MAT 531 SPRING 16 HOMEWORK 5

Due Tuesday, March 1

1. A manifold M whose tangent bundle TM is trivial (i.e. is equivalent to a trivial bundle), is called *parallelizable*. M is called *stably parallelizable* if there is a trivial bundle ε^k of rank k over M such that $TM \oplus \varepsilon^k$ is trivial. Prove that the tangent bundle $T(S^n)$ is stably trivial.

(Hint: Show that the tangent bundle of $S^n \times \mathbb{R}$ is trivial, by considering the standard embedding $S^n \subset \mathbb{R}^{n+1}$).

The same argument shows that the tangent bundle of any compact, oriented surface is stably trivial, (since any compact oriented surface can be embedded in $\mathbb{R}^3$.

2. (a) Given two vector bundles $\pi : E \to M$ and $\pi' : E' \to M$ over M with transition maps $\Phi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})$ and $\Phi'_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})$ over the same cover $\{U_\alpha\}$, prove that the bundle π is equivalent to the bundle π' if and only if there are smooth functions $f_\alpha : U_\alpha \to GL(n, \mathbb{R})$ such that

$$\Phi'_{\alpha\beta}(x) = f_\alpha(x)\Phi_{\alpha\beta}(x)f_\beta^{-1}(x),$$

for $x \in U_\alpha \cap U_\beta$.

2. (b) Let $\pi : E \to M$ be a vector bundle of rank n with transitions maps

$$\Phi_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R}).$$

Prove that E is equivalent to a trivial bundle if and only if there exist smooth maps $\lambda_\alpha : U_\alpha \to GL(n, \mathbb{R})$ such that

$$\Phi_{\alpha\beta}(x) = \lambda_\alpha(x)\lambda_\beta^{-1}(x),$$

for $x \in U_\alpha \cap U_\beta$.

3. In the same setting as (3), suppose $s_\alpha : U_\alpha \to \mathbb{R}^\ltimes$ is a collection of smooth maps such that $s_\alpha(x) = \Phi_{\alpha\beta}(x)s_\beta(x)$ for any $x \in U_\alpha \cap U_\beta$. Prove there is a global section $s : M \to E$ such that $s|_{U_\alpha} = s_\alpha$.

4. A smooth map $F : M \to N$ between manifolds induces a bundle map $F_* = DF : TM \to TN$. Show that in general there is no corresponding map $F^* : T^*N \to T^*M$ of cotangent bundles. Describe conditions under which F^* exists.

5. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $(x, y) \to (x+2y, x-y) = (z, w)$. Let $T = zdz \otimes dw$ and compute $\varphi_* T$.

6. Show that the smooth function

$$det : GL(n, \mathbb{R}) \to \mathbb{R},$$

has differential given by

$$d(det)_A(B) = (detA)tr(A^{-1}B),$$

for $A \in GL(n, \mathbb{R})$ and $B \in T_A GL(n, \mathbb{R}) \simeq M_n(\mathbb{R})$. Here tr is the trace.

(Hint: Using matrix entries A_i^j as global coordinates on $GL(n, \mathbb{R})$ show that

$$\frac{\partial}{\partial A_i^j} det(A) = (detA)(A^{-1})_j^i.$$

To prove this, expand $detA$ by minors along the i^{th} column and use Cramer's Rule.