$. If the curvature K_i of g_i satisfies $K_i \geq 0$ in $D_q(r)$, then standard comparison geometry implies these geodesics have conjugate points at a g_i -bounded distance to $S_q(r)$. This implies $(D_q(r), g_i)$ has uniformly bounded diameter, which contradicts the assumption here that $area_{g_i} D_q(r) \rightarrow \infty$. Thus, $\min_{D_q(r)} K_i < 0$ on some sequence $i \rightarrow \infty$. Let $L_0 = L_0(i) \subset \{K_i = 0\} \subset D_q(r)$, be the collection of curves homologous to $S_q(r)$ such that $K_i \geq 0$ in the region bounded by $S_q(r) \cup L_0$. Now on the one hand, standard comparison geometry for non-negative curvature implies the g_i -length ℓ_{g_i} of L_0 is uniformly bounded. Each component L'_0 of L_0 bounds a small disc $D'_i \subset D_q(r)$ in S^2 . On the other hand, standard comparison geometry for non-positive curvature shows that if $K_i \leq 0$ on D'_i , then $area_{g_i} D'_i \rightarrow \infty$ implies $\ell_{g_i}(L'_0) \rightarrow \infty$; this is a standard isoperimetric-type inequality for non-positive curvature. Since $area_{g_i} D_q(r) \rightarrow \infty$, it follows that for some D'_i , $\max_{D'_i} K_i > 0$ for i large. However, this contradicts the assumption that $K_i \rightarrow K \in \mathcal{C}$. Namely, since $\{K_i\}$ is uniformly within \mathcal{C} , there are no local maxima of K_i with $|K_i|$ sufficiently small.

It is worth noting, and is discussed in detail in Remark 2.7 below, that when $K \notin \mathcal{C}$, such bubble behavior with unbounded area may occur.

Next suppose the total mass or variation of $d\mu_i$ satisfies

$$m(|d\mu_i|) \rightarrow \infty.$$

In this case, we will see that the blow-up behavior of $\{u_i\}$ creates even larger a in (2.50), leading to the same contradiction.

To begin, as above choose base points y_i realizing $\max K_i$. By Proposition 2.1 and Remark 2.2 as above, the sequence (S^2, g_i, y_i) subconverges to a complete conformally flat limit surface (Ω, g, y) ; the convergence is uniform in $C^{2,\alpha}$ (at least) on compact subsets of Ω . As in Remark 2.2, the abstract domain (Ω, g) may be conformally embedded as a planar domain, i.e. a domain in S^2 with complete metric $g = e^{2u} g_{+1}$. As before we let $Z = S^2 \setminus \Omega$ be the singular set of g . By (2.40), there is a sequence of nested neighborhoods $N_m \supset N_{m+1}$ of Z in S^2 with $\cap N_m = Z$ such that $area_{g_i} N_m \rightarrow \infty$ for each fixed m . By completeness $u \rightarrow \infty$ at $\partial\Omega = \partial Z$, and hence by (2.32),

$$(2.52) \quad \partial Z \subset \{K \geq 0\},$$

where $K = \lim K_i$. (It will follow from later arguments below that Z has empty interior, so that $\partial Z = Z$).

The complete surface (Ω, g) may have a finite or infinite number of ends. Suppose first the number of ends is finite and choose any fixed end E . Then E is an annulus in S^2 , with either

finite or infinite extremal length. This dichotomy corresponds to a hyperbolic or parabolic end respectively, in the sense of potential theory, cf. [1].

We claim that the end must be parabolic. This is proved by contradiction, so suppose E is hyperbolic. Let σ_j be a divergent sequence of smooth closed curves in the annular end E disconnecting E . The annulus E is embedded in S^2 and so $\{\sigma_j\}$ is a sequence of Jordan curves in S^2 bounding a nested sequence of domains $D_{j+1} \subset D_j \subset S^2$ with $\sigma_k \subset D_j$ for $k > j$. The limit $D_\infty = \lim D_j = \bigcap D_j$ is a subset of Z in (2.52). Since the end is hyperbolic, D_∞ cannot be a point (since E is parabolic if D_∞ is a point), and hence ∂D_∞ contains a non-trivial arc, i.e. the C^0 image of an interval.

Let r be the g -geodesic distance from a fixed point $p_0 \in E$. By a theorem of Doyle-Grigorian, cf. [19], [23], there is a divergent sequence of points $p_j \in E$, $r(p_j) \rightarrow \infty$, such that $K(p_j) \leq -(1/r^2 \log r)(p_j)$. Passing to a subsequence, we may assume $p_j \rightarrow q$, for some $q \in \partial Z$. If $K(q) < 0$, then one has a contradiction to (2.52), so that $K(q) = 0$. If $K \leq 0$ in a neighborhood of q , then 0 is a local maximum of K , contradicting $K \in \mathcal{C}$. Hence there exist points arbitrarily near q where $K > 0$ and so the level set $K = 0$ containing q locally disconnects S^2 into regions where $K > 0$ and $K < 0$. The closed curves σ_j then contain arcs $\alpha_j^\pm \subset \sigma_j$ with $\alpha_j^+ \subset \{K > 0\}$ while $\alpha_j^- \subset \{K < 0\}$ with α_j^\pm converging to arcs $\alpha^\pm \subset \{K = 0\}$. However, there are then long triangles Δ_j with vertices $\{p_0, \partial(\alpha_j^+)\}$ with $K > 0$ in the region in Δ_j where r is sufficiently large. Such triangles have extremal length diverging to ∞ (positive curvature gives parabolic ends) and hence α^+ is a single point q , giving again a contradiction.

Thus the end E is parabolic, and hence conformally equivalent to a punctured disc $D^2 \setminus q$. As above, we work with the positive measures $|d\mu_i| = |\Delta u_i| dv_{+1}$ restricted to the end E , i.e. a punctured neighborhood of q . Without loss of generality, assume $m_i = m(|d\mu_i|) \rightarrow \infty$. Consider the rescaled measures $d\tilde{\mu}_i = m_i^{-1} d\mu_i$, so the total mass of $d\tilde{\mu}_i$ is 1. It follows that the measures $|d\tilde{\mu}_i|$ subconverge to a positive measure supported at the point q , i.e. to the Dirac measure δ_q . Hence, to leading order, $d\mu_i = m_i \delta_q + dv_i$ where $n_i = m(|dv_i|) \ll m_i$. If $n_i \rightarrow \infty$, one may repeat this process with $d\mu_i - m_i \delta_q$ in place of $d\mu_i$ and conclude that

$$d\mu_i = (1 - \Delta u_i) dv_{+1} = m'_i \delta_q + dv_i$$

with $n_i = m(|dv_i|) \leq N$, for some $N < \infty$. As in (2.48), this implies that on E ,

$$(2.53) \quad u_i = m'_i \log r^{-1} + \beta_i$$

where β_i is bounded near q . Since K_i is thus of the form (2.50) with $a = m'_i > 1$, this contradicts the fact that $K_i \rightarrow K \in \mathcal{C}$.

We note that the same argument applies if there is at least one end of Ω of finite topological type.

Suppose finally all ends are of infinite topological type; in the following we work with just one end E . Such planar ends E have the topological form of an infinite union of pairs of pants P_k ; P_k is diffeomorphic to S^2 with three discs removed. Each leg of P_k is glued to the waist of one of the two successors P_{k+1} or P'_{k+1} of P_k . Such ends have singular set Z equal to a Cantor set in S^2 , cf. [43] for a detailed discussion.

Choose a base point $p_0 \in E$ and let σ be any geodesic ray from p_0 diverging to infinity in E . Let $\{P'_k\} \subset E$ be the collection of pairs of pants intersecting σ . This forms a (connected) sub-end $E' \subset E$ with limit given by a single point $q \in Z$. Next we cap off all boundary components, i.e. circles, of $\{P'_k\}$ not intersecting σ by discs $D_{k'}$, giving then an annular end $\tilde{E} \simeq \sigma \times \mathbb{R}^+$ ending at q . For any fixed disc $D_{k'}$, the functions u_i are unbounded on $D_{k'}$, (since $\text{dist}_{g_i}(D_{k'}, Z) \rightarrow \infty$), but are uniformly bounded on the boundaries $\partial D_{k'} \simeq S^1$. For i sufficiently large, (depending on k'), we then replace $u_i|_{D_{k'}}$ by the harmonic extension \tilde{u}_i of the boundary values of u_i on $\partial D_{k'}$, i.e. the solution of the Dirichlet problem $\Delta \tilde{u}_i = 0$ on $D_{k'}$ with boundary data u_i on $\partial D_{k'}$. In particular, the

maximum and minimum values of \tilde{u}_i on $D_{k'}$ are bounded by the maximum and minimum values of \tilde{u}_i on $\partial D_{k'}$.

This construction gives a parabolic annular end (\tilde{E}, \tilde{g}_i) , $\tilde{g}_i = e^{2\tilde{u}_i} g_{+1}$ limiting at a singular point $q \in Z$. Suppose first the mass \tilde{m}_i of $|d\tilde{\mu}_i|$ remains uniformly bounded. Then the same analysis as that leading to (2.50) gives

$$\tilde{K} = (1 - a - \Delta\beta)(\sin r)^{2a} e^{2\beta}.$$

Of course $\tilde{K} \neq K$, but $\tilde{K} = K$ in a neighborhood of the geodesic ray σ , i.e. in the neighborhood $\cup P'_k$ of q . Since σ may be chosen to be any geodesic ray and since $K \in C^{m-2, \alpha}$, it follows that $K(q) = 0$ and K is of the form (1.5) near q . This brings one exactly to the situation above where (2.50)-(2.51) holds and the same arguments as before lead to the same contradiction. Finally, the same argument as above concerning (2.53) also gives the contradiction when $\tilde{m}_i \rightarrow \infty$. This completes the proof of Proposition 2.6. ■

Propositions 2.4 and 2.6 together complete the proof of Theorem 2.3.

Remark 2.7. It is important to observe that the condition (2.21) that $K \in \mathcal{C}$ is necessary in Theorem 2.3. This follows from the construction of large spherical ‘‘bubbles’’ in a fixed conformal class by Ding-Liu in [18] and later by Borer-Galimberti-Struwe in [8]. In particular, the construction in [8] shows that there is a sequence $\delta_i \rightarrow 0$ and conformal metrics $g_i = e^{2u_i} g_{+1}$ on small spherical discs $D_p(\delta_i) \subset S^2(1)$ with arbitrarily large area, with arbitrarily small but positive Gauss curvature $K_i > 0$, and for which u_i is smoothly bounded near the boundary of the fixed disc $\partial D_p(\delta)$. In particular $u_i \rightarrow \infty$ in $D_p(\delta_i)$. The construction has the property that the Gauss curvature K_i has a local maximum at p with $0 < K_i(p) = \varepsilon_i \rightarrow 0$ but $K_i < 0$ and small near $\partial D_p(\delta)$ for $\delta > 0$ small. Further the functions u_i are smoothly bounded near $\partial D_p(\delta)$. The functions u_i and the associated metrics g_i may then be extended past $\partial D_p(\delta)$ to smoothly bounded conformal metrics on all of S^2 .

This construction shows that $K_i \rightarrow K$ in $C_+^{m-2, \alpha}$ on S^2 does not imply that conformal metrics g_i are uniformly bounded modulo diffeomorphisms, i.e. the condition $K \in \mathcal{C}$ is necessary. Clearly such bubble formation could occur at any assigned finite number of points in S^2 for instance.

Theorem 2.3 implies that the hypotheses (2.9) of Proposition 2.1 hold, for $g_i = e^{2u_i} g_{+1} \in [g_{+1}]$ with $K_i \rightarrow K \in \mathcal{C}$. Proposition 2.1 then shows that convergence of the curvatures implies convergence of the metrics g_i in $C^{1, \alpha} \cap W_*^{2, p}$, modulo diffeomorphism. Next we turn to the role of the diffeomorphisms.

As discussed following the proof of Proposition 2.1, there are diffeomorphisms ψ_i of S^2 such that, (in a subsequence)

$$(2.54) \quad \psi_i^*(e^{2u_i} g_{+1}) \rightarrow e^{2u} g_{+1},$$

in $C^{1, \alpha} \cap W_*^{2, p}$. Further, as in (2.15), there are $\varphi_i \in \text{Conf}(S^2)$ such that

$$(2.55) \quad \eta_i := \varphi_i^{-1} \circ \psi_i \rightarrow Id.$$

The diffeomorphisms ψ_i, η_i (and their inverses) are $(1 + \varepsilon_i)$ quasi-conformal with $\varepsilon_i \rightarrow 0$ as $i \rightarrow \infty$.

The next main step is the following:

Proposition 2.8. *For $g_i = e^{2u_i} g_{+1}$ as above with $K_i \rightarrow K \in \mathcal{C} \subset C_+^{m-2, \alpha}$, the functions $\varphi_i^* u_i$ are uniformly bounded in $C^{1, \alpha} \cap W_*^{2, p}$ for any $\alpha < 1, p < \infty$. Thus $\{u_i\}$ is bounded in $C^{1, \alpha} \cap W_*^{2, p}$ modulo the action of the conformal group on $C^{m, \alpha}$.*

Proof: By the uniform Hölder regularity of locally bounded quasi-conformal maps, cf. [35], [21], the convergence in (2.55) is in $W_*^{1, 2}$ and C^α for any fixed $\alpha < 1$. More significantly, an important

theorem of Astala [3] on the $W^{1,p}$ boundedness of K -quasi-conformal mappings shows that η_i in (2.55) are bounded in $W^{1,p}$ for any $p < \infty$, so that

$$\eta_i \rightarrow Id,$$

in $W_*^{1,p}$, i.e. weakly in $W^{1,p}$. The same statements hold of course for η_i^{-1} .

