ON THE INITIAL BOUNDARY VALUE PROBLEM FOR THE VACUUM EINSTEIN EQUATIONS AND GEOMETRIC UNIQUENESS

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Abstract. We study the initial boundary value problem (IBVP) for the vacuum Einstein equations in harmonic gauge by adding a new field corresponding to the choice of harmonic gauge. For the gauge-type field, both free initial and Dirichlet boundary data as well as initial and boundary data coupled to the metric are analysed and shown to lead to well-posed formulations of the IBVP. In addition, these formulations lead to a solution of the problem of geometric uniqueness, as emphasized by H. Friedrich. In analogy to the solution to the Cauchy problem, we also prove the existence of a unique maximal globally hyperbolic vacuum development of the initial boundary data.

1. Introduction

This article is concerned with the initial boundary value problem (IBVP) for the vacuum Einstein equations on a spacetime M of the form $M \cong I \times S$, where S is a compact 3-manifold with non-empty boundary $\partial S = \Sigma$. The boundary ∂M consists of two parts; the initial surface S and boundary $\mathcal{C} = I \times \Sigma$. These 3-manifolds are glued along their common boundary Σ giving M the structure of a manifold with corner. The initial boundary value problem is the problem of finding Lorentz metrics g on M satisfying the vacuum Einstein equations

$$Ric_g = 0, \tag{1.1}$$

together with prescribed initial conditions along S and boundary conditions along $\mathcal{C}$.

The Cauchy problem for the equation (1.1) has been well-understood since the fundamental work of Choquet-Bruhat [4] and has been extensively studied in the literature, cf. [6], [10], [13] and [21] for example. Given sufficient regularity, the Cauchy problem is well-posed in various choices of gauge and there is a unique maximal globally hyperbolic development of the initial data on S up to isometry. Moreover, the initial data on S are geometric or gauge-invariant; they consist of the specification of a 3-metric and symmetric bilinear form on S, corresponding to the induced metric and second fundamental form induced by the solution g on S.

In comparison, the IBVP has been much less understood, primarily due to gauge problems associated with the choice of boundary data. Ideally, one would like the boundary data to also be geometric, in that it is determined from the gauge-invariant Cauchy data at $\mathcal{C}$, i.e. the induced metric and second fundamental form of $\mathcal{C}$ in (M, g). It remains a basic open problem (not answered here) of whether there is a choice of gauge for which the IBVP is well-posed for some choice of such geometric boundary data.

A number of distinct approaches to the IBVP have been developed. The IBVP was first seriously investigated and proved to have a well-posed solution by Friedrich-Nagy in [9]. This approach is based on the system of Bianchi equations satisfied by the Einstein equations and replaces the metric unknown g by a system comprised of an orthonormal tetrad, associated connection coefficients and Weyl curvature components. The boundary data consist of the mean curvature of the

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boundary, (which is geometric), components of the Weyl curvature at the boundary (which are gauge-dependent) as well as gauge conditions at the boundary.

In [15], cf. also [16], Kreiss-Reula-Sarbach-Winicour prove well-posedness of the IBVP in a harmonic (i.e. wave) coordinate gauge, analogous to the gauge most often used for the Cauchy problem. The boundary conditions here are gauge conditions on the boundary as well as gauge-dependent Sommerfeld-type boundary conditions for the metric. A somewhat simpler set of boundary conditions has been considered by Kreiss-Winicour in [14].

Another approach, well-developed in the literature on numerical relativity is the BSSN formulation, cf. [2]. Further, we mention the recent work of Fournadavlos-Smulevici [7] using a $3+1$ decomposition based on a maximal hypersurface formulation. A comprehensive survey and numerous further references regarding the IBVP for the Einstein equations are given in [22].

