Problem	1	2	3	4	5	**Bonus:**	**Total:**
Points	**6**	**12**	**10**	**10**	**12**	**10**	**50+10**
Scores							

Mat 310 – Linear Algebra – Fall 2004

Name: ______________________ **Id. #:** **Lecture #:**

Test 2 (November 05 / 60 minutes)

There are 5 problems worth 50 points total and a bonus problem worth up to 10 points. Show all work. Always indicate carefully what you are doing in each step; otherwise it may not be possible to give you appropriate partial credit.

1. [6 points] Let W_1 and W_2 be linear subspaces of a vector space V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$. Prove that for each vector $\alpha \in V$ there are *unique* vectors $\alpha_1 \in W_1$ and $\alpha_2 \in W_2$ such that $\alpha = \alpha_1 + \alpha_2$.

2. [12 points] Consider the vectors in $\mathbb{R}^4$ defined by

$$\alpha_1 = (-1, 0, 1, 2), \quad \alpha_2 = (3, 4, -2, 5), \quad \alpha_3 = (1, 4, 0, 9).$$

(a) [8 points] What is the dimension of the subspace W of $\mathbb{R}^4$ spanned by the three given vectors? Find a basis for W and extend it to a basis $\mathcal{B}$ of $\mathbb{R}^4$.

(b) [4 points] Use a basis $\mathcal{B}$ of $\mathbb{R}^4$ as in (a) to characterize all linear transformations $T : \mathbb{R}^4 \to \mathbb{R}^4$ that have the same null space W. What can you say about the rank of such a T? What is therefore the precise condition on the values of T on $\mathcal{B}$?

3. [10 points] Let $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ be the ordered basis for $\mathbb{R}^3$ consisting of

$$\alpha_1 = (1, 0, -1), \quad \alpha_2 = (1, 1, 1), \quad \alpha_3 = (1, 0, 0).$$

What are the coordinates of the vector (a, b, c) in the ordered basis $\mathcal{B}$?

4. [10 points] Let V be the vector space over $\mathbb{R}$ of all real polynomial functions p of degree at most 2. For any fixed $a \in \mathbb{R}$ consider the *shift operator* $T : V \to V$ with $(Tp)(x) = p(x + a)$. Explain why T is linear and find the range and null space of T. Is T an isomorphism? Write down the matrix of T with respect to the ordered basis $\mathcal{B} = \{1, x, x^2\}$.

5. [12 points] Let T be the linear operator on $\mathbb{R}^2$ defined by $T(x_1, x_2) = (-x_2, x_1)$.

(a) [3 points] What is the matrix of T in the standard ordered basis for $\mathbb{R}^2$?

(b) [3 points] Interpret the operation of T geometrically.

(c) [3 points] What is the matrix of T in the ordered basis $\mathcal{B} = \{\alpha_1, \alpha_2\}$, where $\alpha_1 = (1, 2)$ and $\alpha_2 = (1, -1)$?

(d) [3 points] Prove that for every real number c the operator $(T - cI)$ is invertible.

Bonus Problem [up to 10 points] Let $T, U \in L(V, V)$ be linear operators on the finite dimensional vector space V. Prove that the rank of the composition UT is less than or equal to the minimum of the ranks of T and U.