Write then $\psi_i^* = \eta_i^* \varphi_i^*$ and insert in (2.54) to obtain

$$(2.56) \quad \eta_i^*(e^{2\varphi_i^* u_i} g_{+1}) \rightarrow e^{2u} g_{+1},$$

in $C^{1,\alpha}$, bounded in $W^{2,p}$. By Astala's theorem [3], η_i^* and $(\eta_i^{-1})^*$ are uniformly bounded in L^p , for any $p < \infty$, and hence $e^{2\varphi_i^* u_i}$ is uniformly bounded in L^p .

Now from the defining equation (1.1), we have

$$(2.57) \quad \Delta \varphi_i^* u_i = 1 - (K_i \circ \varphi_i) e^{2\varphi_i^* u_i}.$$

The term $K_i \circ \varphi_i$ is bounded in L^∞ while, as noted above, the term $e^{2\varphi_i^* u_i}$ is bounded in L^p . Standard elliptic regularity applied to (2.57) shows that $\varphi_i^* u_i$ bounded in $C^{1,\alpha} \cap W^{2,p}$ modulo constants, i.e. the mean value of $\varphi_i^* u_i$. Since $e^{2\varphi_i^* u_i}$ is bounded in L^p , so is the mean value of $\varphi_i^* u_i$. This proves the result. ■

Given these results, we now restate and prove Theorem 1.1. As in the Introduction, let

$$(2.58) \quad \mathcal{N} = \{K \in C_+^{m-2,\alpha} : |\nabla K|(p) + |\Delta K|(p) > 0, \forall p \text{ s.t. } K(p) > 0\}.$$

It is easy to see that \mathcal{N} is open and dense in $C_+^{m-2,\alpha}$ and is invariant under the action of $\text{Conf}(S^2)$. Let

$$\mathcal{U} = \pi^{-1}(\mathcal{N} \cap \mathcal{C}),$$

and let

$$(2.59) \quad \pi_0 = \pi|_{\mathcal{U}} : \mathcal{U} \rightarrow \mathcal{N} \cap \mathcal{C},$$

be the restriction of π to \mathcal{U} .

Theorem 2.9. *On the domain \mathcal{U} , the curvature map*

$$\pi_0 : \mathcal{U} \rightarrow \mathcal{N} \cap \mathcal{C}$$

in (2.59) is a proper Fredholm map of index 0.

Proof: As noted in the Introduction, π is a Fredholm map of index 0 so the issue is to prove π_0 is proper. Suppose u_i is a sequence in \mathcal{U} with $K_i = K_{e^{2u_i} g_{+1}} = \pi_0(u_i)$ satisfying

$$(2.60) \quad K_i \rightarrow K \text{ in } \mathcal{N} \cap \mathcal{C} \subset C_+^{m-2,\alpha}.$$

Then by Proposition 2.8, there are $\varphi_i \in \text{Conf}(S^2)$ such that the sequence $v_i := \varphi_i^* u_i$ converges (in a subsequence) to a limit in $C^{1,\alpha}$ and weakly in $W^{2,p}$ for any $p < \infty$. Let $v_0 = \lim v_i \in C^{1,\alpha} \cap W^{2,p}$. These functions v_i satisfy

$$(2.61) \quad \Delta v_i = 1 - (K_i \circ \varphi_i) e^{2v_i}.$$

By means of a further (uniformly bounded) sequence of conformal transformations if necessary, we may assume without loss of generality, that $v_i \in \mathcal{S}$ for all i , and $v_0 \in \mathcal{S}$, for \mathcal{S} as in (2.6).

If the sequence of conformal transformations φ_i are bounded in $\text{Conf}(S^2)$, then they converge (smoothly) in a subsequence to a limit and hence $\{u_i\}$ converges in $C^{1,\alpha} \cap W_*^{2,p}$ to a limit u satisfying (1.1) weakly. Standard elliptic regularity theory then implies $u \in C^{m,\alpha}$ and, by (2.60), the convergence $u_i \rightarrow u$ is $C^{m,\alpha}$. This proves the result in this case.

Assume then $\varphi_i \rightarrow \infty$ in $\text{Conf}(S^2)$. This occurs only if the corresponding conformal dilations $[\varphi_i] \rightarrow \infty$ in $\text{Conf}(S^2)/O(3) = \mathbb{R}^3$, cf. (2.2). Without loss of generality, we may then assume that,

in a subsequence, $\varphi_i = \varphi_q(t_i)$ is a sequence of conformal dilations with pole at q and $t_i \rightarrow \infty$. Taking the limit of (2.61), it follows that

$$(2.62) \quad \Delta v_0 = 1 - K(q)e^{2v_0}.$$

Namely $e^{2v_i} \rightarrow e^{2v_0}$ in $C^{1,\alpha}$ while $K \circ \varphi_i \rightarrow K(q)$ strongly in L^p . It follows that v_0 in $W^{2,p}$ is a weak solution of (2.62). As above, elliptic regularity theory implies v_0 is smooth. The equation (2.62) means that the metric $g_0 = e^{2v_0}g_{+1}$ is of constant curvature $K \equiv K(q)$ on S^2 . As is well-known, this means that $v_0 = \varphi^*(const)$ and since $v_0 \in \mathcal{S}$, it follows (cf.(2.7)) that $v_0 = const$ and

$$(2.63) \quad K(q)e^{2v_0} = 1.$$

In particular, note that $\varphi_i \rightarrow \infty$ in $\text{Conf}(S^2)$ is only possible at points q where $K(q) > 0$.

Returning to (2.61), write (2.61) as

$$(2.64) \quad \Delta v_i = 1 - (K_i \circ \varphi_i)e^{2v_i} = 1 - K(q)e^{2v_i} + (K(q) - K_i \circ \varphi_i)e^{2v_i}.$$

In the following, it will simplify the argument to rescale the sequence K_i slightly, $K_i \rightarrow \frac{K(q)}{K_i(q)}K_i$ so that

$$(2.65) \quad K_i(q) = K(q),$$

for i sufficiently large. This is equivalent to shifting v_i by small constants $v_i \rightarrow v_i + \delta_i$ with $\delta_i \rightarrow 0$ as $i \rightarrow \infty$. We assume this has been done, without changing the notation.

Using (2.63), (2.64) may be rewritten as

$$(2.66) \quad \Delta(v_i - v_0) = 1 - e^{2(v_i - v_0)} + (E_i + F_i)e^{2v_i},$$

where

$$(2.67) \quad E_i = (K(q) - K \circ \varphi_i)e^{2v_i} \quad \text{and} \quad F_i = [(K - K_i) \circ \varphi_i]e^{2v_i}.$$

By a careful but straightforward analysis of the asymptotic behavior of $K \circ \varphi_t$ as $t \rightarrow \infty$, Chang-Yang show in [11] that

$$(2.68) \quad |K(q) - K \circ \varphi_t|_{L^2}^2 \leq \chi_q(t) := \begin{cases} c \frac{\log t}{t^2} & \text{if } \nabla K(q) \neq 0, \\ \frac{c}{t^2} & \text{if } \nabla K(q) = 0. \end{cases}$$

Using (2.65), and replacing K above by $K - K_i$, the same argument shows that

$$(2.69) \quad |(K - K_i)(q) - (K - K_i) \circ \varphi_i|_{L^2}^2 \leq \chi_q(t_i).$$

Since v_i is uniformly bounded in $C^{1,\alpha}$, it follows that

$$(2.70) \quad |E_i|_{L^2}^2 + |F_i|_{L^2}^2 \leq \chi_q(t_i).$$

Next for the second term on the right in (2.66), since $v_i \rightarrow v_0$ in $C^{1,\alpha}$, $e^{2(v_i - v_0)} = 1 + 2(v_i - v_0) + \sigma_i$, where $\sigma_i = O((v_i - v_0)^2)$. Then (2.66) gives

$$\Delta(v_i - v_0) + 2(v_i - v_0) = E_i + F_i - \sigma_i.$$

The kernel E_2 of the operator $\Delta + 2$ consists of 1st eigenfunctions of the Laplacian on S^2 , i.e. the restrictions of linear functions ℓ to $S^2(1)$. The tangent space $T_0\mathcal{S}$ to \mathcal{S} at $v_0 = const$ is the orthogonal complement of E_2 . Since $v_i, v_0 \in \mathcal{S}$, and since σ_i is lower order, it follows that $v_i - v_0$ satisfies the elliptic estimate

$$(2.71) \quad |v_i - v_0|_{L^{2,2}}^2 \leq \chi_q(t_i).$$

On other hand, pairing (2.61) with $xe^{-2(v_i - v_0)}$ where x is any linear function on $S^2(1)$ gives

$$xe^{-2(v_i - v_0)}\Delta v_i = xe^{-2(v_i - v_0)} - x \frac{K_i \circ \varphi_i}{K(q)}.$$

Since $e^{-2(v_i-v_0)}\Delta(v_i-v_0) = -\frac{1}{2}\Delta e^{-2(v_i-v_0)} + 2|dv_i|^2 e^{-2(v_i-v_0)}$, we obtain

$$-\frac{1}{2} \int x \Delta e^{-2(v_i-v_0)} + 2 \int x |dv_i|^2 e^{-2(v_i-v_0)} = \int x e^{-2(v_i-v_0)} - \frac{1}{K(q)} \int x K_i \circ \varphi_i.$$

(Here and below, all integrals are with respect to the standard volume form of g_{+1}). Using the self-adjointness of the Laplacian Δ and the fact that x is a first eigenfunction, one sees that the first and third terms above cancel, leaving

$$2 \int x |dv_i|^2 e^{-2(v_i-v_0)} = -\frac{1}{K(q)} \int x K_i \circ \varphi_i.$$

Since $|x| \leq 1$, $v_0 = \text{const}$ and $v_i - v_0 \rightarrow 0$ in C^0 it follows from (2.71) that

$$\left| \int x |dv_i|^2 e^{-2(v_i-v_0)} \right| \ll \chi_q(t_i),$$

and so, for any x ,

$$(2.72) \quad \left| \int x K_i \circ \varphi_i \right| \ll \chi_q(t_i).$$

Now the analysis by Chang-Yang in [11] again shows that for $K \in \mathcal{N}$, i.e. K non-degenerate in the sense of [11], and for a suitable choice of x , one has

$$(2.73) \quad \left| \int x K \circ \varphi_i \right| \geq \psi_q(t_i),$$

where

$$(2.74) \quad \psi_q(t) = \begin{cases} \frac{c}{t} & \text{if } \nabla K(q) \neq 0, \\ c \frac{\log t}{t^2} & \text{if } \nabla K(q) = 0, \Delta K(q) \neq 0. \end{cases}$$

As above, writing $K_i \circ \varphi_i = (K_i - K) \circ \varphi_i + K \circ \varphi_i$, the same analysis by Chang-Yang in [11] as above shows that, for any x ,

$$\left| \int x (K - K_i) \circ \varphi_i \right| \ll \psi_q(t_i),$$

so that

$$(2.75) \quad \left| \int x K_i \circ \varphi_i \right| \geq \psi_q(t_i),$$

Comparing the behavior of $\chi(t)$ and $\psi(t)$ in (2.68) and (2.74), the estimates (2.72) and (2.75) give a contradiction for t_i sufficiently large, i.e. φ_i sufficiently large in $\text{Conf}(S^2)$. It follows that $\{\varphi_i\}$ is bounded in $\text{Conf}(S^2)$, and the proof is then completed above. ■

Remark 2.10. The properness of the map π_0 corresponds to (and is in fact equivalent to) general *a priori* estimates for solutions of (1.1). Thus, as noted in the Introduction, the properness of π_0 implies that for any $K \in \mathcal{N} \cap \mathcal{C}$ with $\text{dist}_{C^{m-2,\alpha}}(K, \partial(\mathcal{N} \cap \mathcal{C})) \geq \varepsilon > 0$, there is a constant $C = C(|K|_{C^{m-2,\alpha}}, \varepsilon, \alpha')$ such that, if u is any solution of (1.1), (i.e. $\pi_0(u) = K$), then

$$(2.76) \quad |u|_{C^{m,\alpha'}} \leq C,$$

for any $\alpha' < \alpha$.

Remark 2.11. We note that the use of Astala's theorem in [3] is crucial in the argument above. If η_i are merely bounded in $W^{1,2}$, one only obtains an L^1 bound on e^{2v_i} from (2.56) and this is not enough to obtain control on v_i from (2.61). In fact obtaining suitable control in this case relates to the exponential Sobolev embedding theorems of Trudinger-Moser-Aubin. On the other hand, one only really needs a bound on η_i in $W^{1,p}$ for some $p > 2$, not any $p < \infty$.

The structure of the domain $\mathcal{K} = \mathcal{N} \cap \mathcal{C}$ plays an important role in the further analysis of the curvature map π . Let $\{\mathcal{K}^i\}$ denote the collection of path components of \mathcal{K} , so that

$$(2.77) \quad \mathcal{K} = \bigcup_i \mathcal{K}^i.$$

As in (1.9), let $\mathcal{U}^i = \pi^{-1}(\mathcal{K}^i)$. The map π (or π_0) restricted to \mathcal{U}^i gives then a proper Fredholm map

$$(2.78) \quad \pi^i : \mathcal{U}^i \rightarrow \mathcal{K}^i,$$

into a connected target space. It is not easy to determine the number of components of \mathcal{K} ; we will see in the next section that \mathcal{K} has in fact infinitely many components.

Since both \mathcal{N} and \mathcal{C} are open and dense in $C_+^{m-2,\alpha}$, the closure $\overline{\mathcal{K}} = C_+^{m-2,\alpha}$. The point-set theoretic boundary

$$(2.79) \quad \partial\mathcal{K} = C_+^{m-2,\alpha} \setminus \mathcal{K} = \partial\mathcal{N} \cup \partial\mathcal{C}$$

consists of two main parts; $\partial\mathcal{N}$ consists of functions K such that $\Delta K(q) = 0$ for some critical point q of K with $K(q) > 0$, while $\partial\mathcal{C}$ consists of functions K for which there exist $p \in \{K = 0\}$ such that (1.5) holds at p and $\Delta K(p) \leq 0$. Note that if p is a non-degenerate critical point of $K \in \partial\mathcal{C}$, then p is necessarily a local maximum. Both of these boundaries will play an important role in the analysis to follow.