A fundamental issue with these approaches to the IBVP, discussed in detail by Friedrich in [11], is the problem of geometric uniqueness. There are two basic related aspects of this problem. The first is the gauge-dependence of the solutions of the IBVP constructed for instance in [9] or [15]; given an arbitrary or random solution $\widetilde{g}$ satisfying given initial and boundary conditions, it is not known in general if such a solution is isometric to a solution constructed in [9] or [15] with the same initial-boundary data. (This is resolved in the case the mean curvature of the boundary cylinder $\mathcal{C}$ is constant in the Friedrich-Nagy framework, cf. [9, Theorem 8.1 and Remark 8.1(i)]). In particular, one does not know whether all vacuum Einstein metrics on M, locally near S, are effectively parametrized by their initial and boundary data. A second, somewhat related, aspect of this problem is the dependence of the boundary data on a choice of unit time-like vector field T on the time-like boundary $\mathcal{C}$. To date, a canonical or geometrically natural choice of T, with corresponding boundary conditions for which the IBVP is well-posed, has not been found. Given two distinct choices T and T' of time-like vector field, it is not clear how to determine whether boundary data associated to T and T' lead to equivalent solutions, i.e. isometric vacuum space-times. This issue cannot be resolved without solving globally for the system and then checking (by inspection).[1]

In this paper, we will only consider formulations of the IBVP using harmonic gauge. Now the choice of a harmonic gauge is not unique; it depends on a suitable choice of boundary data. Thus, working locally for the moment (as in the Cauchy problem), local harmonic or wave coordinates x^α, $\alpha = 0,1,2,3$ with $\Box_g x^\alpha = 0$ and with given natural choices of fixed initial data on S, are only uniquely determined by their boundary data on $\mathcal{C}$. One may choose (for instance) Dirichlet or Sommerfeld-type boundary data for x^α on $\mathcal{C}$. The boundary $\mathcal{C}$ may defined locally as the locus $\{x^1 = 0\}$, so that x^1 is a local defining function for $\mathcal{C}$; this gives a fixed Dirichlet boundary value to x^1. There remain 3 degrees of freedom in the choice of boundary data for x^i, $i = 0,2,3$ on $\mathcal{C}$. This freedom formally corresponds to the freedom in the choice of T above.

There appears to be no general method to remove this freedom by some more canonical choice, (although see the remarks in [11]). Thus, we first take the approach to widen the number of variables or unknowns $g = g_{\alpha\beta}$ and correspondingly increase the number of equations.

To do this, we first formulate the issue globally. As above, let S be a compact connected oriented 3-manifold with nonempty boundary $\partial S = \Sigma$; this will serve as the initial or time zero slice. A typical case is S a 3-ball, with boundary $\Sigma = S^2$. Let $\mathcal{C} = \mathbb{R}^+ \times \Sigma$, with a fixed product structure, and so with a fixed global time function t_0. (This will later serve as the timelike boundary). The initial boundary data set is defined by $T = S \cup \mathcal{C}$, where S and $\mathcal{C}$ are glued along their common boundary Σ. Let M be the space $\mathbb{R}^+ \times S$, so that M is a manifold with corner Σ, and T can be considered in a natural way as the boundary of M. Define the function t_0 on M by $t_0 \circ \pi_1$,

[1]The issue of existence and geometric uniqueness has very recently been resolved by Fournadavlos-Smulevici [8] in the special case of totally geodesic boundary data on $\mathcal{C}$.

where $\pi_1 : \mathbb{R}^+ \times S \to \mathbb{R}^+$ is projection on the first factor. Thus $S = \{t_0 = 0\}$. Throughout the paper, let $S_\tau = \{t_0 = \tau\} \subset M$ and $\Sigma_\tau = S_\tau \cap \mathcal{C}$ denote the level sets of t_0 on M and $\mathcal{C}$ respectively. Throughout the paper, $\mathcal{C}_\tau$ will denote a domain of the boundary manifold of the form $\mathcal{C}_\tau = \{x \in \mathcal{C} : \ 0 \le t_0(x) < \tau\}$, while $\mathcal{T}$ denotes a domain in M, diffeomorphic to M with $\{x \in M : t_0(x) \le \tau_0\} \subset \mathcal{T}$, where τ_0 is a small positive constant. Finally, r_0 will denote a fixed defining function for $\mathcal{C}$ in M, so that $\mathcal{C} = r_0^{-1}(0)$.

