MAT 310 FALL 09 HOMEWORK 11

Due Wednesday, December 9

1. Let $T : V \to V$ be a linear map with characteristic polynomial $q(\lambda) = \lambda^n$, with $n = dimV$, V a complex vector space. Prove that T is nilpotent.

Conversely, if T is nilpotent, prove that its characteristic polynomial is λ^n.

2. As above, suppose $T : V \to V$ is nilpotent. If V has a basis of eigenvectors of T prove that $T = 0$, i.e. T is the zero linear map.

3. Suppose $T : V \to V$ is a linear map with distinct eigenvalues given by $(-5, -3, 0, 2, 4)$ with multiplicities given by $(2, 2, 1, 3, 3)$. Suppose the eigenvalues $(-5, -3, 2)$ have 2 linearly independent eigenvectors, while the remaining eigenvalues $(0, 4)$ have only one eigenvector (up to scalar multiples).

Find the possible Canonical Form I of the linear map T, as in Theorem 8.23. What is the characteristic polynomial of T?

4. Let S and T be two linear maps of a vector space V to itself. The *commutator* $[S, T]$ of S and T is defined to be the linear map

$$[S, T] = ST - TS : V \to V.$$

Recall the product here means composition of linear maps.

(a). Show that

$$tr[S, T] = 0.$$

(b). Use (a) to prove that there do not exist any linear operators S, T from V to V as above such that

$$[S, T] = Id,$$

where Id is the identity map on V.

5.Recall the rank-nullity formula from an earlier chapter:

$$dimnullT + dimrangeT = dimV,$$

for $T : V \to V$ and V finite dimensional. Suppose P is a linear map from V to V satisfying

$$P^2 = P.$$

(P is called a projection operator). Use the rank-nullity formula to find a formula relating trP and $dimrangeT$.