Remark 2.12. As noted above, while \mathcal{N} and $\partial\mathcal{N}$ are invariant under the action of $\text{Conf}(S^2)$, they are not invariant under the natural action $(\psi, K) \rightarrow K \circ \psi$ of $\text{Diff}^{m-2,\alpha}$ on $C_+^{m-2,\alpha}$. (On the other hand, \mathcal{C} and $\partial\mathcal{C}$ are $\text{Diff}^{m-2,\alpha}$ invariant). It is easy to see that the sign of ΔK at a saddle point q may be changed arbitrarily under the action of local diffeomorphisms acting only in a neighborhood of q .

Also, the number of critical points of K (even Morse non-degenerate critical points) may change in a given path component \mathcal{N}^i of \mathcal{N} . For instance, in a neighborhood of a local maximum of K , one may locally perturb K by “pushing down”, creating two local maxima and a saddle point (mountain pass), while keeping $\Delta K < 0$ throughout the process.

It is worthwhile to compare the space \mathcal{K} with the space \mathcal{M} of Morse functions in $C_+^{m-2,\alpha}$. The space \mathcal{M} is also open and dense in $C_+^{m-2,\alpha}$ and is $\text{Diff}^{m-2,\alpha}$ invariant. As in (2.79), one has $\partial\mathcal{M} = C_+^{m-2,\alpha} \setminus \mathcal{M}$ and there is a decomposition into $\text{Diff}^{m-2,\alpha}$ invariant path components

$$(2.80) \quad \mathcal{M} = \bigcup_j \mathcal{M}^j.$$

Now it is clear that there are infinitely many path components \mathcal{M}^j , since for instance the number of non-degenerate critical points, which can be arbitrarily large, is constant on each component \mathcal{M}^j . The boundary or wall $\partial\mathcal{M}$, consisting of functions with at least one degenerate critical point, serves as a birth (or death) region for the creation of non-degenerate critical points. Under perturbation, a degenerate critical point in $\partial\mathcal{N}$ can resolve for instance into either two critical points or zero critical points. For a detailed discussion of the decomposition (2.80), we refer to [2], [38].

We have the following structure of generic points in $\partial\mathcal{K}$.

Proposition 2.13. *The intersection $\mathcal{M} \cap \partial\mathcal{K}$ is open and dense in $\partial\mathcal{K}$. Near a generic $K \in \mathcal{M} \cap \partial\mathcal{K}$, the boundary $\partial\mathcal{K}$ is a smooth, codimension 1 hypersurface of $C_+^{m-2,\alpha}$, locally separating $C_+^{m-2,\alpha}$ into two components. In a neighborhood of any $K \in \mathcal{M} \cap \partial\mathcal{K}$, $\partial\mathcal{K}$ is a region formed from the transverse intersection of a finite number of codimension 1 hypersurfaces.*

Proof: It is obvious that $\mathcal{M} \cap \partial\mathcal{K}$ is open in $\partial\mathcal{K}$. To prove the intersection is dense, we work first with $\partial\mathcal{N}$. Let $K \in \partial\mathcal{N}$ and let K_i be a sequence of Morse functions with $K_i \rightarrow K \in C_+^{m-2,\alpha}$. If in a subsequence $K_i \in \partial\mathcal{N}$, there is nothing more to prove, so suppose i is large and $K_i \notin \partial\mathcal{N}$. For each i large, the function K_i has a finite set Q_i of non-degenerate critical points which are Hausdorff close to the critical locus $Z = \{q : |\nabla K|(q) + |\Delta K|(q) = 0\}$ of K . By means of small local perturbations by diffeomorphisms as discussed in Remark 2.12, one may perturb K_i slightly, keeping K_i Morse, and such that $|\nabla K_i| + |\Delta K_i| = 0$ on the saddle points of Q_i while $|\Delta K_i| \leq \varepsilon_i$ on any local maxima and minima in Q_i ; here $\varepsilon_i \rightarrow 0$ as $i \rightarrow \infty$. Letting $i \rightarrow \infty$ then proves that $\mathcal{M} \cap \partial\mathcal{N}$ is dense in $\partial\mathcal{N}$. The proof that $\mathcal{M} \cap \partial\mathcal{C}$ is dense in $\partial\mathcal{C}$ is similar and straightforward.

Next, define

$$F : C_+^{m-2,\alpha} \times S^2 \rightarrow \mathbb{R}^3$$

$$F(K, p) = (\nabla K(p), \Delta K(p)).$$

Thus $K \in \partial\mathcal{N}$ if there exists p such that $(K, p) \in F^{-1}(0)$. The map F is smooth and the derivative in the first factor is $DF_p(\beta) = \frac{d}{dt}F(K + t\beta)(p) = (\nabla\beta(p), \Delta\beta(p))$, which is clearly surjective since β is arbitrary. Thus, by the implicit function theorem in Banach spaces, $\Sigma = F^{-1}(0)$ is a codimension 3 smooth hypersurface of $C_+^{m-2,\alpha} \times S^2$. Of course $F^{-1}(0)$ may not be connected.

Now consider the projection $p : C_+^{m-2,\alpha} \times S^2 \rightarrow C_+^{m-2,\alpha}$ of Σ . Clearly

$$(2.81) \quad p(\Sigma) = \partial\mathcal{N}.$$

We claim that for $K \in \mathcal{M}$, Σ is transverse to the vertical S^2 factors, in that

$$(2.82) \quad T\Sigma \cap TS^2 = 0.$$

To see this, one has $T\Sigma = \text{Ker}DF$ and for $v \in T_{(K,p)}S^2$, $F_*(v) = \frac{d}{dt}(\nabla K(p + tv), \Delta K(p + tv)) = (D^2K(v), v(\Delta K))$. Thus $v \in T\Sigma \cap TS^2$ if and only if $v \in \text{Ker}D^2K$ and $v \perp \nabla\Delta K$ at p . Since p is a critical point of K and K is a Morse function, it follows that $v = 0$, which proves the claim.

Given any $K \in \mathcal{M} \cap \partial\mathcal{N}$, consider ΔK as a function on the set of critical points p of K . Generically, a point $K \in \mathcal{M} \cap \partial\mathcal{N}$ has exactly one critical point p where $\Delta K = 0$, with $\Delta K \neq 0$ at all other critical points. In this case, the transversality (2.82) implies that $\partial\mathcal{N}$ is a codimension one hypersurface near K . Such local hypersurfaces may have a finite number of intersections, corresponding to the vanishing of ΔK at multiple critical points. Since the critical points are isolated, it is clear that the intersection of j such hypersurfaces is a submanifold of codimension j .

Similarly, a function $K \in \mathcal{M}$ has a finite number of local maxima and by considering the behavior of K on such local maxima, it is easy to see that $\mathcal{M} \cap \partial\mathcal{C}$ is the intersection of a finite number (generically one) of smooth hypersurfaces. ■

The proof of Proposition 2.10 shows that in general $\partial\mathcal{K}$ has the structure of a rectifiable set of codimension one in $C_+^{m-2,\alpha}$. Namely, by definition, a set $Y \subset C_+^{m-2,\alpha}$ is rectifiable of codimension s if Y is contained in a countable union of sets of the form $p(G)$, where $p : C_+^{m-2,\alpha} \times \mathbb{R}^k \rightarrow C_+^{m-2,\alpha}$ is projection on the first factor and G is a codimension $s + k$ submanifold (a graph) in $C_+^{m-2,\alpha} \times \mathbb{R}^k$, cf. [47], [5] for further details. Rectifiable sets of codimension s are preserved under smooth Fredholm maps of index 0 (cf. [5]). The characterization of $\partial\mathcal{N}$ in (2.81) then shows that $\partial\mathcal{N}$ is rectifiable of codimension one and similarly for $\partial\mathcal{C}$.

Although Proposition 2.13 shows that $\partial\mathcal{K}$ locally separates $C_+^{m-2,\alpha}$, it is not easy to see to what extent $\partial\mathcal{K}$ globally separates \mathcal{K} into distinct components. This will be discussed further in Sections 3 and 4.

Regarding \mathcal{C} , it is clear that the space of positive curvature functions

$$(2.83) \quad \mathcal{P} = \{K \in C_+^{m-2,\alpha} : K(x) > 0, \forall x \in S^2\},$$

is path connected and of course contained in \mathcal{C} .

Proposition 2.14. *The space $\mathcal{C} \subset C_+^{m-2,\alpha}$ is path connected.*

Proof: Let $K(r)$, $r \in [-1, 1]$ be a smooth path of functions in $C_+^{m-2,\alpha}$ connecting points $K(-1)$ and $K(+1)$ in \mathcal{C} . Without loss of generality, assume $K(r)$ has only isolated critical points and assume $K(r)$ intersects $\partial\mathcal{C}$ in the single point $K(0)$. Let $K_\pm = K(\pm\varepsilon)$ with ε sufficiently small and assume K_\pm are Morse functions.

We will use the following simple construction from Morse theory. Let K be a Morse function. Then all pairs of local maxima of K are connected by descending gradient flow lines to the collection of saddle points of K . Consider the collection C of local maxima in the region $U_0 = \{K < 0\} \subset S^2$. Each local maximum in C is connected via a saddle point in U_0 to another local maximum in C or to a local maximum (perhaps the absolute maximum) in the region $K > 0$. Here without loss of generality assume that 0 is a regular value of K so that the level set $L_0 = \{K = 0\}$ consists of a finite collection of embedded simple curves, bounding the planar domain $U_0 \subset S^2$.

Then by standard Morse theory there is a smooth curve of functions K_t , leaving K unchanged in the region $K \geq 0$, deforming $K = K_0$ to K_1 and for which K_1 has no local maxima in the region $K_1 < 0$. Namely by pushing down along gradient flow lines, one may inductively cancel off any local maximum with a saddle point in U_0 . (Of course this path passes from one Morse chamber to another, but this is irrelevant here). It follows that K_1 has only local minima and saddle points in the region $K_1 < 0$. The path K_t is contained in \mathcal{C} , i.e. it does not intersect $\partial\mathcal{C}$. One may visualize this construction concretely by viewing K as the height function z of an embedding or immersion of S^2 into \mathbb{R}^3 ; this is always possible by the main result in [34].

Now apply the construction above to the two points $K = K_\pm$. This gives a pair of paths joining K_\pm to points K'_\pm remaining within \mathcal{C} . Next, combine these paths with the paths $K'_\pm(s) = K'_\pm + sc$ for $s \in [0, 1]$ and $c > \max(|\min K'_+|, |\min K'_-|)$. The resulting paths join K'_\pm to points $K'_\pm(1)$ in \mathcal{P} again remaining within \mathcal{C} . The endpoints of these curves may then be connected within \mathcal{P} . ■

Observe that any component \mathcal{N}^i of \mathcal{N} intersects \mathcal{P} ; namely as above, for any $K \in \mathcal{N}^i$, choosing $c > |\min K|$, the path $K(s) = K + sc$, $s \in [0, 1]$ connects K within \mathcal{N}^i to a point in \mathcal{P} .

Proposition 2.15. *Any path component \mathcal{K}^i of $\mathcal{K} = \mathcal{N} \cap \mathcal{C}$ intersects \mathcal{P} . Moreover, $\mathcal{K}^i \cap \mathcal{P}$ is path connected so that a path component \mathcal{K}^i of \mathcal{K} is uniquely determined by the path component $\mathcal{K}^i \cap \mathcal{P}$.*

Proof: The proof is a refinement of the proof of Proposition 2.14. Choose any Morse function $K \in \mathcal{M} \cap \mathcal{K}$. First note that if K has no negative local maxima, i.e. no local maxima p with $K(p) < 0$, then adding a curve of positive constants to K as above connects K to \mathcal{P} within \mathcal{K} , i.e. within a path component of \mathcal{K} .

Next suppose p is a negative local maximum of K which is connected via a gradient flow line to a negative saddle point q , in the sense that $\Delta K(q) < 0$; necessarily $K(q) < 0$. Then as in the proof of Proposition 2.14, one can deform $K = K_0 \in \mathcal{K}$ to $K_1 \in \mathcal{K}$ along a curve by pushing down along gradient flow lines to cancel p and q , while maintaining $\Delta K < 0$ at the curve of saddle points. As above this requires crossing a wall between Morse chambers so that a saddle point becomes Morse degenerate, i.e. $\det D^2 K(q_t) = 0$. However, one may maintain $\Delta K(q_t) < 0$ on the curve q_t of saddle points. Such curves thus remain in \mathcal{K} . One may iterate this process inductively to cancel off all such pairs. This connects K to a point K' to which one may apply the argument above.

Thus suppose K has a negative local maximum p without negative (i.e. $\Delta K < 0$) saddle points in U_0 connected to p . In this case, we first create a new pair of positive local maxima with a new negative saddle point starting from K . Namely the given function K has a positive absolute maximum say at p_0 . By locally pushing down near p_0 as in Remark 2.12, one may create within \mathcal{K} a new function K' with two positive maxima and a saddle q' connecting them, with $\Delta K'(q') < 0$

with $K = K'$ on U_0 , where $U_0 = S^2 \setminus V_0$ and V_0 is a small neighborhood of p_0 . The value $K'(q')$ of the saddle q' may then be pushed down further so that $K'(q') < K(q)$, for any saddle q of K in the region U_0 , again remaining in \mathcal{K} . Replacing K by K' , one may then apply the construction in the previous paragraph to cancel a negative local maximum of $K' = K$ with the negative saddle point q' . Again, this construction may be iterated inductively to cancel all negative local maxima of while remaining within \mathcal{K} . In other words, K is connected by a curve in \mathcal{K} to another Morse function \widehat{K} without negative local maxima. One may then connect \widehat{K} to \mathcal{P} by pushing up by positive constants as above.