On a thickening $\widetilde{M} \supset M$ of M, choose a fixed background smooth complete Riemannian metric g_R. As an example, when $\Sigma = \partial S$ is embedded in $\mathbb{R}^3$ as the boundary of a handlebody, one may choose g_R to be the flat Euclidean metric on $\mathbb{R}^4$. As noted in Remark 4.10 below, the main results described below do not depend on these choices of (t_0, g_R).

Now in addition to Lorentz metrics g, consider wave maps

$$F : (\mathcal{T}, g) \to (\widetilde{M}, g_R) \tag{1.2}$$

coupled to g, i.e. critical points of the Dirichlet energy $\int_{\mathcal{T}} |DF|^2 dV_g$, cf. [12] for instance. Such maps satisfy an equation of the form

$$\Box_g F + \Gamma(F) g(\nabla F, \nabla F) = 0, \tag{1.3}$$

or in local coordinates on the target space,

$$\Box_g F^\alpha + (\Gamma^\alpha_{\beta\gamma} \circ F) g(\nabla F^\beta, \nabla F^\gamma) = 0,$$

where $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of g_R and ∇ is taken with respect to g. (The equation (1.3) expresses the vanishing of the tension field $\tau(F)$ of the map F). Given g, this is a system of semi-linear hyperbolic wave equations for F. For simplicity, we impose Dirichlet boundary conditions for F on the boundary $\mathcal{C} \cap \mathcal{T}$. Moreover, we prescribe initial and boundary data for F so that F is a diffeomorphism onto its image in a small tubular neighborhood of $\mathcal{C} \cap \mathcal{T}$. Thus, locally and near the boundary, F gives a gauge choice of generalized harmonic (or wave) coordinate system depending on the choice of Dirichlet boundary values, as described above with the local chart $\{x^\alpha\}$. We note however that this specification of Dirichlet boundary data for F (which may be considered as gauge source functions in the terminology of [9]) will be removed later, cf. the discussion following Theorem 1.4.

In the bulk $\mathcal{T}$, consider evolution equations for the pair (g, F),

$$\begin{aligned} &Ric_g = 0, \\ &\Box_g F + \Gamma(F) g(\nabla F, \nabla F) = 0. \end{aligned} \tag{1.4}$$

Note that while F is coupled to g, g is not coupled to F, i.e. we are not considering the coupled system of Einstein-wave map equations. The initial data for (g, F) on S are given by

$$\mathbf{I} = \{(g_S, K), (E_0, E_1)\}, \tag{1.5}$$

where g_S is a Riemannian metric and K is a symmetric 2-tensor on S. The pair (g_S, K) is understood as the geometric initial data of g and it satisfies the Hamiltonian and momentum (or Gauss and Gauss-Codazzi) constraint equations:

$$\begin{aligned} &|K|^2 - (\mathrm{tr} K)^2 - R_{g_S} = 0, \\ &\delta[K - (\mathrm{tr} K) g_S] = 0. \end{aligned} \tag{1.6}$$

Here R_{g_S} denotes the scalar curvature of the Riemannian metric g_S on S, while the norm $|.|$, trace tr and divergence δ are all with respect to g_S. The pair (E_0, E_1), assigning initial conditions for F, are given by a map $E_0 : S \to M$ such that $E_0 : S \to S \subset M$ with $E_0|_\Sigma : \Sigma \to \Sigma$ a diffeomorphism, and a vector field $E_1 : S \to (TM)|_S$ transverse to S. The compatibility conditions between the initial and boundary data discussed later then imply that F also induces a diffeomorphism $F|_\Sigma : \Sigma \to \Sigma$.

