MAT 310 Fall 2009
Extra Credit Assignment

This set of projects is intended to help students boost their final grade. Please note the following important rules/guidelines:

- Pick one and only one project. Do not submit anything but the project you have chosen to work on.
- The project is due Wednesday, 12/16 at **12:00** pm in my office (Math building 3-104). Make sure to be on time, as late projects will not be accepted. Of course if you can't come in on Wednesday or have a final at the time this is due, you can always submit early. Staple the assignment and submit a hard copy; please do not email.
- You must type the project, including all formulæ. Write concisely and clearly in complete sentences. Please include a cover page with your name, ID, recitation section and project you have chosen.
- The project must be entirely your own work. You may consult the course textbook and other written textbooks, but you may **not** use the internet or speak to anyone. Include a list of the sources you did use, including page numbers. The only person you may consult about anything related to the project is me, and I'll only answer general questions such as whether you may consult a particular source or whether there is a typo in a problem. Email me any questions: balsam@math.sunysb.edu.
- After finishing the project please print (by hand!) the following statement: *I promise that this assignment is entirely my work. I have followed all guidelines and in particular, have only used the sources I have listed.* Below this, sign and date.
- Treat this as a take-home exam. Infractions are easier to spot than you might think and will be taken seriously! (At the very least if I notice that two assignments are too similar, neither will receive any credit).
- The material covered in the course should be sufficient to answer the questions, but you may need to read/think a bit on your own to answer some of the questions.

The project will be worth up to one-half of a letter grade (e.g. a C becomes a C+, a B+ becomes an A-, etc [1]), but actual credit depends on the quality of the paper. Please note, however, that the final exam is **much** more important than this project and should be your primary focus in the next week. Do not think that a failing score on the exam can be saved by a decent project. In fact this is not true. A respectable final exam score and no project is far better than a good project and a terrible score on the exam, so spend your time wisely.

Good luck and have a wonderful break!!!

[1] This doesn't mean however an F becomes a D unless you are very close to passing

Project 1: Unitary Operators[2]

1. Let W be a complex inner product space, and A be a self-adjoint linear operator on W. Show that:

1. $\|\alpha + iA\alpha\| = \|\alpha - iA\alpha\|$ for every $\alpha \in W$.
2. $\alpha + iA\alpha = \beta + iA\beta$ if and only if $\alpha = \beta$
3. $I + iA$ is invertible.
4. $I - iA$ is invertible.
5. If W is finite-dimensional, show $B = (I - iA)(I + iA)^{-1}$ is a unitary operator.

2. If V is an inner-product space, a rigid motion $T : V \to V$ is any function (not necessarily linear) such that $\|T\alpha - T\beta\| = \|\alpha - \beta\|$ for all $\alpha, \beta \in V$.One example of a rigid motion is a linear unitary operator. Another is translation by a fixed vector γ:

$$T_\gamma(\alpha) = \alpha + \gamma \tag{1}$$

1. Let V be $\mathbb{R}^2$ with the standard inner product. If T is a rigid motion on V such that $T(0) = 0$, show that T is a linear map and that it is unitary.
2. Use the result you just proved to show that every rigid motion of $\mathbb{R}^2$ is composed of a translation, followed by a unitary operator.
3. Now show that a rigid motion of $\mathbb{R}^2$ is either a translation followed by a rotation, or a translation followed by a reflection, followed by a rotation.

[2]These problems are taken from *Linear Algebra* by Hoffman and Kunze, 2^{nd} edition

Project 2: The Painter Problem

This project consists of one nonstandard problem on linear algebra. To solve it, you probably only need to know the very basics, but you need to think a lot and to be inventive. Good luck!

A house painter has 7 cans with 7 different paints, but does not have any spare cans. Each can is $\frac{9}{10}$ full. The only way to mix paints is to pour some amount of paint from one can to another and stir thoroughly. The painter wants to prepare a blend containing equal amounts of all 7 paints. Is it possible?

Project 3: Jordan Normal Form

1. Classify up to similarity all $n \times n$ complex matrices A such that $A^n = I$.

2. Find all eigenvectors of an operator $A : \mathbb{C}^2 \to \mathbb{C}^2$ with the following matrix:

$$\begin{pmatrix} 6 & -1 \\ 16 & 2 \end{pmatrix} \tag{2}$$

   What is its Jordan form? Give an example of a basis in $\mathbb{C}^2$ such that the matrix of A in this basis is the Jordan form of A.

3. Recall that the sequence of Fibonacci numbers a_n is given by the following recurrence relation:

$$a_{n+2} = a_{n+1} + a_n \tag{3}$$

   where $a_0 = a_1 = 1$. (So the first few terms are 1,1,2,3,5,8,13...). Find an explicit formula for a_n by completing the following steps:

   - Consider an operator A on the vector space $\mathbb{R}^2$ such that A maps the vector with coordinates (x, y) to the vector $(y, x + y)$. Show that A maps (a_{n-2}, a_{n-1}) to (a_{n-1}, a_n), where a_n is the Fibonacci sequence. Write the matrix A_β of A with respect to the standard basis β and prove that

$$A_\beta^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} \tag{4}$$

   - Diagonalize the operator A (i.e. find its eigenvalues, eigenvectors and show that eigenvectors form a basis in $\mathbb{R}^2$. Denote this basis by δ (So A_δ is diagonal!). Find the transition matrix from β to δ. (Recall P is a transition matrix from A_β to A_δ if $PA_\beta P^{-1} = A_\delta$, i.e. P implements a change-of-basis).
   - Find A_δ^n. Then argue that $A_\beta^n = P^{-1}A_\delta^n P$. Find A_β^n and write an explicit formula for the n-th Fibonacci number.