The second statement follows from the stability or continuity of the construction above. Thus, suppose K_1 and K_2 are two points in $\mathcal{K}^i \cap \mathcal{P}$ connected by a smooth path $K(s)$ in $\mathcal{K}^i \subset \mathcal{N} \cap \mathcal{C}$. The process above applied at K_1 may be applied continuously along $K(s)$ to deform the path $K(s)$ into $\mathcal{K}^i \cap \mathcal{P}$.

■

3. DEGREE COMPUTATIONS

The purpose of this section is to prove Theorems 1.2 and 1.3.

The map π_0 in (1.8) or (2.59) is a proper Fredholm map of index 0 between open subsets of Banach spaces. Proper Fredholm maps $F : X \rightarrow Y$ of index 0 between oriented Banach spaces or oriented Banach manifolds X, Y with Y path connected have a \mathbb{Z} -valued degree defined as

$$\deg F = \sum_{x \in F^{-1}(y)} \text{sign}(D_x F) \in \mathbb{Z},$$

where y is any regular value of F and the sign of $DF(x)$ is ± 1 according to whether $DF(x)$ preserves or reverses orientation at x , cf. [20] or more recently, [6], [7] for example. A point $y \in Y$ is a regular value of F if $D_x F$ is an isomorphism, for all $x \in F^{-1}(y)$. Note that the mod 2 reduction of $\deg_{\mathbb{Z}} F$ is the (unoriented) Smale degree [44]. The domain \mathcal{U} of π_0 is an open subset of a Banach space, and so is clearly orientable, as is the target space $\mathcal{N} \cap \mathcal{C} \subset C_+^{m-2, \alpha}$.

Proof of Theorem 1.2.

To calculate the degree of

$$\pi^i : \mathcal{U}^i \rightarrow \mathcal{K}^i,$$

as in (2.78), recall that Proposition 2.15 shows that each component \mathcal{K}^i of \mathcal{K} intersects \mathcal{P} and \mathcal{K}^i itself is path connected. The degree of π^i is then the same as the degree of the restricted map

$$\tilde{\pi}^i : \mathcal{U}_{\mathcal{P}}^i \rightarrow \mathcal{K}^i \cap \mathcal{P},$$

where \mathcal{P} is the space of positive curvature functions as in (2.83) and $\mathcal{U}_{\mathcal{P}}^i = (\pi^i)^{-1}(\mathcal{K}^i \cap \mathcal{P})$.

The degree of $\tilde{\pi}^i$ is calculated by Chang-Gursky-Yang in [12], by an elegant argument based on the Poincaré-Hopf theorem and is given by the formula (1.11). Hence (1.11) holds for π^i .

■

Remark 3.1. Note that the formula for $\deg \pi^i$ given by (1.11) changes, generically by one, when passing through the boundary region $\partial \mathcal{C} \cap \mathcal{N}$, i.e. on a curve $K(t)$ on which a local maximum of $K(t)$ passes through zero. Similarly it changes generically by one on curves $K(t)$ passing through the boundary region $\partial \mathcal{N} \cap \mathcal{C}$, so that $\Delta K(t)$ changes sign at a saddle point of $K(t)$. Note that the sign of ΔK can be changed or altered just by means of local diffeomorphisms of S^2 supported near the saddle points, as in Remark 2.12.

We note that the results above in Sections 2 and 3 do not use any variational formulation of the curvature equation (1.1). We now turn to the proof of Theorem 1.3 which gives another calculation of $\deg \pi^i$ using the functional J from (1.2).

Thus consider

$$(3.1) \quad J : C^{m,\alpha} \times C_+^{m-2,\alpha} \rightarrow \mathbb{R},$$

$$J(u, K) = \int_{S^2} (|du|^2 + 2u) dv_0 - \log \left(\int_{S^2} K e^{2u} dv_0 \right).$$

Note that J is invariant under the translations $u \rightarrow u + c$. The variation $D_1 J$ of J at (u, K) with respect to the first variable is given by

$$(3.2) \quad \frac{1}{2} D_1 J = -\Delta u + 1 - \kappa K e^{2u},$$

where $\kappa = \left(\int K e^{2u} dv_0 \right)^{-1}$. Hence, critical points of J with respect to variations of u , subject to the constraint $\kappa = 1$, are exactly the solutions of (1.1). It is worth noting that the structure, for example the topology, of the constraint manifold

$$\mathcal{C}_K = \left\{ u \in C^{m,\alpha} : \int_{S^2} K e^{2u} dv_0 = 1 \right\},$$

depends on $K \in C_+^{m-2,\alpha}$. Of course there are no critical points with respect to variation in the second or K -variable. By Theorem 1.1, the set of critical points of J_K (satisfying the constraint $\kappa = 1$) is compact for any fixed $K \in \mathcal{K}$. If K is a regular value of π_0 , there are only finitely many critical points of J_K and all are non-degenerate, (cf. Proposition 5.3 below).

Given a critical point u of J_K , the index ind_u of J_K at u is the maximal dimension on which the second variation $D^2 J_K$ with respect to u is negative definite. It is easy to see that ind_u is finite, for any fixed u . We note that it is well-known [27] the infimum of J_K is never achieved unless $K = \text{const}$.

Proof of Theorem 1.3.

This result is by now an essentially standard result for variational problems satisfying compactness conditions as in Theorem 1.1. In the following, we adapt an elegant approach of White [47], [48] to this setting.

The right-hand side of (1.12) is defined for each regular value K and the main issue is to prove the expression is well-defined, i.e. independent of the choice of $K \in \mathcal{K}^i$. Thus let $K(0)$ and $K(1)$ be any two regular values of π^i and let $K(t)$, $t \in [0, 1]$, be an oriented smooth curve in \mathcal{K}^i joining them. Let $u(t) = (\pi_0)^{-1}(K(t))$. As shown in Theorem 5.5 below (which is independent of the considerations here), the set of singular points $\Sigma \subset \mathcal{U}^i$ is a stratified space of codimension at least one. Hence, if necessary, $K(t)$ may be perturbed slightly so that $K(t)$ is transverse to π^i and thus $u(t)$ is a collection of smooth disjoint curves in \mathcal{U}^i connecting regular points over $K(0)$ and $K(1)$, cf. also [44]. Define an orientation on $u(t)$ by defining π_0 to be orientation preserving near any regular point $u(t)$ if the index $\text{ind}_{u(t)}$ is even, orientation reversing if the index is odd. Given this orientation on curves, the mapping degree of

$$\pi_0 : u(t) \rightarrow K(t),$$

between these 1-dimensional manifolds is then given exactly by (1.12). It suffices then to show that this definition of orientation is well-defined, i.e. consistent, in that when $u(t)$ passes through a critical point of π_0 , the local orientation of π_0 changes according to the change of the index. In the following, we assume without loss of generality that $u(t)$ is a connected curve.

Suppose $u(t_0)$, $t_0 \in (0, 1)$ is a critical point of π_0 , so that the Hessian $D^2 J_K$ at $u(t_0)$ has a non-trivial null-space N , i.e. the nullity $\dim N$ of J_K at $u(t_0)$ is non-zero, cf. also Proposition 5.3

below. Without loss of generality, assume $\dim N = 1$; the arguments below apply to each line in N . The following computations take place for t near t_0 .

It is trivial to see that the map $F : C^{m,\alpha} \times C_+^{m-2,\alpha} \rightarrow C^{m-2,\alpha}$ given by $F(u, K) = \frac{1}{2}D_1J = 1 - \Delta u - \kappa K e^{2u}$ is a submersion. Hence $F^{-1}(0)$ is a closed submanifold; clearly $F^{-1}(0) = \text{graph } \pi$. The submanifold $F^{-1}(N)$ is a 1-dimensional thickening of $F^{-1}(0)$ and one may thicken the curve $(u(t), K(t))$ to a 2-parameter family

$$\sigma(t, s) = (u(t, s), K(t, s)) \in F^{-1}(N) \quad \text{with} \quad K(t, s) = K(t).$$

By construction, one has

$$(3.3) \quad \frac{d}{ds} J(\sigma(t, s)) = \int_{S^2} \langle F(\sigma(t, s)), \frac{du}{ds} \rangle dv_{+1}.$$

Clearly $\frac{d}{ds} J(\sigma(t, s))|_{s=0} = 0$. By construction $F(\sigma(t, s)) \in N$ and $\frac{du}{ds} \in N$ are non-zero for $s \neq 0$. In the (t, s) plane P , the locus where $\partial J(t, s)/\partial s = 0$ (with $J(t, s) := J(\sigma(t, s))$) is given locally by the graph $s = u(t)$ of the curve $u(t)$. In particular, $(t, u(t))$ is locally the boundary of the open domain $U = \{\partial J/\partial s > 0\}$ in P :

$$u(t) = \partial U.$$

If ind_{u_t} is even for t near t_0 , $t \neq t_0$, $u(t)$ is given the boundary orientation induced by U while if $\text{ind}_{u(t)}$ is odd near t_0 , give $u(t)$ the opposite orientation. It is now simple to see that this choice agrees with the definition above. Namely, the point t_0 is a critical point for the map $\pi_2 \circ u : I \rightarrow \mathbb{R}$, where π_2 is projection onto the s -factor. If this critical point is a folding singularity for $\pi_2 \circ u$, then the index of $u(t)$ changes by 1 in passing through $u(t_0)$ and reverses the orientation of $\pi_2(u(t))$, (exactly as in the case of the standard folding singularity $x \rightarrow x^2$). If on the other hand $\pi_2 \circ u$ does not fold with respect to π_2 (so that one has an inflection point) then the index of $u(t)$ does not change through t_0 and π_2 maps $u(t)$ to $\pi_2 \circ u(t)$ in an orientation preserving way.

This shows that the degree (1.12) is well-defined. The fact that it equals the degree in (1.12) up to an overall sign follows from the basic uniqueness properties of such \mathbb{Z} -valued mappings, cf. [6]. This completes the proof. ■

4. DISCUSSION ON THE IMAGE OF π .

In this section we summarize some of the main results of the previous sections and prove several further results on the behavior of π near the boundary $\partial\mathcal{K}$. We also discuss the structure of the components of \mathcal{K} . These issues provide a bridge to issues discussed in the next section.

It follows from the results in Sections 2 and 3 that the boundary $\partial\mathcal{K}$ is a closed rectifiable set of codimension 1, separating $C_+^{m-2,\alpha}$ into an infinite collection of components \mathcal{K}^i on which the proper maps

$$\pi^i : \mathcal{U}^i \rightarrow \mathcal{K}^i$$

have a well-defined degree. For any K in a component \mathcal{K}^i of non-zero degree, (1.1) is solvable (since π^i is surjective), and there is an effective, signed, count on the number of solutions for generic K . There are no restrictions on the structure of the set of critical points for K in such components \mathcal{K}^i ; for example, critical points of K need not be isolated.

The only possible regions where (1.1) may not have a solution are the components of degree zero and the boundary region $\partial\mathcal{K} = \partial\mathcal{N} \cup \partial\mathcal{C}$.

First we discuss some general structural results for \mathcal{K} .

Lemma 4.1. *The degree of π_0 may take on any value in \mathbb{Z} , i.e. for all $n \in \mathbb{Z}$, there exists a component \mathcal{K}^i of \mathcal{K} such that*

$$\deg \pi^i = n.$$

Proof: Let K be a Morse function in \mathcal{K} . Let M and m denote the number of local maxima and minima respectively, and let s^+ , s^- denote the number of saddle points with $\Delta K > 0$ and $\Delta K < 0$ respectively. Then the Morse Lemma gives $M - (s^+ + s^-) + m = \chi(S^2) = 2$, while the degree formula (1.11) gives $\deg \pi_0 = M - s^- - 1$.

If all saddles have $\Delta K > 0$, so $s^- = 0$, then $\deg \pi_0 = M - 1$, while if all saddles have $\Delta K < 0$, then $\deg \pi_0 = M - (s^+ + s^-) - 1 = 1 - m$. It is clear that for any M there is a Morse function with M local maxima and similarly for any m , there is a Morse function with m local minima.

Finally, as in Remark 3.1, at any given saddle point q , by using local diffeomorphisms supported near q , one may change the sign of ΔK arbitrarily. It follows that $\deg \pi_0$ takes on all values in \mathbb{Z} . ■

Remark 4.2. It is unknown whether the set of components \mathcal{K}^i of a fixed degree is connected, (although this seems unlikely). In view of Proposition 2.15, consider the collection of components $\mathcal{K}^i \cap \mathcal{P}$, i varying, of a fixed degree, say $n = M - s^- - 1$ and let $K_i = K$ be a Morse function in $\mathcal{K}^i \cap \mathcal{P}$. Each such K has a finite number of critical points and the flow lines of ∇K flow between these critical points. Any pair of local maximum points p_1, p_2 of K connected to a saddle point q of K with $\Delta K(q) < 0$ can be canceled by a path $K(t)$, $t \in [0, 1]$, of Morse functions staying in the given component $\mathcal{K}^i \cap \mathcal{P}$, leaving a single maximum point. Thus both M and s^- can be reduced by one along such paths in $\mathcal{K}^i \cap \mathcal{P}$. In the same way, any pair of local minima of K can be canceled within $\mathcal{K}^i \cap \mathcal{P}$ with a saddle point with $\Delta K > 0$ leaving a single minimum point. Thus, within each component $\mathcal{K}^i \cap \mathcal{P}$, one may find Morse functions $K'_i \in \mathcal{K}^i \cap \mathcal{P}$ with the minimal numbers M_{min} , m_{min} of local maxima and minima respectively - among Morse functions in $\mathcal{K}^i \cap \mathcal{P}$. The numbers M_{min} and m_{min} , and so also the minimal number of saddle points s_{min} , are thus the same for the collection of components $\mathcal{K}^i \cap \mathcal{P}$. Note also that the space of Morse functions with a fixed number of local maxima and minima is path-connected in \mathcal{M} , cf. [34] for example. However it is not clear that there must exist paths connecting such points in $\mathcal{K}^i \cap \mathcal{P}$, i.e. paths on which the sign of ΔK at the saddle points remains fixed.