The boundary data for F are chosen as (free) Dirichlet data. Thus, let $G : \mathcal{C} \to M$ be a map with $G : \mathcal{C} \to \mathcal{C}$ which is the restriction of a diffeomorphism in a thickening of $\mathcal{C}$; we assume $F|_{\mathcal{C}\cap\mathcal{T}} = G|_{\mathcal{C}\cap\mathcal{T}}$ so that $r_0 \circ F = 0$ on $\mathcal{C}\cap\mathcal{T}$. The main issue is then the choice of boundary data for g. This data is coupled to F in that the boundary geometry of g is expressed in terms of the wave map F above. Namely, the level sets $(\Sigma_G)_\tau = G^{-1}(\Sigma_\tau) \subset \mathcal{C}$ and correspondingly $(S_F)_\tau = F^{-1}(S_\tau) \subset M$ induce foliations of $\mathcal{C}$ and $\mathcal{T}$ in a neighborhood of Σ, and the geometry of g is expressed in terms of these foliations, cf. §2 for further details. This is equivalent to decomposing the pull-back metric $(F^{-1})^*g$ with respect to the foliation S_τ near T. As will be seen below, there are a number of natural choices of boundary data for the system (1.4); in this paper, we concentrate on two main classes.

The first collection of free data on the boundary $\mathcal{C}$ consists of data

$$\mathbf{B} = \{G, [\gamma], \Theta\}, \tag{1.7}$$

where as above, $G : \mathcal{C} \to \mathcal{C} \subset M$ is a diffeomorphism on $\mathcal{C}$ which maps the edge Σ to Σ, Θ is a vector field on M restricted to $\mathcal{C}$ and $[\gamma]$ defines a conformal class of Riemannian metrics on the level sets Σ_τ of $\mathcal{C}\cap\mathcal{T}$.The geometric meaning of the vector field Θ is given in (1.10) below. It will always be assumed that G is both an orientation and time-orientation preserving diffeomorphism.

A rough version of the first main theorem is as follows; a more precise statement is given in Theorem 4.1. cf. also Theorem 5.2 and Remark 5.9.

Theorem 1.1. *(Well-Posedness I) The IBVP for the system* (1.4) *with initial data* $\mathbf{I}$ *as in* (1.5) *and boundary data* $\mathbf{B}$ *as in* (1.7) *is well-posed in* C^∞*. Thus, given smooth initial data* (g_S, K) *satisfying the constraint equations* (1.6)*, smooth initial data* (E_0, E_1) *and smooth boundary data* $(G, [\gamma], \Theta)$ *as in* (1.5)-(1.7)*, all defined on* T *and satisfying smooth compatibility conditions at the corner* Σ*, there exists a smooth triple* $(\mathcal{T}, g, F)$*, where* g *is a spacetime metric on a solid cylinder* $\mathcal{T} \subset M$ *and* $F : \mathcal{T} \to \widetilde{M}$ *is a wave map with respect to* g*, such that*

$$\begin{cases} Ric_g = 0 \\ \Box_g F + \Gamma(F) g(\nabla F, \nabla F) = 0 \end{cases} \quad \text{in } \mathcal{T} \tag{1.8}$$

$$\begin{cases} g|_S = g_S, \;\; K_g|_S = K \\ F = E_0, \;\; F_*(T_g) = E_1 \end{cases} \quad \text{on } S \tag{1.9}$$

and

$$\begin{cases} F = G \\ [g_F^t] = [\gamma] \\ F_*(T_g + \nu_g) = \Theta \end{cases} \quad \text{on } \mathcal{C}\cap\mathcal{T}. \tag{1.10}$$

Further, the constructed solution $(\mathcal{T}, g, F)$ *depends continuously on the initial and boundary data.*

In (1.9) the initial conditions for g are that the induced metric $g|_S$ on S equals the prescribed g_S while the second fundamental form $K_g|_S$ of $S \subset (\mathcal{T}, g)$ equals the prescribed K. The initial conditions for F are that F induces the map E_0 on S and the push-forward vector field $F_*(T_g)$ equals the prescribed vector field E_1. Here and thoughout the paper, T_g denotes the future-pointing unit time-like normal of the level set $(S_F)_\tau$ in $\mathcal{T}$ with respect to g. Thus in the initial condition T_g is simply the unit normal to S in $\mathcal{T}$ since $F : S \to S$.

