**HOMEWORK I, MAT 568, FALL 2010**

Due: Tuesday, Oct 12.

If you've not already done so, read and understand all of Chapter 1 of the Petersen text. Ask if you have any questions. Read also some aspects of Chapter 5, namely Sections 5.2-5.3, 5.5 and also the proof of the Hopf-Rinow theorem in §5.8. Either ignore concepts used in those sections we haven't covered yet, or learn about them from the text, (e.g. the definition of "geodesic segment").

1. Problem 1, p.17: Given Riemannian metrics $g_M$ and $g_N$ on manifolds $M$, $N$, the product metric on $M \times N$ is the metric $g_M + g_N$.

(a). Show that $(\mathbb{R}^n, g_{Eucl}) = (\mathbb{R}, dt^2) \times \cdots \times (\mathbb{R}, dt^2)$.

(b). Show that the flat square torus

$$T^2 = \mathbb{R}^2/\mathbb{Z}^2 = (S^1, (\tfrac{1}{2\pi})^2 d\theta^2) \times (S^1, (\tfrac{1}{2\pi})^2 d\theta^2).$$

(c). Show that

$$F(\theta_1, \theta_2) = \frac{1}{2}(\cos\theta_1, \sin\theta_1, \cos\theta_2, \sin\theta_2),$$

is a isometric embedding of $T^2$ into $\mathbb{R}^4$.

The next 3 problems are from Petersen, Ch. 5.

2. Show that any homogeneous manifold $(M, g)$, (i.e. the isometry group acts transitively on $M$), is necessarily geodesically complete.

3. Show that any Riemannian metric $(M, g)$ may be conformally changed to a complete Riemannian metric, i.e. there is a smooth positive function $\lambda$ such that the metric $\lambda^2 g$ is a complete Riemannian metric.

4. Show that in any Riemannian manifold $(M, g)$, one has

$$d(exp_p(tv), exp_p(tw)) = |t||v - w| + O(t^2),$$

where $d$ is the distance function and $|v|$ is the $g$-norm of $v$.

The following (technical) problem is needed for Problem 6.

5. Let $\gamma : [a, b] \to M$ be a geodesic in $(M, g)$ and let $p : [\alpha, \beta] \to [a, b]$ be a diffeomorphism, so that $c = \gamma \circ p$ is a reparametrization of $\gamma$. Show that $c$ satisfies

$$\frac{d^2c^k}{dt^2} + \sum_{ij} \Gamma^k_{ij}(c(t)) \frac{dc^i}{dt}\frac{dc^j}{dt} = \frac{dc^k}{dt}\frac{p''(t)}{p'(t)}.$$

Conversely, show that if $c$ satisfies this equation, then $\gamma$ is a geodesic.

6. The Poincaré half-plane is the manifold $(\mathbb{R}^2)^+ = \{(x, y) : y > 0\}$, with Riemannian metric

$$g = \frac{1}{y^2}(dx^2 + dy^2).$$

(a). Compute that

$$\Gamma^2_{11} = \Gamma^1_{12} = \Gamma^1_{21} = -\frac{1}{y},\ \Gamma^1_{11} = \frac{1}{y},\quad \text{and all other } \Gamma = 0.$$

(b). Let $c(t) = (t, y(t))$ be a semicircle in the half-plane with center at $(0, y_0)$ of radius $R$. Show that

$$\frac{d^2y}{dt^2} = -\frac{y'(t)}{t - y_0} - \frac{y'(t)^2}{y(t)}.$$

(c). Using Problem 5, show that all geodesics in the Poincaré half-plane are reparametrizations of semi-circles with center on the $x$-axis, together with straight lines parallel to the $y$-axis.

(d). Show that these geodesics have infinite length in either direction, so that the upper half plane is complete in this metric. Is this true for the Euclidean metric?

(e). Finally, show that the linear fractional transformations

$$f(z) = \frac{az + b}{cz + d}$$

mapping the upper half plane to itself, are isometries of the Poincaré metric. Conclude that this metric is homogeneous.