On the other hand, note that the argument above proves that there is only one component \mathcal{K}^0 of degree zero, since $M_{min} = m_{min} = 1$, $s_{min} = 0$ on \mathcal{K}^0 .

Next we show that there are solutions of (1.1) in each of the regions \mathcal{K}^0 (the degree zero component) and $\partial\mathcal{K}$ mentioned above.

Proposition 4.3. *The domain \mathcal{K}^0 is non-empty, i.e. there exist u with $K_u = \pi(u) \in \mathcal{K}^0$.*

Proof: It is well-known, cf. [33], that given any function $K \in C_+^\infty$ there is a smooth metric g on S^2 such that $K_g = K$. By the uniformization theorem $g = \psi^*(e^{2u}g_{+1})$, for some diffeomorphism ψ and function u . The metric $\tilde{g} = e^{2u}g_{+1}$ is conformal to g_{+1} with Gauss curvature $K_{\tilde{g}} = K \circ \psi^{-1}$.

Now choose K to be any Morse function in \mathcal{K}^0 with exactly two critical points, necessarily a maximum and a minimum. This property is invariant under diffeomorphisms, and necessarily $\tilde{K} = K_{\tilde{g}} \in \mathcal{K}^0$. Then $\pi(u) = \tilde{K}$. ■

Although Remark 4.2 shows that \mathcal{K}^0 is only one component among infinitely many others on which π_0 is onto, the next result shows that, in a certain sense, the degree zero component \mathcal{K}^0 is the largest.

Proposition 4.4. *For any given K , the set of linear functions ℓ such that*

$$(4.1) \quad K + \ell \in \text{Im } \pi$$

is a bounded set in \mathbb{R}^3 . Hence, for any given K , for any ℓ sufficiently large, $K + \ell \in \mathcal{K}^0$ and $K + \ell$ is not solvable for u in (1.1).

Proof: The proof is (naturally) based on the Kazdan-Warner obstruction (1.4). First if $K = \text{const}$, then $K + \ell \notin \text{Im } \pi$ for any $\ell \neq 0$ by (1.4). Thus we assume $K \neq \text{const}$. Any given ℓ may be written in the form $\ell = c\langle p, \cdot \rangle$, where $p \in S^2$, $c > 0$ and $\langle \cdot, \cdot \rangle$ is the Euclidean dot product. As noted following (2.5), $K \in \text{Im } \pi$ if and only if $K_\varphi = K \circ \varphi \in \text{Im } \pi$ for any $\varphi \in \text{Conf}(S^2)$. Given p as above, choose φ such that $\min K_\varphi$ is achieved at $-p$ while $\max K_\varphi$ is achieved at p ; this is always possible since $K \neq \text{const}$. Now choose the conformal vector field $X = \nabla \ell_0$ and let $\ell_\varphi = \ell \circ \varphi$.

Then if $K_\varphi + \ell_\varphi \in \text{Im } \pi$, one must have

$$\int_{S^2} X(K_\varphi + \ell_\varphi) dv_g = \int_{S^2} [X(K_\varphi) + cX(\ell_0 \circ \varphi)] dv_g = 0,$$

for some volume form dv_g . Note that $X(\ell_0 \circ \varphi) = \langle D\varphi(\nabla \ell_0), X \rangle$ with respect to g_{+1} . Since φ is conformal, $X(\ell_0 \circ \varphi)_x = \chi(x)X(\ell_0)_{\varphi(x)}$ where $\chi = \sqrt{\det D\varphi} > 0$. One has $X(\ell_0) = \xi = \sin^2 r \geq 0$. This gives

$$\int_{S^2} X(K_\varphi + \ell_\varphi) dv_g = \int_{S^2} [X(K_\varphi)(x) + c\chi(x)\xi(\varphi(x))] dv_g = 0.$$

Using the change of variables formula, this may be rewritten as

$$(4.2) \quad \int_{S^2} X(K_\varphi + \ell_\varphi) dv_g = \int_{S^2} [X(K_\varphi)(x) + c\tilde{\chi}(x)\xi(x)] dv_g = 0.$$

where $\tilde{\chi} = \chi \circ \varphi^{-1} \det D(\varphi^{-1})$ is a positive functions on S^2 . One has $0 \leq \xi \leq 1$, with ξ vanishing only at the poles $\pm p$. If c is sufficiently large, then $X(K_\varphi) + c\tilde{\chi}\xi > 0$ away from the poles $\pm p$, since the term $X(K_\varphi)$ is fixed. On the other hand, at and sufficiently near the poles, $X(K_\varphi) \geq 0$ by construction and so again $X(K_\varphi) + c\tilde{\chi}\xi \geq 0$. This contradicts (4.2) (for any area form dv_g) if c is sufficiently large. ■

Next, even though π_0 is not necessarily proper near $\partial\mathcal{K}$, we show there are solutions in both parts of $\partial\mathcal{K}$.

Proposition 4.5. *The domain $\partial_1\mathcal{U} := \pi^{-1}(\partial\mathcal{N})$ is non-empty, i.e. there exist u with $K_u = \pi(u) \in \partial\mathcal{N} \subset \partial\mathcal{K}$.*

Proof: There may be many proofs of this result, but we work here with a particular and simple, natural construction of a large class of functions $K \in \text{Im } \pi \cap \partial\mathcal{N}$.

Namely, consider the behavior of K on the space of u which are eigenfunctions of the Laplacian, so that $\Delta u = -\lambda u$, $\lambda > 0$. These are the spherical harmonics, or equivalently, restrictions of harmonic polynomials in x, y, z on \mathbb{R}^3 to S^2 . From (1.1), for such u one has

$$(4.3) \quad K = e^{-2u}(1 + \lambda u).$$

Hence

$$(4.4) \quad \nabla K = e^{-2u}((\lambda - 2) - 2\lambda u)\nabla u,$$

so that the locus of critical points of K is the critical locus of u , together with the level set $u = (\lambda - 2)/2\lambda$. At the critical locus, one has

$$(4.5) \quad \Delta K = -e^{-2u}((\lambda - 2) - 2\lambda u)\lambda u.$$

For any harmonic homogeneous polynomial u of degree at least 2, it is easy to see that 0 is always a critical value of u . Hence, for K generated from arbitrary eigenfunctions of the Laplacian, one always has

$$K \in \partial\mathcal{N}.$$

Thus if E_λ denotes the λ -eigenspace of the Laplacian on $S^2(1)$, then for any λ ,

$$\pi(E_\lambda) \subset \partial\mathcal{N}.$$

Note that (4.3) and (4.5) imply that

$$\pi(E_\lambda) \subset \mathcal{C}.$$

■

Remark 4.6. If $\lambda > 2$, one may restrict u to the open ball in the eigenspace E_λ where $\max u < (\lambda - 2)/2\lambda$. For such u , the critical points of K are exactly the critical points of u .

It is worth discussing a simple, specific example in more detail. Thus let $u = \frac{1}{4}(x^2 - y^2)$. This is an eigenfunction with $\lambda = 6$; note that $\frac{1}{4} < \frac{1}{3} = \frac{\lambda-2}{2\lambda}$. The critical points of u , equal to the critical points of K , come in three pairs.

The first pair is $x = \pm 1, y = z = 0$, giving $u = \frac{1}{4}$ and $\Delta K < 0$. This gives a pair of maxima of u and K . The second pair is given by $y = \pm 1, x = z = 0$, giving $u = -\frac{1}{4}$ and a pair of minima of u and K with $\Delta K > 0$. The third pair $z = \pm 1, x = y = 0$ has $u = 0$ and gives a pair of saddle points of u and K with $\Delta K = 0$. A simple computation shows that D^2K is non-degenerate at $z = \pm 1$, while D^2K is degenerate at the other critical points.

Now consider perturbations of u inducing perturbations of K into \mathcal{N} . Let $v = \eta w$ where η is a cutoff function supported near $z = \pm 1$ with $\eta \equiv 1$ in a small neighborhood U of the poles, and w is another eigenfunction of Δ with eigenvalue $\lambda = 6$, say $w = z^2 - y^2$. The computation of K remains the same for $u + \varepsilon v$ in U and the poles $z = \pm 1$ are still critical points. By choosing $\varepsilon > 0$ or $\varepsilon < 0$, the poles resolve independently into non-degenerate saddles with $\Delta K \neq 0$; one may arrange $\Delta K > 0$ or $\Delta K < 0$ at a given pole by choosing $\varepsilon > 0$ or $\varepsilon < 0$ suitably. Outside of U , the εv perturbation does not change the structure of the other critical points of K .

This shows that for suitable perturbations of u , the curvature $K \in \partial\mathcal{N}$ may be perturbed into a degree 0 component of π_0 , (choosing one perturbation of the saddle to $\Delta K < 0$ and the other to $\Delta K > 0$), or into a degree 1 component, (choosing both saddle perturbations to $\Delta K > 0$), or into a degree -1 component, (choosing both saddle perturbations to $\Delta K < 0$).

Although K here is not a Morse function, consider instead the eigenfunctions $u = x^2 - (1 + \varepsilon)y^2 + \varepsilon z^2$. The same considerations apply as above and now $K = K(u)$ is a Morse function for $\varepsilon \in (0, 1)$, so $K \in \mathcal{M} \cap \partial\mathcal{N}$. In connection with Proposition 2.13, such K lie in the intersection of two regular local hypersurfaces of $\partial\mathcal{N}$ and the structure of \mathcal{N} near K is described by the wall-crossing above.

A similar structure holds for general eigenfunctions of the Laplacian, for any λ .

A stronger existence result holds in the region $\partial\mathcal{C} \subset \partial\mathcal{K}$. The following result corresponds to Theorem 1.5. Write

$$\partial\mathcal{C} = \cup_{n,m} \partial\mathcal{C}_m^n,$$

where $\partial\mathcal{C}_m^n = \partial\mathcal{C} \cap \bar{\mathcal{K}}^m \cap \bar{\mathcal{K}}^n$ with $m < n$ and $\mathcal{K}^m, \mathcal{K}^n$ are components of degree m, n respectively. Thus a curve $K(t)$ passing from $\mathcal{K}^m \cap \mathcal{N}$ to $\mathcal{K}^n \cap \mathcal{N}$ passes through $\partial\mathcal{C}_m^n$, increasing the degree by $n - m$.

Proposition 4.7. *The map π is onto $\partial\mathcal{C}_m^n \cap \mathcal{M}$ for any (m, n) with $m \neq 0, m < n$.*

Proof: As discussed in Remark 2.7, the map π is not proper at $\partial\mathcal{C}$, due to the formation of (unbounded) bubble solutions as a local maximum of K passes from small positive values to small negative values. This corresponds to a decrease in the degree of π^0 by one.

However, such bubbles do *not* form when passing in the reverse direction, from small negative to small positive values where the degree increases by one, at least within the space of Morse functions $K \in \mathcal{M}$. To prove this, suppose $K \in \partial\mathcal{C} \cap \mathcal{M}$ so that there exists x with $K(x) = 0$ and x is a local maximum of K . Thus there exists $\delta > 0$ such that $K(y) \leq K(x) = 0$ for all $y \in D_x(\delta)$ where $D = D_x(\delta)$ is the disc of radius δ in $(S^2(1), g_{+1})$. Without loss of generality, assume there exists $\varepsilon_0 > 0$ such that $K \leq -\varepsilon_0 < 0$ on ∂D . Suppose there is a sequence $K_i \in \mathcal{C} \cap \mathcal{N}$ with $K_i = \pi(u_i) \rightarrow K$ in $C_+^{m-2, \alpha}$ such that x is a local maximum of K_i with $K_i(x) = -\varepsilon_i \rightarrow 0$ and $K_i(y) \leq 0$ for $y \in D_x(\delta)$. We claim that $area_{g_i} D$ remains bounded as $i \rightarrow \infty$. Namely, since the curvature K_i is non-positive on the simply connected disc D , the geodesic circles $(S_i(r), g_i)$ of radius r about x with respect to g_i are convex. Letting $\ell_i(r)$ denote the g_i -length of $S_i(r)$, it follows that $\ell_i'' \geq 0$ where the derivative is with respect to geodesic g_i -distance r . If $area_{g_i} D \rightarrow \infty$, it follows that the distance function r_i becomes unbounded on D as $i \rightarrow \infty$ and hence also

$$(4.6) \quad \ell_i \rightarrow \infty \text{ on } D.$$

Now by (2.31), the area of g_i is uniformly bounded near the boundary ∂D and in particular the g_i -length of $S = \partial D$ is uniformly bounded. This contradicts (4.6), which proves the existence of a bound on $area_{g_i} D$. It then follows from Theorem 2.3 and the proof of Theorem 2.9 that the sequence g_i is bounded and, in a subsequence, $g_i \rightarrow g$, $u_i \rightarrow u$, in $C^{m, \alpha}$. This proves the claim above.

Now if $K_i \in \mathcal{K}^m$ with $m \neq 0$, then since π^m is onto, $K_i = \pi(u_i)$, for some sequence $u_i \in \mathcal{U}^m$. For $K = \lim K_i \in \partial\mathcal{C}$ as above, suppose that $K \in \partial\mathcal{C} \cap \bar{\mathcal{K}}^n$, for $n > m$, so that $K \in \partial\mathcal{C} \cap \bar{\mathcal{K}}^m \cap \bar{\mathcal{K}}^n$. This corresponds to an increase of the degree and the argument above proves that $K \in Im \pi$. ■

5. DOMAIN AND RANGE STRUCTURE

In this section, we discuss a number of aspects of the structure of the set of regular and singular points of π . The discussion is independent of the global results discussed in previous sections, until the analysis beginning with and following Proposition 5.7, which discuss global issues again.