The first boundary condition in (1.10) states that the map $F|_{\mathcal{C}\cap\mathcal{T}} : \mathcal{C}\cap\mathcal{T} \to \mathcal{C} \subset \widetilde{M}$ induced by F equals the restriction $G|_{\mathcal{C}\cap\mathcal{T}}$ of the prescribed diffeomorphism G. The metric g_F denotes the pull-back

$$g_F = (F^{-1})^*g,$$

which is well-defined at least in a neighborhood of $\mathcal{C}$ in $F(\mathcal{T})$. Let g_F^t denote the induced 2-dimensional metric on the level set $\Sigma_\tau \subset \mathcal{C} \cap \mathcal{T}$. Then the second equation in (1.10) states that g_F^t is in the same conformal class as the given conformal class $[\gamma]$. The last boundary equation means the push forward of the vector field $T_g + \nu_g$ by F equals the prescribed vector field Θ. Here and throughout the paper ν_g denotes the outward unit spacelike normal to $(\mathcal{C} \cap \mathcal{T}) \subset (\mathcal{T}, g)$. Observe the last equation in (1.10) can be equivalently written as

$$T_{g_F} + \nu_{g_F} = \Theta,$$

where T_{g_F}, ν_{g_F} denote the timelike unit normal to S_τ and spacelike unit normal to $\mathcal{C}$ in $(\mathcal{T}, g_F)$ respectively.

As noted above, locally F may be viewed as a (generalized) harmonic or wave coordinate chart near Σ. Thus, Theorem 1.1 gives the existence of vacuum Einstein metrics g satisfying boundary conditions which locally are expressed in this chart. This IBVP will be shown to be locally well-posed in Sobolev spaces H^s in Theorem 4.1 below.

Let $\mathrm{Diff}(M)$ be the group of diffeomorphisms ψ of M. A map $\psi : M \to M$ is a diffeomorphism if it extends to a diffeomorphism of an open neighborhood of M into itself. Note that ψ induces diffeomorphisms $\psi_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, $\psi_S : S \to S$ and $\psi_\Sigma : \Sigma \to \Sigma$. The same notion applies to $\mathrm{Diff}(\mathcal{T})$. The group $\mathrm{Diff}(\mathcal{T})$ acts on solutions $(\mathcal{T}, g, F)$ by pull-back:

$$(\psi, (g, F)) \to (\psi^* g, \psi^* F), \tag{1.11}$$

where $\psi^* F = F \circ \psi$. It is important to understand how $\mathrm{Diff}(\mathcal{T})$ acts on the initial and boundary data. Using the fact that $T_{\psi^* g} = (\psi^{-1})^* g$ and similarly for ν, one easily verifies the following transformation rule:

$$\psi^* \mathbf{I} = \{(\psi^* g_S, \psi^* K), (E_0 \circ \psi, E_1\}, \quad \psi^* \mathbf{B} = \{G \circ \psi, [\gamma], \Theta\}. \tag{1.12}$$

It is crucial for the uniqueness results discussed below that the boundary data $([\gamma], \Theta)$ are invariant under the action of the gauge group $\mathrm{Diff}(\mathcal{T})$, i.e. $\mathrm{Diff}(\mathcal{T} \cap \mathcal{C})$. On the other hand, the Dirichlet boundary data G should be regarded as a choice of gauge, in that it transforms non-trivially (although simply) under the action of $\mathrm{Diff}(\mathcal{T} \cap \mathcal{C})$.

Let $\mathrm{Diff}_0(M)$ be the group of diffeomorphisms ψ of M equal to the identity on $T = S \cup \mathcal{C}$ and similarly for $\mathrm{Diff}_0(\mathcal{T})$. The action of $\mathrm{Diff}_0(\mathcal{T})$ preserves both the solutions $(\mathcal{T}, g, F)$ as well as the initial and boundary data, i.e. via (1.12),

$$\psi^*(\mathbf{I}, \mathbf{B}) = (\psi^* \mathbf{I}, \psi^* \mathbf{B}) = (\mathbf{I}, \mathbf{B}) \ \ \forall \ \psi \in \mathrm{Diff}_0(\mathcal{T}).$$

In Theorem 1.3 below, we prove a natural converse statement that solutions $(\mathcal{T}, g, F)$ are uniquely determined by their initial and boundary data on a suitable sub-domain $\mathcal{T}' \subset \mathcal{T}$, up to the action of $\mathrm{Diff}_0(\mathcal{T}')$. More generally, the action of the full group $\mathrm{Diff}(\mathcal{T})$ on the initial boundary data $(\mathbf{I}, \mathbf{B})$ as in (1.12) is equivalent to its action on the space of solutions $(\mathcal{T}, g, F)$; again cf. Theorem 1.3 below for the precise statement.

