MAT 341 (HW 4 SOLNS)

2.4

2) Here

$$f(x) = u(x,0) = T_0 + T_1\left(\frac{x}{a}\right)^2 \quad (0<x<a)$$

$$a_0 = \frac{1}{a}\int_0^a f(x)\,dx$$

$$= \frac{1}{a}\int_0^a \left(T_0 + \frac{T_1 x^2}{a^2}\right)dx$$

$$= T_0 + \frac{1}{3}T_1$$

$$a_n = \frac{2}{a}\int_0^a f(x)\cos\left(\frac{n\pi x}{a}\right)dx$$

$$= \frac{2}{a}\int_0^a \left(\underbrace{T_0\cos\left(\frac{n\pi x}{a}\right)}_{\text{vanishes}} + \frac{T_1}{a^2}\cos\left(\frac{n\pi x}{a}\right)x^2\right)dx$$

computed using integration by parts

$$= \frac{4T_1\cos n\pi}{n^2\pi^2}$$

So, $$u(x,t) = \frac{3T_0+T_1}{3} + \sum_{n=1}^{\infty}\frac{4T_1\cos(n\pi)}{n^2\pi^2}\cos\lambda_n x\, e^{-\lambda_n^2 kt}$$

4) $f(x) = \begin{cases} T_1 & 0 < x < a/2 \\ T_2 & \frac{a}{2} < x < a \end{cases}$

$$a_0 = \frac{1}{a}\int_0^a f(x)\,dx = \frac{1}{a}\int_0^{a/2} T_1 + \frac{1}{a}\int_{a/2}^a T_2$$

$$= \frac{T_1 + T_2}{2}$$

$$a_n = \frac{2}{a}\int_0^a f(x)\cos\left(\frac{n\pi x}{a}\right)$$

$$= \frac{2}{a}\int_0^{a/2} T_1 \cos\left(\frac{n\pi x}{a}\right) + \frac{2}{a}\int_{a/2}^a T_2 \cos\frac{n\pi x}{a}$$

$$= \frac{2}{n\pi}\left[T_1 \sin\left(\frac{n\pi}{2}\right) - T_2 \sin\left(\frac{n\pi}{2}\right)\right]$$

So,

$$u(x,t) = \frac{T_1 + T_2}{2} + \sum_{n=1}^{\infty} \frac{2(T_1 - T_2)}{n\pi} \sin\left(\frac{n\pi}{2}\right)\cos(\lambda_n x)$$

$$\exp(-\lambda_n^2 kt)$$

5)

$V(x)$ is linear since $\frac{d^2 v}{dx^2} = 0$

a) Assume $V(x) = ax + b$

We have $\frac{dv}{dx} = a$

· but $\frac{\partial u}{\partial x}(0,t) = S_1 \Rightarrow \lim_{t \to \infty} \frac{\partial u}{\partial x}(0,t) = S_1$

$\Rightarrow a = \frac{dv}{dx} = S_1$

Similarly $a = S_2$

$\therefore S_1 = S_2$

Conversely if $S_1 = S_2$ the

Steady state $V(x) = S_1 x + c$ (c is a constant)

Observe that the condition

$S_1 = S_2$ is physically necessary

since heat flux q is

proportional to $\partial u / \partial x$

$q_0 = S_2 k_2$  $\hat{q} = S_1 k$

So if the system ends up in

a steady state then $S_1 = S_2$

If stead state soln exist then

b) $W(x,t) = u(x,t) - V(x)$

$$\frac{\partial W(x,t)}{\partial x}\Big|(0,t) = \frac{\partial u}{\partial x}(0,t) - \frac{\partial V}{\partial x} = 0$$

Similarly

$$\frac{\partial W(x,t)}{\partial x}\Big|(a,t) = 0$$

c) $u(x,t) = A(kt + x^2/2) + Bx$

$$\frac{\partial^2 u}{\partial x^2} = A = \frac{1}{k}\frac{\partial u}{\partial t} \quad , \quad u(x,0) = Ax^2/2 + Bx$$

So u satisfies heat eqn.

$$\frac{\partial u}{\partial x}(0,t) = B = S_0$$

$$\frac{\partial u}{\partial x}(a,t) = Aa + B = S_1$$

$$\Longleftrightarrow$$

$$\therefore \quad A = \frac{S_1 - S_0}{a} \quad , \quad B = S_0$$

$\therefore$ so A, B can be chosen.