Given a Fredholm map $F : X \rightarrow Y$ of index 0 between Banach manifolds, recall that x is a regular point of F if the linearization $D_x F$ is an isomorphism. The inverse function theorem implies that F is a local diffeomorphism in a neighborhood of a regular point. A point $y \in Y$ is a regular value if all points $x \in F^{-1}(y)$ are regular points. By definition, any $y \notin Im F$ is a regular value. A point $x \in X$ is a singular point if it is not a regular point.

Let

$$\Sigma \subset C^{m, \alpha}$$

be the closed set of singular points of π , and \mathcal{O} the open set of regular points of π , so that

$$C^{m, \alpha} = \mathcal{O} \cup \Sigma.$$

If u is a singular point of π , so that $Ker D_u \pi \neq 0$, then there exists a deformation u' of u which preserves the Gauss curvature to first order: $K' = \frac{d}{dt} K(u + tu')_{t=0} = 0$. Recall the basic defining equation (1.1), i.e.

$$e^{2u} K = 1 - \Delta u.$$

Differentiating in the direction $u_t = u + tu'$ and setting $v = u'$, one has $e^{2u}(K' + 2vK) = -\Delta v$. This gives:

Lemma 5.1. *A point $u \in C^{m, \alpha}$ is a critical point of π if and only if there exists a variation $u' = \omega$ such that*

$$(5.1) \quad \Delta \omega + 2Ke^{2u}\omega = 0.$$

■

It is worth noting that by (2.5), the sets of regular and singular points, as well as regular and singular values, are invariant under the action of $\text{Conf}(S^2)$.

Remark 5.2. Since $\Delta_{\mu^2 g} = \mu^{-2} \Delta_g$ in dimension two, one easily finds

$$(5.2) \quad D_u \pi(v) = -\Delta_g v - 2Kv,$$

for $g = e^{2u} g_{+1}$. In case $K > 0$, this can be rewritten as

$$(5.3) \quad D_u \pi = -K(\Delta_{\tilde{g}} v + 2v),$$

where $\tilde{g} = Kg$. Thus u is a critical point of π if and only if 2 is an eigenvalue of the conformal Laplacian $\Delta_{\tilde{g}}$. By a well-known theorem of Hersch [26], for any metric $\tilde{\gamma}$ on S^2 with $\text{area}(S^2, \tilde{\gamma}) = 4\pi$, one has

$$\lambda_1 \leq 2,$$

with equality if and only if $\tilde{\gamma}$ is of constant curvature 1. Thus the conformal metrics \tilde{g} in (5.3) have 2 as a 2nd (or higher) eigenvalue of the Laplacian except when g has constant curvature.

Proposition 5.3. *The function K is a regular value of π if and only if all critical points of the functional J_K in (1.2) are non-degenerate.*

Proof: If u is a critical point of J_K giving a solution of (1.1), then (3.2) implies that $D^2 J_K(v) = -\Delta v - 2Ke^{2u}v = e^{2u}(-e^{-2u}\Delta v - 2Kv) = e^{2u}K'(v)$. Thus, $v \in \text{Ker} D^2 J_K$ if and only if $DK(v) = 0$, which proves the result.

■

At a critical point u of π , let N denote the solution space of (5.1) and let $d = d_u = \dim N$. The space $N = N_u$ equals the null-space of $D^2 J_K$ at u , and d is the nullity of the critical point u . By (5.2), $\text{Im} D_u \pi$ consists of functions of the form $\Delta_g v + 2Kv$ where g and K are fixed and v varies over $C^{m,\alpha}$. In particular

$$(5.4) \quad N \perp \text{Im} D_u \pi,$$

with respect to the $L^2(g)$ inner product, so that N spans the g -normal space to $\text{Im} D_u \pi$.

Let $(C_+^{m-2,\alpha})' \subset C_+^{m-2,\alpha}$ denote the subspace of functions χ such that

$$(5.5) \quad \int_{S^2} \chi dv_{+1} = 4\pi.$$

Of course for $K = \pi(u)$, $\chi := Ke^{2u} \in (C_+^{m-2,\alpha})'$ by the Gauss-Bonnet theorem. Similarly let $(C^{m,\alpha})' = \{u \in C^{m,\alpha} : \int u dv_{+1} = 0\}$.

Lemma 5.4. *The map*

$$(5.6) \quad \begin{aligned} C : (C^{m,\alpha})' &\rightarrow (C_+^{m-2,\alpha})', \\ C(u) = \mu &:= Ke^{2u} = \pi(u)e^{2u}, \end{aligned}$$

is a smooth diffeomorphism.

Proof: We first observe that C is surjective, i.e. $\mu = Ke^{2u}$ may be arbitrarily specified. One needs to show that given any μ with $\int \mu dv_{+1} = 4\pi$, there is u such that

$$\Delta u - 1 = -\mu.$$

This is an elementary consequence of the Fredholm alternative.

Next we claim that $\text{Ker} DC = 0$ everywhere. To see this, suppose $\mu' = 0$. Then $\Delta u' = 0$, so that $u' = \text{const}$. Since $u \in (C^{m,\alpha})'$, $\int u' dv_{+1} = 0$, and hence $u' = 0$. It is easy to see that C is of index 0, and hence it follows that C is a local diffeomorphism onto $(C_+^{m-2,\alpha})'$. Finally, to see that C is

one-to-one, suppose $\mu_1 = \mu_2$. Then $\Delta u_1 = \Delta u_2$, which implies $u_1 = u_2 + \text{const}$, so again $u_1 = u_2$. This proves the result. ■

We next show that the singular set Σ of π is a stratified space; this is essentially Theorem 1.4. For the purposes of this paper, a subset X of $C^{m,\alpha}$ will be called a stratified space if $X = \cup S_i$, where each S_i is a smooth manifold of codimension i in $C^{m,\alpha}$ such that the point-set theoretic closure satisfies

$$\bar{S}_k \setminus S_k \subset \bigcup_{i>k} S_i.$$

Theorem 5.5. *The singular set $\Sigma \subset C^{m,\alpha}$ of π is a closed stratified space, with strata Σ_d consisting of submanifolds of codimension $d = \dim N \geq 1$.*

In particular, the regular points of π are open and dense in $C^{m,\alpha}$.

Proof: Consider the map

$$(5.7) \quad L : (C_+^{m-2,\alpha})' \times (C^{m,\alpha} \setminus \{0\}) \rightarrow C^{m-2,\alpha},$$

$$L(\mu, v) = \Delta v + 2\mu v.$$

Note that by (5.2), $L \circ C = -e^{-2u} D\pi$, so at a critical point of π ,

$$D^2\pi = -e^{2u} D(L \circ C).$$

We claim that 0 is a regular value of L , so that the inverse image $S = L^{-1}(0)$ is a smooth properly embedded submanifold of $(C_+^{m-2,\alpha})' \times (C^{m,\alpha} \setminus \{0\})$. To do this, one needs to show the linearization is surjective and the kernel splits.

The map L is linear in the second component v , and D_2L maps onto $Im(\Delta + 2\mu)$ equal to the orthogonal complement to the kernel $Ker(\Delta + 2\mu)$. On the other hand, one has

$$(5.8) \quad D_1L(v) = 2\mu'v,$$

where D_1 is the derivative in the first or μ -direction. We claim that the projection of $\mu'v$ onto the kernel $Ker(\Delta + 2\mu)$ is surjective as μ' ranges over $T(C_+^{m-2,\alpha})'$. The component of $\mu'v$ in the kernel is given by

$$(5.9) \quad \int_{S^2} \mu'v\omega dv_{g+1},$$

for some unit vector $\omega \in Ker(\Delta + 2\mu)$, so the claim is that for any v , there exists μ' such that (5.9) is non-zero. The only condition on μ' is the linearization of (5.5) which gives

$$(5.10) \quad \int_{S^2} \mu' dv_{g+1} = 0.$$

There exists μ' satisfying (5.10) with (5.9) non-zero provided ωv is not identically one and v is not identically zero. The latter holds by assumption (see (5.7)), and the former holds since for instance $\omega = 0$ somewhere. This shows that DL is surjective.

The kernel of DL at (μ, v) is given by pairs (μ', w) such that

$$(5.11) \quad \Delta w + 2\mu w = -2\mu'v.$$

This is solvable for w if and only if $\mu'v \perp Ker(\Delta + 2\mu)$ and the solution is unique modulo $Ker(\Delta + 2\mu)$. Thus $Ker DL$ is a closed subspace, naturally isomorphic to $(Ker(\Delta + 2\mu))^\perp \oplus Ker(\Delta + 2\mu)$. It has a closed complement, given by $\mu'v \in Ker(\Delta + 2\mu)$ with $w \in Ker(\Delta + 2\mu)^\perp$.

This shows that the set $S = \{(\mu, \omega)\} \subset \text{domain } L$ such that $\Delta\omega + 2\mu\omega = 0$, is a smooth properly embedded submanifold of the domain of L . The tangent space TS at (μ, w) is given by $Ker DL$, i.e. pairs (μ', w) satisfying (5.11). Of course S may have a number of distinct components. The

reason for excluding $\{0\}$ in the domain of L in (5.7) is to remove a trivial region $(\mu, 0)$ of S , diffeomorphic to $(C_+^{m-2,\alpha})'$.

The structure of the singular set Σ is now obtained by studying the projection of S into the base space $(C^{m,\alpha})'$. Observe that the vertical fiber of S where μ is fixed and ω varies is the linear space N of dimension $d(\mu)$, so that S is ruled by linear spaces, equal to the fibers $S \cap pr_1^{-1}(pt)$ of the projection pr_1 onto the first factor in (5.7). The function

$$(5.12) \quad d : S \rightarrow \mathbb{Z}^+, \quad d = \dim N(\mu)$$

is upper semi-continuous; it may increase in limits, but not decrease. Let $\Sigma = pr_1(S)$.

Let $S_d = \{\mu \in S : d(\mu) = d\}$. One should be aware in the following that any given S_d may be empty. For instance $S_0 = \emptyset$, since $(\mu, v) \in S_0$ implies $v = 0$, which is not in the domain of L . We proceed inductively on d and work on a fixed component of S . First, the upper semi-continuity of d implies that S_1 is open in S . Since S_1 is ruled by lines, the projection

$$pr_1 : S_1 \rightarrow \Sigma_1 \subset (C_+^{m-2,\alpha})'$$

with $\Sigma_1 = pr_1(S_1)$ is a vector bundle of rank 1 (a line bundle with the zero-section removed) over the manifold Σ_1 . By (5.11), the space Σ_1 is of codimension 1 in $(C_+^{m-2,\alpha})'$. The upper semi-continuity of d implies that $\partial S_1 = \bar{S}_1 \setminus S_1 \subset \cup_{k \geq 2} S_k$.

Next, in the closed complement $S \setminus S_1$, S_2 is an open (possibly empty) subset in the induced topology, again by the upper semi-continuity of d . The projection

$$pr_1 : S_2 \rightarrow \Sigma_2 \subset (C_+^{m-2,\alpha})',$$

with $\Sigma_2 = pr_1(S_2)$ is a vector bundle of rank 2 over the manifold Σ_2 . Again by (5.11), the manifold Σ_2 is of codimension 2 in $(C_+^{m-2,\alpha})'$.

Continuing inductively in the same way gives manifolds Σ_d of codimension d in $(C_+^{m-2,\alpha})'$ such that the point-set theoretic boundary of each Σ_d is contained in $\cup_{d' > d} \Sigma_{d'}$. This gives Σ the structure of a stratified space.

Via the diffeomorphism C from Lemma 5.4, we view $\Sigma = pr_1(S)$ as a subset of \mathcal{U}' . By scaling (i.e. the addition of constants to u), Σ becomes a stratified subset of \mathcal{U} with strata Σ_d of codimension d in $C^{m,\alpha}$. ■

Next we turn to discussion of the singular value set $\mathcal{Z} \subset C_+^{m-2,\alpha}$. By definition,

$$\mathcal{Z} = \pi(\Sigma).$$

The singular set Σ in the domain $C^{m,\alpha}$ is a stratified space, with strata Σ_d given by submanifolds of codimension d in $C^{m,\alpha}$ with (point-set theoretic) boundary satisfying $\partial \Sigma_d \subset \Sigma_{d+1}$. The map π maps Σ onto the set of singular values \mathcal{Z} in $C_+^{m-2,\alpha}$ so that \mathcal{Z} is a singular ‘‘variety’’ of codimension at least one in $C_+^{m-2,\alpha}$. More precisely, \mathcal{Z} is a rectifiable set in $C_+^{m-2,\alpha}$ of codimension at least one, (cf. the definition following Proposition 2.13).

Let

$$\mathcal{Z}_d = \pi(\Sigma_d),$$

so that, if non-empty, \mathcal{Z}_d is rectifiable of codimension d in $C_+^{m-2,\alpha}$. It is not known if \mathcal{Z} is stratified as is Σ . The following result gives a sufficient criterion for the strata \mathcal{Z}_d to be locally given by submanifolds of codimension d in $C_+^{m-2,\alpha}$.

Lemma 5.6. *Suppose $u \in \Sigma_d$ with $K = \pi(u) \in \mathcal{Z}_d$ and let N be the d -dimensional kernel of $\Delta + 2Ke^{2u}$. If for some $\tilde{\omega} \in N$, the symmetric bilinear form $S_{\tilde{\omega}} : N \times N \rightarrow \mathbb{R}$,*

$$(5.13) \quad S_{\tilde{\omega}}(\omega_1, \omega_2) = \int_{S^2} K \omega_1 \omega_2 \tilde{\omega} dv_g,$$

is non-degenerate, then \mathcal{Z}_d is a submanifold of codimension d near K .