Note that one has a natural short exact sequence of groups

$$0 \to \mathrm{Diff}_0(\mathcal{T}) \to \mathrm{Diff}(\mathcal{T}) \to \mathrm{Diff}(T \cap \mathcal{T}) \to 0. \tag{1.13}$$

The quotient space $\mathrm{Diff}(T \cap \mathcal{T}) = \mathrm{Diff}(\mathcal{T})/\mathrm{Diff}_0(\mathcal{T})$ has equivalence classes represented by maps $\psi : T \cap \mathcal{T} \to T \cap \mathcal{T}$ which extend to diffeomorphisms $\psi : \mathcal{T} \to \mathcal{T}$ which are either not equal to the identity on $S \subset T$ or not equal to the identity on $\mathcal{C} \cap \mathcal{T} \subset \mathcal{T}$. In particular, one has (non-canonical) inclusions $\mathrm{Diff}(S) \hookrightarrow \mathrm{Diff}(\mathcal{T})$ and $\mathrm{Diff}(\mathcal{C} \cap \mathcal{T}) \hookrightarrow \mathrm{Diff}(\mathcal{T})$.

Focusing on the boundary data on $\mathcal{C} \cap \mathcal{T}$ as in (1.10), if one fixes a boundary gauge $G = G_0$ at $\mathcal{C}$, there are 6 remaining degrees of freedom in the choice of boundary data for the metric $g_{\alpha\beta}$. The group $\mathrm{Diff}(\mathcal{C} \cap \mathcal{T}) \subset \mathrm{Diff}(\mathcal{T})$ has 3 degrees of freedom, leaving then $6 - 3 = 3$ degrees of freedom, roughly corresponding to isometry classes of solutions. Two of these parameters, e.g. the conformal class $[g_F^t]$, correspond to the "gravitational degrees of freedom", leaving then one degree of freedom for evolution of the location of the boundary $\mathcal{C}$.

The equations in (1.8) comprise 14 coupled equations for the 14 unknowns (g, F), $((g_{\alpha\beta}, F^\alpha)$ in components). There are only 10 boundary conditions in (1.10); 4 Dirichlet conditions on F and 6 on g coupled to F; these latter will primarily be viewed as conditions for g (given F). This discrepancy corresponds to the fact that the equation $Ric_g = 0$ is degenerate hyperbolic. As is common and carried out in §2, one adds a gauge term $\delta^* V$ to make the equations (1.8) hyperbolic, giving the gauge reduced Einstein equations. This requires adding the 4 extra boundary conditions $V = 0$ at $\mathcal{C}$ to ensure that solutions of the gauge reduced Einstein equations are actually solutions of the vacuum Einstein equations. Similarly, there are only 20 initial conditions in (1.9) for the 14 unknowns (g, F). For the gauge reduced Einstein equations, the 8 extra components $g_{0\alpha}$ and $\partial_t g_{0\alpha}$ are added to the initial data, subject to the constraint $V = 0$ on S which consists of 4 equations; the action of the diffeomorphism group $\mathrm{Diff}_0(M)$ then accounts for the remaining 4 degrees of freedom; this is described in detail in §2.

Observe that the 6 boundary data $([\gamma], \Theta)$ are Dirichlet boundary conditions on the metric on $\mathcal{C}$, given the local chart F. We note here that there are a number of possible modifications to the boundary conditions $\mathbf{B}$ for which Theorem 1.1 remains valid locally. As a trivial example, one may change the last boundary condition for Θ in (1.10) locally to

$$g_{0\alpha} + g_{1\alpha} = \theta_{\tilde{\alpha}}, \tag{1.14}$$

where $\theta_{\tilde{\alpha}}$ are the components of Θ expressed in some chart χ of $\mathcal{T}$ and $g_{\alpha\beta}$ are the components of g expressed in the chart $\chi \circ F$, cf. Remark 4.7. However, for many or most of these possible modifications, it may not be possible to extend the local existence to existence of solutions in a full domain $\mathcal{T}$ containing S; it is the invariance of the boundary conditions $([\gamma], \Theta)$ in (1.10) which makes this possible.

