PRACTICE FINAL: MAT 310 FALL 09

These problems range from easy to hard. This "exam" is more difficult than the actual final. Some problems from this list will be on the final; how many depends on how many are discussed during the review on Friday.

1. If $S, T : V \to V$ are linear maps from an inner product space to itself, prove that

$$(ST)^* = T^* S^*.$$

2. If T is self-adjoint, prove that

$$||(T \pm i)x||^2 = ||Tx||^2 + ||x||^2.$$

3. State the spectral theorem for self-adjoint linear maps $T : V \to V$. Find an orthonormal basis of eigenvectors for the linear map given by the matrix

$$\begin{pmatrix} 2 & 4 \\ 4 & 3 \end{pmatrix}$$

What is the characteristic polynomial of this linear map.

4. If $T : V \to V$ is a linear map on an inner product space satisfying $\langle Tv, v \rangle > 0$ for all v, prove that T is invertible.

5. Let $U \subset \mathbb{C}^3$ be the subspace generated by the vectors $v_1 = (1,1,0)$, $v_2 = (0,-1,1)$, $v_3 = (1,0,1)$.
Find an orthogonal basis for U with respect to the usual dot product.
Find a subspace $W \subset \mathbb{C}^3$ such that $\mathbb{C}^3 = U \oplus W$.

6. Let $\mathcal{F}$ be the vector space of functions $f : \mathbb{R} \to \mathbb{R}$ and let V be the subspace generated by the functions e^x, xe^x, x^2e^x. Let $T : V \to V$ be the operator defined by $T(f) = f' - f$. Choose a basis for V and write the matrix for T in that basis. Is T invertible?

7. Suppose V is an inner product space. Prove that

$$\langle S, T \rangle = trace(ST^*)$$

defines an inner product on $\mathcal{L}(V)$ = the space of linear maps $V \to V$.
Show that in this inner product

$$||T||^2 = \sum ||T_{e_i}||^2,$$

where e_i is any orthonormal basis of V.

8. Given an inner product space V and vectors $v, w \in V$, define $T : V \to V$ by $T(u) = \langle u, v \rangle w$. What is T^*?. Find a formula for $traceT$.

9. Is the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

diagonalizable, i.e. does it have a basis of eigenvectors?

10. Find the eigenvalues and the associated generalized eigenvectors of the linear map associated to the matrix

$$\begin{pmatrix} 2 & 4 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$