Homework 6
Solutions

(1) We compute the second derivatives

$$\frac{\partial^2 u_n}{\partial x^2} = -\lambda_n^2 u_n$$

$$\frac{\partial^2 u_n}{\partial t^2} = -\lambda_n^2 c^2 u_n$$

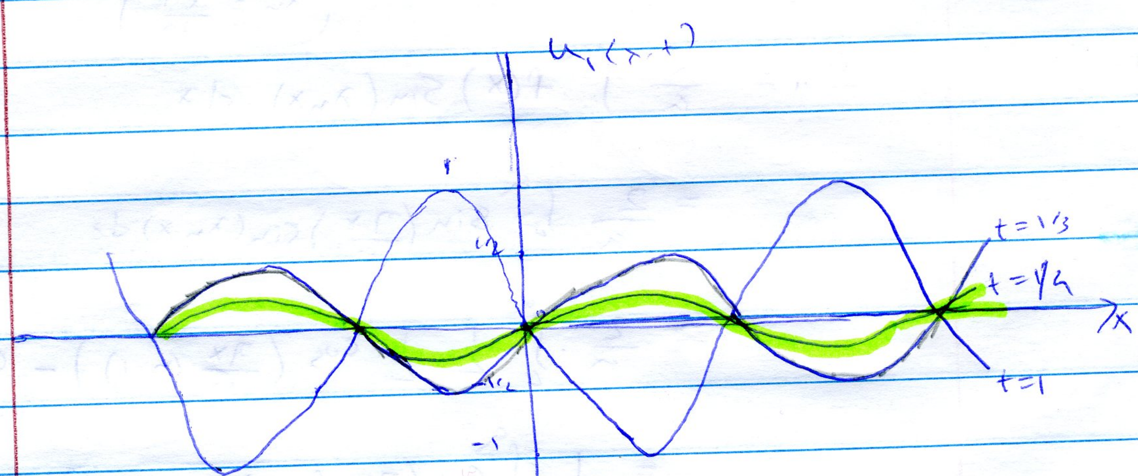
So u_n satisfies the wave equation. For the boundary conditions substitute for $x = 0, a$

$x = 0 \Rightarrow \sin(0) = 0 \quad \therefore u_n(0,t) = 0$

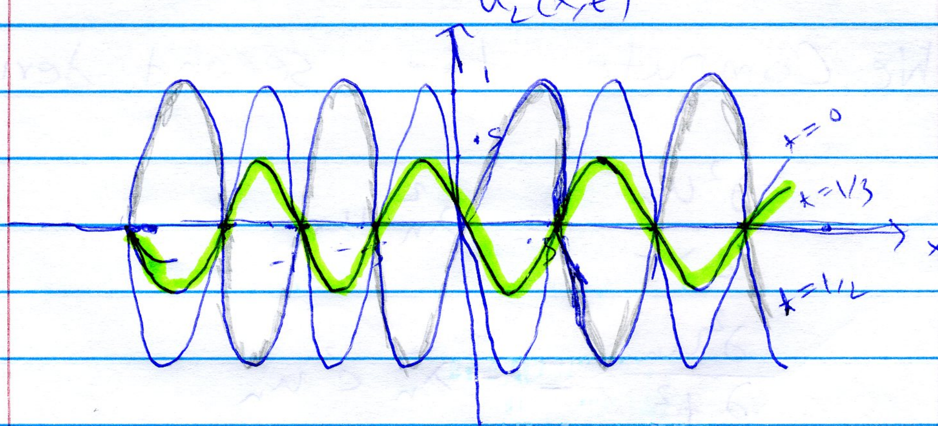
$x = a \Rightarrow \sin(\lambda_n a) = \sin(n\pi) = 0$

making $u_n(a,t) = 0$

(2)



$\boxed{u_1(x,t) = \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi c t}{a}\right)}$ $\quad u_2(x,t) = \sin\left(\frac{2\pi x}{a}\right)\cos\left(\frac{2\pi}{a}\right.$



(4)