Proof: For $u \in \Sigma_d$, consider the straight line $u_t = u + tu'$, where $u' = \omega \in N$, so ω is normal to $ImD_u\pi$ in $L^2(g)$. Then $\sigma_t = \pi(u_t) = K(u_t)$ is a smooth curve in $C_+^{m-2,\alpha}$, with vanishing tangent vector at $t = 0$, $\sigma'(0) = D\pi(u') = 0$. Calculating the second derivative from (5.2) gives, since $u'' = 0$,

$$(5.14) \quad D^2\pi(\omega, \omega) = 2e^{-2u}\Delta\omega = -4K\omega^2.$$

Thus taking the $L^2(g)$ pairing with the normal vector $\tilde{\omega}$ as in (5.4) gives

$$\langle D^2\pi(\omega, \omega), \tilde{\omega} \rangle = -4 \int_{S^2} K\omega^2 \tilde{\omega} dv_g.$$

The result is then a straightforward consequence of the implicit function theorem, cf. Theorem 3.7.2 of [39] for instance. ■

Next we consider relations between the singular set \mathcal{Z} and the existence of solutions of (1.1). As discussed in Sections 2 and 4, the boundary $\partial\mathcal{K}$ is a closed rectifiable set of codimension 1, separating $C_+^{m-2,\alpha}$ into an infinite collection of domains partially distinguished by the degree of π_0 . There are only three possible regions where (1.1) may not have solutions. The first region is the degree zero component \mathcal{K}^0 , cf. Remark 4.2. By Proposition 4.7, (i.e. Theorem 1.5), a second region is a portion of the boundary region $\partial\mathcal{C} \subset \partial\mathcal{K}$, while the third region is the boundary $\partial\mathcal{N} \subset \partial\mathcal{K}$. We explore existence and non-existence in these regions in more detail.

We begin first with \mathcal{K}^0 . For $\pi^0 : \mathcal{U}^0 \rightarrow \mathcal{K}^0$ as in (2.78), let

$$\mathcal{E} = Im\pi^0 \subset \mathcal{K}^0.$$

Proposition 5.7. *The image \mathcal{E} of π^0 is a non-empty, closed subset of \mathcal{K}^0 .*

Proof: It follows from Proposition 4.3 that $\mathcal{E} \neq \emptyset$. Since π^0 is proper, it has closed range, which proves the second statement. ■

Let \mathcal{E}_{int} denote the interior of \mathcal{E} , as a set in $C_+^{m-2,\alpha}$; this is the union of the set of regular values of π^0 with the singular values (if any) contained in the interior, i.e. $(\mathcal{E}_{int} \cap \mathcal{Z})$. Obviously \mathcal{E}_{int} is open in \mathcal{E} . More significantly, it follows from Theorem 5.5 and the fact that π^0 is smooth that \mathcal{E}_{int} is dense in \mathcal{E} ,

$$(5.15) \quad \bar{\mathcal{E}}_{int} = \mathcal{E}.$$

Let $\mathcal{B} = \partial\mathcal{E}_{int}$ be the boundary of \mathcal{E}_{int} , so that $\mathcal{E} = \mathcal{E}_{int} \cup \mathcal{B}$. One thus has a decomposition of the target

$$(5.16) \quad \mathcal{K}^0 = \mathcal{E}_{int} \cup \mathcal{B} \cup \mathcal{D},$$

where $\mathcal{D} = \mathcal{K}^0 \setminus \mathcal{E}$ is the non-existence region where there are no solutions to the Nirenberg problem. The linear functions $K = \ell$ are in \mathcal{K}^0 but are not in $Im\pi^0$ by the Kazdan-Warner obstruction (1.4) so that

$$\mathcal{D} \neq \emptyset.$$

Both \mathcal{E}_{int} and \mathcal{D} are non-empty open sets, so that the decomposition (5.16) is non-trivial. Of course the boundary satisfies

$$\mathcal{B} \subset \mathcal{Z}.$$

Proposition 5.8. *The wall \mathcal{B} separating \mathcal{E}_{int} from \mathcal{D} satisfies*

$$(5.17) \quad \mathcal{B} \subset \bar{\mathcal{Z}}_1.$$

In particular, $\mathcal{Z}_1 \neq \emptyset$. An open-dense subset \mathcal{B}' of $\mathcal{B}_1 = \mathcal{B} \cap \mathcal{Z}_1$ is a smooth hypersurface in $\mathcal{K}^0 \subset C_+^{m-2,\alpha}$.

Proof: The singular strata Σ_k are of codimension k in the domain \mathcal{U}^0 . Hence their images \mathcal{Z}_k under the smooth map π are of codimension at least k in $C_+^{m-2,\alpha}$. Closed subsets of codimension at least two do not locally separate the space $C_+^{m-2,\alpha}$, and since \mathcal{B} does locally separate, one must have $\mathcal{Z}_1 \neq \emptyset$ and (5.17) follows.

We use Lemma 5.6 to determine which regions in \mathcal{B}_1 are codimension 1 submanifolds. First, define a point $K \in \mathcal{B}_1$ to be *path-accessible* if \mathcal{E}_{int} is locally path connected near K . These are points K such that for any open neighborhood V of K in the target $C_+^{m-2,\alpha}$, there is a neighborhood $U \subset V$ of K such that $U \cap \mathcal{E}_{int}$ is path connected. This fails for instance for points K at which two local regions of \mathcal{B}_1 intersect at K , separating \mathcal{E}_{int} into two local components near K . Let \mathcal{B}' denote the set of path-accessible points in \mathcal{B}_1 .

By construction, \mathcal{B}' is open in \mathcal{B}_1 . We claim that \mathcal{B}' is also dense in \mathcal{B}_1 . Namely for any point $K \in \mathcal{B}_1$ and for any neighborhood U of K in $C_+^{m-2,\alpha}$, one has $U \cap \mathcal{D} \neq \emptyset$. It follows from this that $U \cap \mathcal{B}' \neq \emptyset$, so that \mathcal{B}' is dense.

Next we prove that the path-accessible points \mathcal{B}' form (locally) a smooth hypersurface in $C_+^{m-1,\alpha}$. To do this, it suffices to show that the form S in (5.13) is non-degenerate on \mathcal{B}' .

Suppose the form $S = D^2\pi$ is degenerate at u with $\pi(u) = K \in \mathcal{B}'$. Then there exists $u' = \omega \in N$ and $\tilde{\omega} \in N$ such that $\langle D^2\pi(\omega, \omega), \tilde{\omega} \rangle = 0$. We then calculate the third derivative $D^3\pi(\omega, \omega, \omega) = \frac{d^3(K_{u_t})}{dt^3}$. Taking the derivative of (5.14) along u_t , using the fact that $K'(\omega) = 0$ and $u'' = 0$ gives

$$D^3\pi(\omega, \omega, \omega) = -4e^{-2u}\omega^2\Delta\omega = 8K\omega^3,$$

so that

$$(5.18) \quad \langle D^3\pi(\omega, \omega, \omega), \tilde{\omega} \rangle = 8 \int_{S^2} K\omega^3\tilde{\omega}dv_g.$$

When $\dim N = 1$, $\tilde{\omega}$ is a multiple of ω . Using (5.2) (with $v = \tilde{\omega} = \omega$) gives

$$\langle D^3\pi(\omega, \omega, \omega), \omega \rangle = -4 \int_{S^2} \omega^3\Delta_g\omega dv_g = 12 \int_{S^2} \omega^2|d\omega|^2 dv_g > 0.$$

Now as in the proof of Lemma 5.6, suppose $\sigma(0) = \pi(u_0) = K \in \mathcal{B}_1$, so that $u \in \Sigma_1$. The normal space to $Im D\pi$ at K is 1-dimensional and spanned by ω . The term (5.18) is then strictly positive and hence the third derivative of $\sigma(t)$ in the direction of the normal vector ω is non-vanishing. It follows that the function $\langle \sigma(t), \omega \rangle$ behaves as a cubic, and so changes signs. Moreover, by construction $\sigma(t) \in \mathcal{E}$ and since $\sigma'(t) \neq 0$ for $t \neq 0$, it follows that $\sigma(t) \in \mathcal{E}_{int}$ for $t \neq 0$ with t small. Now if K is path-accessible, then $\sigma(t)$ is in a single local path component of \mathcal{E}_{int} , which contradicts the fact that $\sigma(t)$ behaves as a cubic. This contradiction shows that the form (5.13) is non-degenerate on \mathcal{B}' , which proves the result. ■

Remark 5.9. The open, dense set $\mathcal{B}' \subset \mathcal{B}$ where \mathcal{B} is a submanifold of codimension 1 in $C_+^{m-2,\alpha}$ is a classical *bifurcation locus* for the Nirenberg problem, cf. [39] for instance. Namely for $K \in \mathcal{B}'$ with (outward) normal vector ω , consider the line $K(t) = K + t\omega$ in $C_+^{m-2,\alpha}$. For $t > 0$ sufficiently small, $K(t) \notin Im \pi$, i.e. there are no solutions of (1.1) for such $K(t)$. For $t < 0$ sufficiently small, there are locally exactly two distinct solutions of (1.1), i.e. the map π is locally $2 - 1$ onto $K(t)$,

$t < 0$. These two solutions merge at $t = 0$ to give a local unique solution to the Nirenberg problem. Thus the map π is locally a $2 - 1$ fold map onto the curve $K(t)$, $t \leq 0$.

A large open set in the non-existence region \mathcal{D} is given by the curvature functions from Proposition 4.4. A further specific example of non-existence of solutions in $\mathcal{D} \subset \mathcal{K}^0$ near the boundary with the region \mathcal{K}^1 (a degree one component) is constructed and analysed in detail by Struwe in [45].

Remark 5.10. As an application of the results above, we discuss the behavior of the map π^0 near the special point $K = 1$ (or $K = \text{const}$). Consider the region \mathcal{P} of positive curvature functions in $C_+^{m-2,\alpha}$, as in (2.83). Then $\mathcal{P} \subset \mathcal{C}$ and as noted in [12], the domains $\mathcal{K}^i \cap \mathcal{P}$ are star-shaped about $K = 1$, i.e. if $K \in \mathcal{K}^i \cap \mathcal{P}$, then the line segment $t + (1 - t)K$, $t \in [0, 1)$ lies in $\mathcal{K}^i \cap \mathcal{P}$. Thus the structure of $\partial\mathcal{K}$ becomes very complicated near $K = 1$.

In the component \mathcal{K}^0 where

$$\text{deg}\pi^0 = 0,$$

any regular value $K \in \mathcal{E}_{\text{int}}$ near $K = 1$ has an even number of solutions of (1.1). On the other hand, it is easy to see that the point $u = 0$ with $\pi(u) = K_u = 1$ is a regular point for the restricted map $\pi_1 = \pi|_{\mathcal{S}}$ for \mathcal{S} as in (2.6), i.e. $D_0\pi_1$ is injective. Points $u \in \mathcal{S}$ near $u = 0$ are then also regular and so in a neighborhood V of 0 in \mathcal{S} , the map

$$(5.19) \quad \pi_1 : V \rightarrow H,$$

is a diffeomorphism onto its image H ; H is a local smooth hypersurface of codimension 3 in $C_+^{m-2,\alpha}$.

Consider then a regular value $K \in H \cap \mathcal{K}^0$. The value K has a unique inverse image (i.e. solution of (1.1)) $u_1 \in V \subset \mathcal{S}$ and so there must be another solution $u_2 \notin V$ of (1.1). (In fact with further work it can be proved that $u_2 \notin \mathcal{S}$). Moreover, the solutions u_1 and u_2 have J_K -indices with opposite (even-odd) parity. By (2.7), there exists a conformal dilation φ such that

$$\varphi^*u_2 \in \mathcal{S},$$

and so

$$\pi(\varphi^*u_2) = K \circ \varphi.$$

Suppose that $\tilde{u}_2 = \varphi^*u_2 \in V$, so that $K \circ \varphi \in H$. Then u_1 and \tilde{u}_2 are two regular points of π^0 in $V \subset \mathcal{S}$ with index of opposite parity. However, this is not possible. Namely, one may take a curve $K(t)$ from K to $K \circ \varphi$ in H with $K(t)$ a regular value of π for all t . The curve $K(t)$ has a unique lift to a curve $u(t) \subset V$. As discussed in the proof of Theorem 1.3 in Section 3, the index of $u(t)$ is then independent of t , giving a contradiction. It follows that

$$(5.20) \quad \tilde{u}_2 \notin V.$$

This argument proves that if $K \in \mathcal{S}$ is sufficiently close to 1, i.e. $|K - 1|_{C^{m-2,\alpha}} \leq \varepsilon$ for ε sufficiently small, then

$$|\varphi|_{C^1} \geq C = C(\varepsilon),$$

where $C(\varepsilon) \rightarrow \infty$ as $\varepsilon \rightarrow 0$. For if C were uniformly bounded as $\varepsilon \rightarrow 0$, then the solution \tilde{u}_2 above would necessarily be in V , contradicting (5.20).

Thus, pairs of solutions to (1.1) with $K \in \mathcal{K}^0$ sufficiently close to 1 are (very) far apart in $C^{m,\alpha}$, differing from each other by (very) large conformal factors. This illustrates concretely the effect of the non-compactness of the conformal group.

Proposition 5.8 describes the general structure of the wall separating the regions of existence and non-existence in \mathcal{K}^0 . We next consider briefly the issue of existence and non-existence near the second region $\partial\mathcal{K}$. Recall that

$$\partial\mathcal{K} = \partial\mathcal{C} \cup \partial\mathcal{N}.$$

By Proposition 4.7, for any (m, n) with $m < n$ and $m \neq 0$, one has

$$\partial\mathcal{C}_m^n \subset \text{Im } \pi,$$

so that for any $K \in \partial\mathcal{C}_m^n$ there is a solution u of (1.1) with $K_u = K$. From the proof of Proposition 4.7, for any such (m, n) , say $n = m + 1$, there are (large families of) solution curves $u(t)$, $t \in [0, 1]$ with $K_{u(t)} \in \mathcal{K}^m$ for $t < \frac{1}{2}$, $u(\frac{1}{2}) \in \partial\mathcal{C}_m^n$ and $K_{u(t)} \in \mathcal{K}^n$ for $t > \frac{1}{2}$. Thus, curves of solutions do not necessarily “blow-up” on approach to $\partial\mathcal{C}$ and π is “proper” along such curves.

