Solutions to Homework 8

4.1

$\boxed{2}$

(i) $$u(x,y) = x^2 + (-y^2) = x^2 - y^2$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(2x) = 2$$

Siml/ $$\frac{\partial^2 u}{\partial y^2} = -2$$

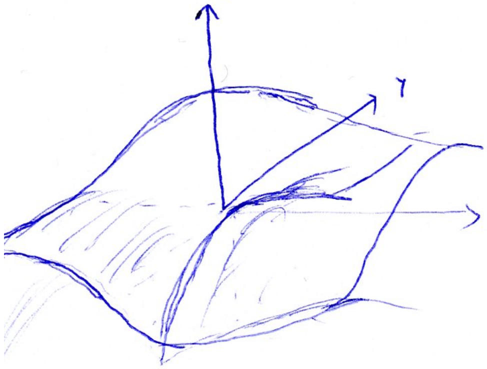
$$\nabla u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(ii) $$u(x,y) = xy$$

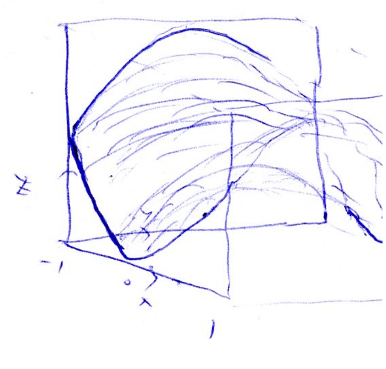
$$\frac{\partial u}{\partial x} = y, \quad \frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{\partial^2 u}{\partial y^2} = 0$$

$$\therefore \nabla u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



at $x=0$
$z = -y^2$
at $y=a$, $z = a^2 - y^2$



$$\frac{d^2u}{dx^2} = 0$$

$$\Rightarrow \quad u(x) = Ax + B \qquad \text{(By Integrating)}$$

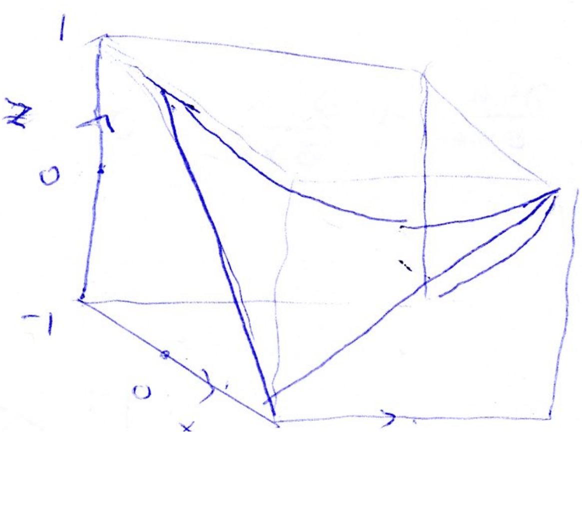
Dirichlet conditions

$$B = u(0) = a_0$$

$$Aa + B = u(a) = b_0 \qquad \text{Where } a_0, b_0 \in \mathbb{R}$$

Nuemann Conditions

$$A = u'(0) = c$$



4.2

5 $u(0,y)=0$, $u(a,y)=0$; So,

We know that the general solution is given by $\left(\lambda_n = ?\right.$

$$u(x,y) = \sum_{n=1}^{\infty} \left(a_n \cosh(\lambda_n y) + b_n \sinh(\lambda_n y)\right) \sin(\lambda_n x$$

We need to compute a_n's and b_n's

We have

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\{(\lambda_n x)\} = 0 \quad (0<$$

$$\Rightarrow a_n = 0 \quad \forall n \in \mathbb{N}$$

$$u(x,b) = f(x) = \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi b}{a}\right) \sin\left(\frac{n}{\ }\right.$$

$$\therefore b_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right)$$

Where
$$f(x) = \begin{cases} \frac{2x}{a} & (0<x< \\ 2\left(1-\frac{x}{a}\right) & \left(\frac{a}{2}<x<a\right) \end{cases}$$