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \quad 0<x<a, \quad 0<t$$

$$u(0,t) = 0 \qquad u(a,t)=0, \quad 0<t$$

$$u(x,0) = f(x) = \sin\left(\frac{\pi x}{a}\right), \quad 0<x<a$$

and,

$$\frac{\partial u}{\partial t}(x,0) = g(x) = 0$$

Now,

$$\left(\lambda_n = \frac{n\pi}{a}\right)$$

$$a_n = \frac{2}{a}\int_0^a f(x)\sin(\lambda_n x)\,dx$$

$$= \frac{2}{a}\int_0^a \sin\left(\frac{\pi x}{a}\right)\sin(\lambda_n x)\,dx$$

$$= \frac{2}{a}\cdot\int_0^a \frac{1}{2}\left[\cos\left(\frac{\pi x}{a}(n-1)\right) - \cos\left(\frac{\pi x}{a}(n+1)\right)\right]dx$$

$$= \frac{1}{a}\int_0^a \cos\left(\frac{\pi x}{a}(n-1)\right)dx - \frac{1}{a}\int_0^a \cos\left(\frac{\pi x}{a}(n+1)\right)dx$$

$$= \frac{1}{a}\left|\frac{a}{\pi(n-1)}\sin\left(\frac{\pi x}{a}(n-1)\right)\right|_0^a - \frac{1}{a}\left|\frac{a}{\pi(n+1)}\sin\left(\frac{\pi x}{a}(n+1)\right)\right|_0^a$$

$$= 0$$

Since $b_n = \frac{2}{n\pi c} \int_0^a g(x) \sin \lambda_n x$

b_n is also 0

(5) Here $f(x) = 0$ and $g(x) = 1$

So again $a_n = 0$

$$b_n = \frac{2}{n\pi c} \int_0^a \sin\left(\frac{n\pi x}{a}\right) dx$$

$$= \frac{2}{n\pi c} \left| \frac{-a}{n\pi} \cos\left(\frac{n\pi x}{a}\right) \right|_0^a$$

$$= \frac{2}{n^2\pi^2 c} \left(a - a\cos(\pi n)\right)$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} \sin(\lambda_n x)\, b_n \sin(\lambda_n t)$$

where $\lambda_n = \frac{n\pi}{a}$

(14) Again we assume seperation

$u(x,t) = \phi(x)\, T(t)$

Here constant has to be negative

we have

$T'' + \lambda^2 c^2 T = 0$ —— (1) $t > 0$

$\varphi'' + \lambda^2 \varphi = 0$ —— (2) $a > x > 0$

$\therefore\ \varphi(0)\,T(t) = 0 \quad , \varphi'(a)\,T(t) = 0$

We must have $\varphi(0) = 0$, $\varphi'(a) = 0$ as $T \neq 0$

We have

$$\varphi(x) = c_1 \cos(\lambda \cancel{x}) + c_2 \sin(\lambda x)$$

and $\varphi'(x) = -c_1 \lambda \sin(\lambda(x)) + c_2 \lambda \cos(\lambda x)$

$\varphi(0) = 0 \Rightarrow c_1 = 0$

$\varphi'(a) = 0 \Rightarrow c_2 \lambda \cos(\lambda a) = 0 \Rightarrow \cos(\lambda a) = 0$

$\Rightarrow \lambda a = (2n+1)\pi/2$

$\therefore$ Let $\lambda_n = \frac{(2n+1)}{a}\frac{\pi}{2}$

Then $\varphi(x) = \sin(\lambda_n x)$ for some n

and

$T(t) = a_n \cos(\lambda c t) + b_n \sin(\lambda c t)$

$f(x) = 0 \Rightarrow a_n = 0$,

$g(x) = 1$ so by a similar calculati

we get $b_n = \frac{8a}{(2n+1)^2 \pi^2 c}$

$$\therefore \boxed{u(x,t) = \sum_{n=1}^{\infty} \sin(\lambda_n x)\, b_n \sin(\lambda_n c t)}$$

3 (8)

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \qquad 0 < x < a, \ 0 < t$$

$u(0,t) = 0$, $u(a,t) = 0$

$u(x,0) = f(x) = 0$, $\frac{\partial u}{\partial t}(x,0) = g(x)$

We know $g(x) = \begin{cases} 0 & 0 < x < 0.4a \\ 5c & 0.4a < x < 0.6a \\ 0 & 0.6a < x < a \end{cases}$

$G(x) = \frac{1}{c} \int_0^x g(x)\, dx$

$$\boxed{\therefore G(a) = \frac{1}{c}\left[\int_0^{0.4a} g(x)\,dx + \int_{0.4a}^{0.6a} g(x)\,dx + \int_{0.6a}^{a} g(x)\,dx\right] = a}$$