We expect similar behavior holds at $\partial\mathcal{N}$ but this remains unknown in general. The eigenfunction examples discussed in Proposition 4.5 and Remark 4.6 show that

$$Im \pi \cap \partial\mathcal{N} \neq \emptyset.$$

Moreover, Remark 5.10 holds on any component \mathcal{K}^n of degree n and not just on \mathcal{K}^0 ; for any such \mathcal{K}^n , there are solutions u with $u \in \mathcal{K}^n \cap H$, for H as in (5.19). This follows for instance from the work of Chang-Gursky-Yang [12], where $deg \pi^n$ is shown to be the intersection number of the orbit $\{K \circ \varphi\}$, $\varphi \in \text{Conf}(S^2)$ with H , for K sufficiently close to 1. As above with $\partial\mathcal{C}$, there are (large families of) solution curves $u(t) \subset V \subset \mathcal{S}$ with $K_{u(t)}$ passing smoothly from one component of $\mathcal{K}^m \cap \mathcal{P}$ to another component $\mathcal{K}^{m+1} \cap \mathcal{P}$ within H .

Remark 5.11. Since the degree of π_0 changes on passing from one component of $\partial\mathcal{K}$ to a neighboring or bordering component, π_0 cannot extend to a proper map on any open set containing a portion of $\partial\mathcal{K}$. In particular, the *a priori* estimates (2.76) do not extend to such larger domains. On curves $K(t)$ passing through $\partial\mathcal{C} \subset \partial\mathcal{K}$, there must exist curves of solutions $u(t)$ which blow-up as $K_{u(t)}$ approaches $\partial\mathcal{C}$. The analysis given in the proof of Proposition 4.7 shows that in a passage from \mathcal{K}^{m+1} to \mathcal{K}^m , $m \neq 0$, it is likely that only one solution curve blows up, while the remaining m solution curves pass continuously through $\partial\mathcal{C}_m^{m+1}$, i.e. a single blow-up curve is the reason for the drop in degree from $m + 1$ to m . Similarly, in passing from \mathcal{K}^1 to \mathcal{K}^0 through $\partial\mathcal{C}$ one expects generically there is only one solution curve, and such a solution curve blows up on approach to $\partial\mathcal{C}_0^1$.

One could expect similar behavior on passing through the $\partial\mathcal{N}$ region of $\partial\mathcal{K}$.

6. SYMMETRY BREAKING AND EXISTENCE

In this section, we turn to the question of existence of solutions of (1.1) with extra symmetry. A key point of view is to find conditions which break the symmetry of the (non-proper part of the) action of the conformal group $\text{Conf}(S^2)$ on the domain and target spaces of the map π .

To begin, we give a new proof of Moser’s theorem [37], that any even function K on S^2 ($K(-x) = K(x)$) with $K > 0$ somewhere is the Gauss curvature of a conformal metric on S^2 . The proof below does not use any sharp Sobolev inequality of Moser-Trudinger-Aubin type.

Let A be the antipodal map $x \rightarrow -x$ of S^2 . Let $C_{ev}^{m,\alpha} \subset C^{m,\alpha}$ be the subspace of even functions, so $u(-x) = u(x)$ and similarly consider $C_{ev}^{m-2,\alpha} \subset C_+^{m-2,\alpha}$. The restriction of π in (1.3) to $C_{ev}^{m,\alpha}$ gives a smooth Fredholm map

$$(6.1) \quad \pi_{ev} : C_{ev}^{m,\alpha} \rightarrow C_{ev}^{m-2,\alpha}.$$

Let $\mathcal{C}_{ev} = \mathcal{C} \cap C_{ev}^{m-2,\alpha}$.

Theorem 6.1. *Any even function $K \in C_+^{m-2,\alpha}$ is the Gauss curvature of a metric $\gamma = e^{2u}\gamma_{+1}$, with u an even function on S^2 . Moreover, the restriction of π_{ev} in (6.1) to $\mathcal{U}_{ev} = \pi_{ev}^{-1}(\mathcal{C}_{ev})$ is proper and the degree (up to sign) of the map*

$$(6.2) \quad \pi_{ev} : \mathcal{U}_{ev} \rightarrow \mathcal{C}_{ev},$$

satisfies

$$(6.3) \quad deg \pi_{ev} = 1.$$

Proof: As before, via (1.1), it is easy to see that π_{ev} in (6.1) is Fredholm of index 0. Recall from the work in Section 2 that the non-properness of the map π has two sources. First, the non-compactness of the conformal group $\text{Conf}(S^2)$, which gives rise to the division $\partial\mathcal{N}$ into the regions \mathcal{N} and second the ‘bubble formation’ at $\partial\mathcal{C}$, giving rise to the region \mathcal{C} . We will deal with these in turn.

To begin, the conformal group of \mathbb{RP}^2 is *compact*, equal to its isometry group. Propositions 2.1, 2.8 and Theorem 2.3 apply equally well to \mathbb{RP}^2 in place of S^2 and the proof of Theorem 2.9 for the map π_{ev} in (6.2) becomes much simpler. The compactness of the conformal group implies that the diffeomorphisms ψ_i of \mathbb{RP}^2 in (??), (which lift to even diffeomorphisms of S^2) themselves converge (in a subsequence) to a limit ψ , so that the renormalized diffeomorphisms ψ_i in (2.54) converge to the identity in $C^{m+1,\alpha}$. This shows that π_{ev} in (6.2) is indeed proper. In effect, $\partial\mathcal{N} = \emptyset$ and $\mathcal{N} = C_{ev}^{m-2,\alpha}$ in this situation.

Proposition 2.14 shows that \mathcal{C} is connected and the same proof shows that \mathcal{C}_{ev} is connected. Hence $\deg \pi_{ev}$ is well-defined on \mathcal{U}_{ev} .

To compute the degree of π_{ev} , consider the value $K = 1$. The solutions of (1.1) with $K = 1$ are given by $u = \log \chi$, for χ as in (2.3). The only such u which is even, and so descends to u on \mathbb{RP}^2 is $u = 0$. Thus, $\pi_{ev}^{-1}(1)$ is the function $u = 0$. Moreover, the kernel $\text{Ker} D\pi$ of π at $u = 0$ is given by the linear functions ℓ , of which the only even function is $\ell = 0$. Thus $\text{Ker} D\pi_{ev} = 0$. This shows that $K = 1$ is a regular value of π_{ev} , uniquely realized by the regular point $u = 0$. This proves (6.3).

It follows that π_{ev} is onto \mathcal{C}_{ev} . Theorem 1.5, (or more precisely its proof in Proposition 4.7), shows that π_{ev} is also onto $\partial\mathcal{C}_{ev}$. (There are no components of $\mathcal{K} = \mathcal{N} \cap \mathcal{C} = \mathcal{D}$ of degree zero. This completes the proof. ■

Remark 6.2. It is not asserted (and is not true) that π_{ev} in (6.1) is proper. The formation of bubbles as discussed in Remark 2.7 holds on \mathbb{RP}^2 just as well as on S^2 . It follows in particular that the number of even solutions of (1.1) as $K \in \mathcal{C}_{ev}$ approaches a given function in $\partial\mathcal{C}$ from the degree decreasing side (so a local maximum of K transitions from positive to negative values) is often different than the number of even solutions as K approaches a given function from the degree increasing side of $\partial\mathcal{C}$; compare with the proof of Proposition 4.7.

It is not difficult to see that the result above generalizes, with essentially the same proof, to the action of any compact group $\Gamma \subset O(3) = \text{Isom}(S^2)$ of isometries of S^2 which breaks the non-compact action of $\text{Conf}(S^2)$ on the target space $C_+^{m-2,\alpha}$. Thus, we assume $\Gamma \subset O(3)$ satisfies the following: for any conformal dilation $\varphi = \varphi_{p,t}$, $\varphi \neq \text{Id}$ in $\text{Conf}(S^2)$, there exists $\gamma \in \Gamma$ such that

$$(6.4) \quad \gamma \circ \varphi \neq \varphi \circ \gamma,$$

as elements in $\text{Conf}(S^2)$. For example the subgroup $\mathbb{Z}_2 \subset O(3)$ generated by the antipodal map satisfies (6.4).

Let $C_\Gamma^{m-2,\alpha}$ be the space of Γ -invariant functions in $C_+^{m-2,\alpha}$, i.e. $K(\gamma(x)) = K(x)$, for all $x \in S^2$, $\gamma \in \Gamma$ and similarly for $C_\Gamma^{m,\alpha}$. As before, π restricts to a smooth Fredholm map of index 0,

$$\pi_\Gamma : C_\Gamma^{m,\alpha} \rightarrow C_\Gamma^{m-2,\alpha}.$$

As in (6.2), consider also

$$(6.5) \quad \pi_\Gamma : \mathcal{U}_\Gamma \rightarrow \mathcal{C}_\Gamma,$$

We then have:

Theorem 6.3. *If a finite group action $\Gamma \subset O(3)$ as above breaks the non-compact action of the conformal group, then any Γ -invariant function $K \in C_+^{m-2,\alpha}$ is the Gauss curvature of a conformal metric $\gamma = e^{2u}\gamma_{+1}$ with u also Γ -invariant. Further, the map π_Γ in (6.5) is proper and (up to sign)*

$$(6.6) \quad \deg \pi_\Gamma = 1.$$

Proof: The proof is identical to the proof of Theorem 6.1, with one distinction. Let $S = S^2/\Gamma$ be the orbit space of the Γ -action on S^2 . Then S is an orbifold, in fact a “good” orbifold in the sense of Thurston. A classification of orbifolds in the special case when $\Gamma \subset SO(3)$ is given for instance in [49]. As in the proof of Theorem 6.1, the metrics g_i descend to orbifold metrics \tilde{g}_i on S , and one needs an orbifold or Γ -equivariant version of Propositions 2.1, 2.8 and Theorem 2.3 to complete the proof as before.

However, since the Γ -action is fixed for all metrics g_i , this is essentially standard. Namely, recall that the proof of the Gromov convergence theorem (Proposition 2.1) is fundamentally based on the construction of suitable harmonic coordinate charts of uniform size in which the metric is well-controlled, cf. [22] or [41]. Since Γ acts by isometries on (S^2, g_i) , for each i one may choose a suitable Γ -equivariant finite atlas \mathcal{A} of harmonic coordinates on (S^2, g_i) , i.e. $\{x_i^k\} : U \subset S^2 \rightarrow B \subset \mathbb{R}^2$ is a g_i -harmonic coordinate chart in \mathcal{A} if and only if $\{x_i^k \circ \gamma\}$ is in \mathcal{A} , for each $\gamma \in \Gamma$. The collection of transition maps between charts in \mathcal{A} are thus also Γ -equivariant. The geometric center of mass construction used in [22] or [41] to pass from local to global maps is also Γ -equivariant. Given this, it is readily verified by inspection that the proof in [22] or [41] implies the Γ -equivariant convergence of the metrics modulo diffeomorphism, i.e. the diffeomorphisms ψ_i as in (??) satisfy

$$\gamma \circ \psi_i = \psi_i \circ \gamma,$$

for all $\gamma \in \Gamma$ and so are orbifold diffeomorphisms. The condition (6.4) ensures that the group of Γ -equivariant conformal diffeomorphisms is trivial, so that by the same arguments used before in the proof of Theorem 6.1, π_Γ in (6.5) is proper. The remainder of the proof then proceeds exactly as in the proof of Theorem 6.1. ■

As a simple example, one may take Γ to be the group generated by rotation by angle π (or any other non-zero rational angle) along one axis and rotation by angle π (or any other non-zero rational angle) along a linearly independent axis.

Remark 6.4. The surjectivity statements in Theorems 6.1 and 6.3 also hold for thickenings of $C_\Gamma^{m-2,\alpha}$, i.e. open sets in $C_+^{m-2,\alpha}$ containing $C_\Gamma^{m-2,\alpha}$. Namely, first observe that $C_\Gamma^{m-2,\alpha}$ is not contained in the singular value set \mathcal{Z} . To see this, suppose $K \in C_\Gamma^{m-2,\alpha} \cap \mathcal{Z}$ and let ω be an outward normal vector to $T\mathcal{Z}$ in $C_+^{m-2,\alpha}$, i.e. $\omega \in N$; ω is generically unique up to scalar multiples. Choose a Γ -invariant function χ such that

$$\langle \chi, \omega \rangle = \int_{S^2} \chi \omega dv_{+1} \neq 0.$$

Then there exists ε small such that $K_\varepsilon = K - \varepsilon\chi \in C_\Gamma^{m-2,\alpha} \setminus \mathcal{Z}$.

Thus, although $C_\Gamma^{m-2,\alpha}$ is of infinite codimension in $C_+^{m-2,\alpha}$, there are open neighborhoods \mathcal{V} of $C_\Gamma^{m-2,\alpha}$ contained in $Im \pi$. Hence the full $Conf(S^2)$ orbit of \mathcal{V} is contained in $Im \pi$.

Remark 6.5. Although $\deg \pi_{ev}$ or $\deg \pi_\Gamma$ in (6.3) or (6.6) equals one (up to sign), this does not imply that the image $C_\Gamma^{m-2,\alpha}$ is contained in the degree one component \mathcal{N}^1 (or \mathcal{N}^{-1}) of π_0 . For instance, for $\Gamma = \mathbb{Z}_2$ it is easy to see from the degree formula (1.11) that for any odd $n \in \mathbb{Z}$, there exist $K \in C_{ev}^{m-2,\alpha}$ such that $K \in \mathcal{N}^n$. Hence, for such K , there are at least n distinct solutions of (1.1); one expects that often only one these n solutions is an even function.

Finally, Remark 6.2 also applies to the map π_Γ .

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