HOMEWORK I, MAT 568, FALL 2014

Due: Thursday, Oct 9.

If you've not already done so, read and understand all of Chapter 1 of the Petersen text. Read also all of Chapter 5, from 5.1 - 5.6. We've covered most all of this in class. (You can skip 5.1, which is kind of "weird"). Read also 5.8 - the Hopf-Rinow theorem.

Petersen Text, 2nd Edition.

Chapter 1, Problems: 1, 2, 5.

Chapter 5, Problems: 3, 6, 14

The following (technical) problem is needed for Problem 8.

7. Let $\gamma : [a, b] \to M$ be a geodesic in (M, g) and let $p : [\alpha, \beta] \to [a, b]$ be a diffeomorphism, so that $c = \gamma \circ p$ is a reparametrization of γ. Show that c satisfies

$$\frac{d^2c^k}{dt^2} + \sum_{ij} \Gamma^k_{ij}(c(t)) \frac{dc^i}{dt}\frac{dc^j}{dt} = \frac{dc^k}{dt}\frac{p''(t)}{p'(t)}.$$

Conversely, show that if c satisfies this equation, then γ is a geodesic.

8. The Poincaré half-plane is the manifold $(\mathbb{R}^2)^+ = \{(x, y) : y > 0\}$, with Riemannian metric

$$g = \frac{1}{y^2}(dx^2 + dy^2).$$

(a). Compute that

$$\Gamma^2_{22} = \Gamma^1_{12} = \Gamma^1_{21} = -\frac{1}{y},\ \Gamma^2_{11} = \frac{1}{y}, \quad \text{and all other } \Gamma = 0.$$

(b). Let $c(t) = (t, y(t))$ be a semicircle in the half-plane with center at $(0, y_0)$ of radius R. Show that

$$\frac{d^2y}{dt^2} = -\frac{y'(t)}{t - y_0} - \frac{y'(t)^2}{y(t)}.$$

(c). Using Problem 7, show that all geodesics in the Poincaré half-plane are reparametrizations of semi-circles with center on the x-axis, together with straight lines parallel to the y-axis.

(d). Show that these geodesics have infinite length in either direction, so that the upper half plane is complete in this metric. Is this true for the Euclidean metric?

(e). Finally, show that the linear fractional transformations

$$f(z) = \frac{az + b}{cz + d}$$

mapping the upper half plane to itself, are isometries of the Poincaré metric. Conclude that this metric is homogeneous.