G can be extended with period $2a$

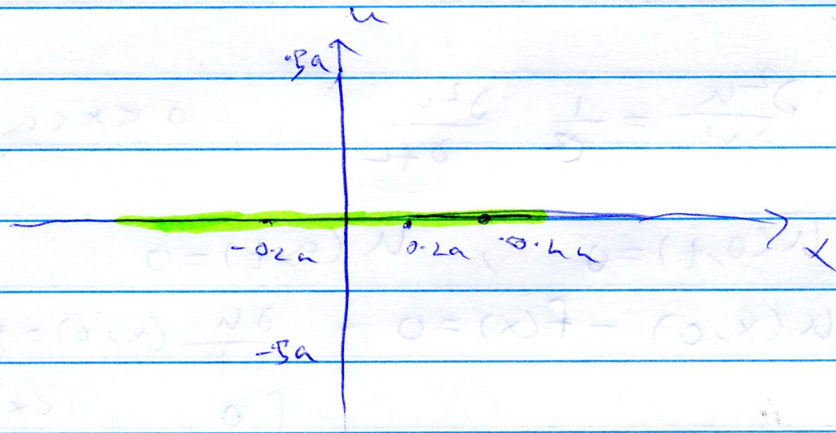
We see

$G(x) = \begin{cases} 0 & 0 \le x \le 0.4a \\ 5(x-0.4a) & 0.4a \le x \le a \\ a & 0.6a \le x \le 1.4a \\ 5(1.6a - x) & 1.4a \le x \le 1.6a \\ 0 & 1.6a \le x \le 2a \end{cases}$

$\therefore \quad u(x,t) = \frac{1}{2}\left[G(x+ct) - G(x-ct) \right.$

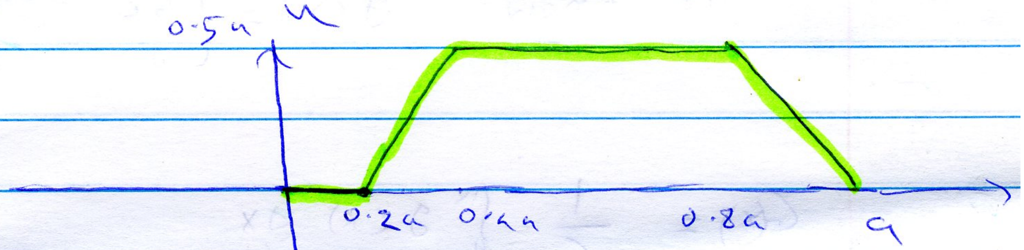
Case 1.

ct=0



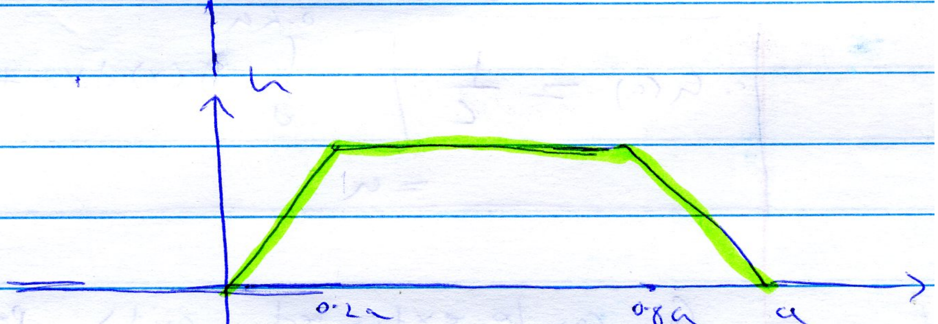
Case 2.

ct=0.2a



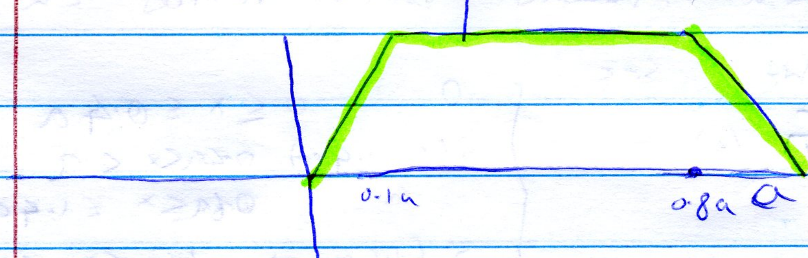
Case 3.

ct=0.4a



Case 4

ct=0.5a



Case 5

ct=a