MAT 322 SPRING 18 HOMEWORK 7

Due Tuesday, April 3

The first two problems are taken from M. Spivak, Calculus on Manifolds.

1. If M is a k-dimensional manifold-with-boundary in $\mathbb{R}^n$, prove that the boundary ∂M is a $(k-1)$-dimensional manifold and $M \setminus \partial M$ is a k-dimensional manifold, both in $\mathbb{R}^n$.

2. (a). Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be self-adjoint with matrix $A = (a_{ij})$, so $a_{ij} = a_{ji}$. Let

$$f(x) = \langle T(x), x\rangle = \sum a_{ij} x^i x^j.$$

Show that

$$\partial_k f(x) = 2 \sum_{j=1}^{n} a_{kj} x^j.$$

By considering the maximum of $\langle T(x), x\rangle$ on $S^{n-1} \subset \mathbb{R}^n$, show that there is an $x \in S^{n-1}$ and $\lambda \in \mathbb{R}$ such that

$$T(x) = \lambda x. \tag{0.1}$$

(b). For x as in (0.1), if $V = \{y \in \mathbb{R}^n : \langle x, y\rangle = 0\}$, show that $T(V) \subset V$ and $T : V \to V$ is self-adjoint.

(c). Show that T has an orthonormal basis by eigenvectors of T.

3. Do problems 1-5 of Section 29, p. 251 of Munkres text.