MAT 362 Spring 2015 Final Exam

This is an open book exam, based on the honor system. You can use any books, lecture notes, etc. to assist you in solving the problems. However, you cannot talk or discuss any issues related to the exam with someone else. The exam must express your own, and only your own, understanding.

Show all your relevant work: Partial credit will not be given without justification or reasoning of your solutions.

The exam is due:

Thursday, May 14, 2:30 pm.

Please bring your exam to my office - Math Tower 4-110 at that time. Alternately, you may put the exam under my office door any time prior to the due time. Please be sure to notify me that you have done so!

If you have any questions, you can e-mail me at: anderson@math.sunysb.edu or call at 358-8194.

Total Points: 140

1.(20pts) (a). Show that the map

$$F : \mathbb{R}^2 \to \mathbb{R}^2, \quad F(x) = c \cdot x,$$

is a geodesic mapping, i.e. F maps geodesics in $S_1 = \mathbb{R}^2$ to geodesics in $S_2 = \mathbb{R}^2$. Here c is a positive constant and $\mathbb{R}^2$ is the flat xy plane in $\mathbb{R}^3$. Prove that F is *not* an isometry.

(b). Suppose

$$F : S_1 \to S_2$$

is a smooth geodesic mapping between geodesically complete surfaces. Show that if F also preserves the lengths of geodesics, then F is an isometry.

2.(25pts) (a) Suppose S is a geodesically complete surface, diffeomorphic to $\mathbb{R}^2$, with Gauss curvature $K \le 0$ everywhere. Prove that there exists a unique geodesic between any two points p and q in S.

(b). Prove that the exponential map exp_p based at any p is a diffeomorphism onto S.

(c). Give a counterexample to this statement when S is not diffeomorphic to $\mathbb{R}^2$.

3. (20pts) (a). Prove that the set of points in $\mathbb{R}^3$ satisfying the equation

$$4x^2 + 16y^4 + 64z^6 = 1$$

is a compact surface S in $\mathbb{R}^3$.

(b). Compute the total Gauss curvature $\int_S K dA$ of S.

4. (20pts) Let $x(u^1, u^2)$ and $\tilde{x}(\tilde{u}^1, \tilde{u}^2)$ be two charts (local parametrizations) of a fixed surface S in $\mathbb{R}^3$. The first fundamental form in these charts are thus the same in the common domain of the charts, so that

$$\tilde{I} = \sum_{i,j} \tilde{g}_{ij} d\tilde{u}^i d\tilde{u}^j = \sum_{k,\ell} g_{k\ell} du^k du^\ell = I.$$

(This means that $\tilde{I}(v, w) = I(v, w) = v \cdot w$, for any tangent vectors v, w. Of course $E = g_{11}$, etc. in the E, F, G notation). Find the relation between the coefficients $\tilde{g}_{ij}$ and $g_{k\ell}$. Here $\tilde{u}^i = \tilde{u}^i(u^1, u^2)$.

5. (20pts) (a) Consider the paraboloid

$$z = x^2 + y^2.$$

This is a smooth surface S in $\mathbb{R}^3$. It is geodesically complete. Prove that the exponential map of S based at the origin $0 = (0,0,0)$ is a diffeomorphism $T_0S \to S$.

(b). For any $v \in T_0S$ with $|v| = 1$, show that the complete geodesic $\gamma_v(t)$, $t \in \mathbb{R}$, is not globally length minimizing. More precisely, show that for R sufficiently large, the geodesic segment $\gamma_v[-R, R]$ joining the points $p = \gamma_v(-R)$ with $q = \gamma_v(R)$ is not length minimizing.

6. (35 pts) The surface obtained by rotating the curve

$$y = \cosh x,$$

about the x-axis is called the catenoid C_1.

(a). Compute the Gauss curvature of the catenoid.

(b). Next, determine the area element dA and compute, from (a), the total Gauss curvature

$$\int_{C_1} K dA.$$

(You may use the fact that the antiderivative (integral) of $1/\cosh^2$ is tanh.

(c). Describe the image of the Gauss map of the catenoid on the sphere $S^2(1)$, and explain how you can recover your result from (b) almost directly, without computation.

(d). Prove (or at least show why) the catenoid is diffeomorphic to the cylinder - call it C_2, obtained by rotating the line $y = 1$ about the x-axis.

(e). Determine, by any method, the total Gauss curvature, i.e.

$$\int_{C_2} K dA$$

for the cylinder.

(f). Show that the Euler characteristic of the cylinder is 0. You can use the fact that a finite interval $I = (0, 1)$ is diffeomorphic to the whole line $\mathbb{R}$, so the cylinder is diffeomorphic to $I \times S^1$.

Conclude from your work above that the Gauss-Bonnet theorem does not hold for non-compact (geodesically complete) surfaces in the same way that it holds for compact surfaces.