MAT 362 SPRING 09 HOMEWORK 9

Due Thursday, May 7

1. Suppose S is a compact, oriented surface in $\mathbb{R}^3$ with Gauss curvature K satisfying $K > 0$ everywhere. Prove that S is diffeomorphic to S^2 - the 2-sphere. Is the converse true? - thus if S is diffeomorphic to S^2, is necessarily $K > 0$ everywhere?

2. For S as above, suppose $K(x) < 0$ for all $x \in S$. Prove that S cannot be diffeomorphic to the sphere S^2 or the torus T^2.

3. For S any compact oriented surface in $\mathbb{R}^3$, prove there must exist a point x_0 on S where

$$K(x_0) > 0.$$

Hint: Choose the smallest r such that the sphere $S^2(r)$ of radius r centered at the origin contains the surface S, and let x_0 be a point where $S^2(r)$ touches S. Then argue that at this point, $K(x_0) > 0$.

Note: this result is false for surfaces in $\mathbb{R}^4$ for example.

4. Let T be a geodesic triangle on a compact oriented surface S which bounds a disc in S. Thus, the 3 edges of T are geodesics in S and the interior is homeomorphic to a disc. Using the fact that the geodesic curvature of any geodesic curve is 0, prove:

(a). If $K > 0$ everywhere on T, then the sum of the interior angles of T is $> \pi$.
(b). If $K < 0$ everywhere on T, then the sum of the interior angles of T is $< \pi$.
(c). If $K = 0$ everywhere on T, then the sum of the interior angles of T is π.

Again, all these results are false (in general) if the triangle is not homeomorphic to a disc.

5. Prove there is no compact minimal surface in $\mathbb{R}^3$. A surface is minimal if its mean curvature $H = \kappa_1 + \kappa_2 = 0$, everywhere, where κ_i are the principal curvatures.