If $S_1 \neq S_0$ $\quad A \neq 0$

$\therefore \quad \lim_{t \to \infty} u(x,t) = \pm\infty$

8) If $\phi''(x) = p^2 \phi(x)$, $\phi'(0) = \phi'(a) = 0$

The general soln is given by

$$\phi(x) = c_1 e^{px} + c_2 e^{-px}$$

$$\phi'(0) = c_1 p - c_2 p = 0 \Rightarrow c_1 = c_2$$

$$\phi'(a) = c_1 (pe^{pa} - pe^{-pa}) = 0 \quad a \neq 0$$

$$\Rightarrow c_1 = 0$$

$$\therefore c_1 = c_2 = 0 \Rightarrow \phi(x) = 0$$

9)

Use Theorem 7 (Section 1.5)

$\sum_{n=1}^{\infty} |A_n(t_1)|$ converges

an $|A_n(t_1)| = |a_n| \exp(-\lambda_n^2 k t)$

$a_n = \frac{1}{a}\int_0^a f(x) \cos \lambda_n x \, dx$ so $|a_n| \leq \frac{1}{a}\int_0^a |f(x)| dx = c_1$

a_n's are bounded and $\exp(-\lambda_n^2 k t)$ $\leq c_2^{-n^2}$

So $|A_n(t_1)| \le c_1 c_2^{-n^2}$

as a geometric series

converges, the series

$\sum |A_n(t_1))|$ converges

$\therefore$ So the given series converges uniformly in $0 \le x \le a$ By Theorem 7.
to $u(x, t_1)$

Similar argument shows $\left(\frac{\partial^2 u}{\partial x^2}\right)$

2.7

(3 d)

$$\varphi(x) = c_1 \sin \lambda x + c_2 \cos \lambda x$$

$$\varphi'(x) = \lambda c_1 \cos \lambda x - \lambda c_2 \sin \lambda x$$

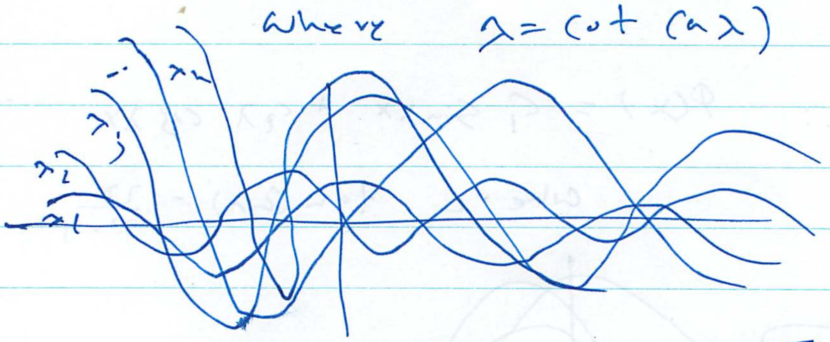
So $\varphi(0) = c_2 = \varphi'(0) = \lambda c_1$.

$$\varphi'(a) = \lambda c_1 \cos(a\lambda) - \lambda c_2 \sin a\lambda$$

$$\Rightarrow \lambda = \cot(a\lambda)$$

$$\varphi(x) = c_1 \sin \lambda x + c_1 \lambda \cos \lambda x$$

where $\lambda = \cot(a\lambda)$



For large n The graph looks like cosine graph

(3e)

Here we get

$\varphi(0) = \varphi'(0)$ so $c_2 = \lambda c_1$

and $\varphi(a) = -\varphi'(a)$

so $c_1 \sin(a\lambda) + \lambda c_1 \cos(a\lambda)$

$= \left(-\lambda^2 c_1 \sin(a\lambda) + \lambda c_1 \cos(a\lambda)\right) \times (-1)$

$\therefore \tan(a\lambda) = \frac{2\lambda}{\lambda^2 - 1}$

$\therefore \varphi(x) = c_1 \sin\lambda x + c_2 \lambda \cos\lambda x$

where $\tan(a\lambda) = \frac{2\lambda}{\lambda^2-1}$