MAT 341

HOMEWORK 5 SOLUTIONS

2.10

(1)

$$f(x) = \begin{cases} 0 & 0<x<a \\ T & a<x<b \\ 0 & b<x \end{cases}$$

By defn. $B(\lambda) = \frac{2}{\pi}\int_0^\infty f(x) \sin\lambda x \, dx$

$$= \frac{2}{\pi}\int_a^b T \sin\lambda x \, dx$$

$$= \frac{2T}{\lambda}\left(\cos(a\lambda) - \cos(b\lambda)\right)$$

and $u(x,t) = \int_0^\infty B(\lambda) \sin(\lambda x) \exp(-\lambda^2 kt)\, d\lambda$ $(0<t)$ $(0<x<\infty)$

(2)

First of all,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\int_0^\infty B(\lambda)\sin\lambda x \exp(-\lambda^2 kt)\,d\lambda\right)$$

(Taking $\frac{\partial}{\partial x}$ inside the integral)

$$= \frac{\partial}{\partial x}\left(\int_0^\infty \lambda B(\lambda)\cos\lambda x \exp(-\lambda^2 kt)\,d\lambda\right)$$

$$= -\lambda^2 \int_0^\infty B(\lambda)\sin\lambda x \exp(-\lambda^2 kt)\,d\lambda$$

(This can be done since the 'inside' is 'nice')

bounded is sufficient here

So $\frac{1}{k}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0<x<\infty$

also,

$u(0,t) = 0$ $\quad 0<t$ $\quad$ (as $\sin(0)=0$)

$$u(x,0) = \int_0^\infty B(\lambda)\sin(\lambda x)\,d\lambda$$

$$= f(x)$$

$$\lim_{t\to\infty} u(x,t) = 0$$

So the steady state solution is 0.

(7) This Problem has a Steady State Solution $\lim_{t \to \infty} u(x,t) = T_0$ (Proved in Same way as in [illegible] (

Then $w(x,t) = u(x,t) - T_0$ Satisfies the problem in Equation (1) and (2), and equation (3) is replaced by $w(x,0) = f(x) - T_0$ So The Solution given in equation (9) requires that

$$B(\lambda) = \frac{2}{\pi} \int_0^\infty (f(x) - T_0) \sin(\lambda x)\, dx$$

$$\left(\int_0^\infty |f(x) - T_0|\, dx \to \infty \right)$$

and $u(x,t) = T_0 + \int_0^\infty B(\lambda) \sin(\lambda x) e^{-\lambda^2 k t}\, d\lambda$

So The formula fo u is

$$u(x,t) = T_0 + \int_0^\infty B(\lambda)\, \sin(\lambda x)\, \exp(-\lambda^2 k t)\, d\lambda$$

where

$$B(\lambda) = \frac{2}{\pi} \int_0^\infty f(x) - T_0 \, \sin(\lambda x)\, dx$$

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(5) By Differenting u we get

$$\frac{\partial u}{\partial t} = \frac{1}{\sqrt{4k\pi}}\left[\frac{1}{\sqrt{t}} e^{\frac{-x^2}{4kt}} \frac{x^2}{4kt^2} - \frac{1}{2t^{3/2}} e^{-x^2/4kt}\right]$$

(By e^{x} I mean $\exp(x)$)

$$= \frac{e^{-x^2/4kt}}{\sqrt{4k\pi}}\left[\frac{x^2}{4kt^2\sqrt{t}} - \frac{1}{2t\sqrt{t}}\right]$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{4k\pi}} \frac{1}{\sqrt{t}} e^{-x^2/4kt}\left(\frac{-2x}{4kt}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{e^{-x^2/4kt}}{\sqrt{4k\pi}}\left[\frac{-1}{2kt\sqrt{t}} + \frac{x^2}{4k^2t\sqrt{t}}\right]$$

So, $\therefore$ $\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}}$

⑦ Substitute directly into equation ①, ② and ③, We Observe that $u(x,t)=1$ is a solution

Now If we put (the solution) $F(x')=1$ in equation ⑦ We get

$$1 = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\left[\frac{-(x'-x)^2}{4kt}\right] dx'$$