Following the example solved in the book

$$b_n = \frac{8\sin\left(\frac{n\pi}{2}\right)}{\pi^2 n^2 \sinh\left(\frac{n\pi b}{a}\right)}$$

$u(x,0)=0, \quad u(x,b)=0 \qquad 0<x<a$

The general solution here is $\boxed{\lambda'_n = \frac{n\pi}{b}}$

$$u(x,y) = \sum_{n=1}^{\infty} \left(A_n \cosh(\lambda'_n x) + B_n \sinh(\lambda'_n x)\right) \sin(\lambda'_n y)$$

5 $u(0,y)=0$ Just like $\boxed{5}$ we have

$$\sum_{n=1}^{\infty} A_n \sin(\lambda'_n y) = 0 \Rightarrow A_n = 0 \quad \forall n \in \mathbb{N}$$

$$u(a,y) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi b}{a}\right) \sin(\lambda'_n y)$$

$$= 1$$

$$\Rightarrow B_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{b}\int_0^b (1) \sin(\lambda'_n y)\, dy$$

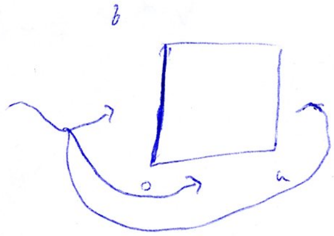
$$= \frac{2}{b}\int_0^b \sin\left(\frac{n\pi y}{b}\right) dy$$

$$= \frac{2}{b}\left| -\cos\left(\frac{n\pi y}{b}\right)\frac{b}{n\pi}\right|_0^b$$

$$= \frac{2}{n\pi}\left(1-(-1)^n\right)$$

$$\Rightarrow B_n = \frac{2}{\sinh\left(\frac{n\pi b}{a}\right) n\pi}\left(1-(-1)^n\right)$$

The outward normal is 0



$\nabla^2 u = 0$

$\frac{\partial u}{\partial y}(x,0) = 0$, $u(x,b) = 100$
(A) (B)

$\frac{\partial u}{\partial x}(0,y) = 0$, $\frac{\partial u}{\partial x}(a,y) = 0$

Since $\nabla^2(u-100) = 0$

$u-100$ satisfies $\frac{\partial(u-100)}{\partial y}(x,0) = 0$ $(u-100)(x,b) = 0$
(A') (B')

$\frac{\partial(u-100)}{\partial x}(0,y) = 0$, $\frac{\partial(u-100)}{\partial x}(a,y)$

Let $u' = u - 100$

Following <u>example</u> 1.

$$u'(x,y) = a_0 + b_0 y + \sum_{n=1}^{\infty} \left(a_n \cosh(\lambda_n x) + b_n \sinh(\lambda_n y)\right) \cos(\lambda_n x)$$

~~(A')~~ ~~implies~~ (A')
$$\Rightarrow b_0 b + \sum_{n=1}^{\infty} \left(b_n \sinh(\lambda_n b)\right) \cos(\lambda_n x) = 0$$

$\Rightarrow b_0 b = 0$, $b_n \sinh(\lambda_n b) = 0 \Rightarrow b_0 = 0, b_n$

(B') $\Rightarrow a_0 + \sum a_n \cosh(\lambda_n b) \cos(\lambda_n x) = 0$

$\Rightarrow a_0 = 0$ $a_n \cosh(\lambda_n b) = 0$ $\Rightarrow a_0 = a_n = 0$

$\Rightarrow u' = 0$ is th only solution

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$\nabla^2 P = -H$ H is a const.

$P(x,y) = A + Bx + Cy + Dx^2 + Exy + Fy^2$

$\frac{\partial P}{\partial x} = B + 2Dx + Ey \rightarrow \frac{\partial^2 P}{\partial x^2} = 2D$

$\frac{\partial P}{\partial y} = C + Ex + 2Fy \rightarrow \frac{\partial^2 P}{\partial y^2} = 2F$

$\nabla^2 P = -H$

$\Leftrightarrow \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = -H$

$\Leftrightarrow 2D + 2F = -H$

$\Leftrightarrow D + F = -\frac{H}{2}$

A, B, C, E can le any real numbe