HOMEWORK 2, MAT 568, FALL 2014

Due: Thursday, Oct 30.

1. Suppose g_1, g_2 are metrics defined on the unit ball $B^n \subset \mathbb{R}^n$. If g_1 is isometric to g_2, prove that in exponential normal coordinates based at $\{0\}$,

$$g_1 = g_2$$

on some smaller ball $B \subset B^n$, modulo an isometry of $T_0 B^n$, i.e. an element in $O(n)$.

(This means that exponential normal coordinates are a local slice for the action of the diffeomorphism group on the space of local metrics).

2. Let (M, g) be a Riemannian manifold and $f : M \to M$ a diffeomorphism. Let ∇^{f^*g} be the Levi-Civita connection for the metric f^*g and ∇^g the Levi-Civita connection for the metric g. Prove that

$$f_* \nabla^{f^*g}_X Y = \nabla^g_{f_*X} f_* Y.$$

From this, deduce that the $(3, 1)$ Riemann curvature tensor transforms naturally under pullback:

$$R^{f^*g} = f^* R.$$

3. Let $\widetilde{g} = \lambda^2 g$, where λ is a positive constant, so that $\widetilde{g}$ is a rescaling of g. Show that (for the Levi-Civita connection)

$$\widetilde{\nabla} = \nabla$$

and hence, if R denotes the $(3, 1)$ Riemann curvature tensor, then

$$\widetilde{R}(X, Y)Z = R(X, Y)Z,$$

so the connection and curvature tensor are "scale-invariant".

On the other hand, deduce that the sectional curvature, Ricci curvature on unit vectors, and scalar curvature transform under rescaling as λ^{-2}. This is the way derivatives transform under rescaling.

4. Now let $\widetilde{g} = u^2 g$ be a conformal change of the metric g. Prove that

$$\widetilde{\nabla}_X Y = \nabla_X Y + X(\log u)Y + Y(\log u)X - g(X, Y)\nabla \log u.$$

Extra Credit: Deduce the formula for the transformation of the curvature under conformal changes.

5. Let G be a Lie group with a bi-invariant Riemannian metric, i.e. the metric is invariant under right and left translations of the group. From HW

I, inner automorphisms $i_h(g) = h^{-1}gh$ are isometries of the metric and hence the adjoint action of the Lie algebra $\mathcal{L}(G)$ is skew-symmetric, i.e.

$$ad_U : \mathcal{L}(G) \to \mathcal{L}(G), \quad ad_U(X) = [X, U]$$

satisfies

$$\langle [X, U], Y\rangle = -\langle X, [Y, U]\rangle.$$

Use this to prove that for the Levi-Civita connection on left-invariant vector fields:
(a). $\nabla_X Y = \frac{1}{2}[X, Y]$.
(b). $R(X, Y)Z = \frac{1}{4}[Z, [X, Y]]$.
(c). $\langle R(X, Y)Z, W\rangle = -\frac{1}{4}\langle [X, Y], [Z, W]\rangle$.
Conclude that all the sectional curvatures are non-negative. Show that $Ric(X) = 0$ if and only if X commutes with all other left-invariant vector fields, i.e. X is in the center of $\mathcal{L}(G)$.

6. Let M^{n-1} be a hypersurface in $\mathbb{R}^n$ with the induced metric, and suppose a local chart of M is given as the graph of a function $f : \mathbb{R}^{n-1} \to \mathbb{R}$. Suppose that $f(0) = 0$, so that the origin $0 \in M$ and $Df(0) = 0$, so that the tangent space to M at 0 is $\mathbb{R}^{n-1}$.

Show that the 2nd fundamental form of M at 0 is proportional to the Hessian of f:

$$A = \frac{1}{|\nabla f|} D^2 f.$$

7. Consider the hypersurface M in $\mathbb{R}^{n+1}$ given by

$$x^{n+1} = (x^n)^2,$$

with the induced Riemannian metric.

Prove that M is isometric to $\mathbb{R}^n$ with the flat metric, i.e. it is flat.

On the other hand, prove that the 2nd fundamental form A of M does not vanish anywhere.