It is of basic interest to understand if the boundary conditions (1.7) can be made more geometric, in particular whether the 4 degrees of freedom in the choice of Θ can be reduced to 3 (or less). Using harmonic gauges, it appears to be unlikely that they can be made "fully geometric", (in that the boundary data is expressed completely in terms of the induced metric and second fundamental form of the boundary $\mathcal{C}$), but we present below a class of more geometric formulations based on the mean curvature of various slices of $\mathcal{C}$.

Prescribe then a collection of boundary data on $\mathcal{C}$

$$\mathbf{B}_{\mathcal{C}} = \{G, [\gamma], H, \Theta_{\mathcal{C}}\}, \tag{1.15}$$

where G, $[\gamma]$ have the same meaning as in the $\mathbf{B}$ boundary data, H is a scalar field on $\mathcal{C}$ and $\Theta_{\mathcal{C}}$ is a vector field tangent to $\mathcal{C}$. The function H prescribes a combination of various mean curvature terms at $\mathcal{C}$; we refer to Proposition 3.4 for the exact definition. The vector field $\Theta_{\mathcal{C}}$ is a projection of Θ to $T\mathcal{C}$, so that $\Theta_{\mathcal{C}}$ is intrinsic to the boundary $\mathcal{C}$, in contrast to the boundary condition Θ in (1.7).

Pairing with $\mathbf{B}_{\mathcal{C}}$, we define a collection of initial data on S

$$\mathbf{I} = \{(g_S, K), (E_0, E_1)\}, \tag{1.16}$$

of exactly the same type as in (1.5).

Theorem 1.2. *(Well-Posedness II) The IBVP for the system* (1.4) *with boundary data* $\mathbf{B}_{\mathcal{C}}$ *as in* (1.15) *and initial data* $\mathbf{I}$ *as in* (1.16) *is well-posed in* C^∞*. Thus, given smooth initial data* (g_S, K) *satisfying the constraint equations* (1.6)*, smooth initial data* (E_0, E_1) *and smooth boundary data* $(G, [\gamma], H, \Theta)$ *as in* (1.16)-(1.15)*, all defined on* T *and satisfying smooth compatibility conditions at* *the corner Σ, there exists a smooth pair (g, F), where g is a spacetime metric on the solid cylinder $\mathcal{T}$ and $F : \mathcal{T} \to \widetilde{M}$ is a wave map with respect to g, such that*

$$\begin{cases} Ric_g = 0 \\ \Box_g F + \Gamma(F) g(\nabla F, \nabla F) = 0 \end{cases} \quad \text{in } \mathcal{T} \tag{1.17}$$

$$\begin{cases} g|_S = g_S, \;\; K_g|_S = K \\ F = E_0, \;\; F_*(T_g) = E_1 \end{cases} \quad \text{on } S \tag{1.18}$$

and

$$\begin{cases} F = G \\ [g_F^t] = [\gamma] \\ H_{g_F} = H \\ F_*(T_g + \nu_g)^T = \Theta_{\mathcal{C}} \end{cases} \quad \text{on } \mathcal{C} \cap \mathcal{T}. \tag{1.19}$$

Further, the constructed solution $(\mathcal{T}, g, F)$ depends continuously on the initial and boundary data.

Here the notation is the same as in (1.8)-(1.10), where in addition the boundary data H_{g_F} is a linear combination of different types of mean curvature measured on the boundary (cf. (3.35) and Proposition 3.4). In the last boundary equation the superscript $(\cdot)^T$ denotes the projection of a vector to $\mathcal{C}$ with respect to g_R, i.e.

$$F_*(T_g + \nu_g)^T = F_*(T_g + \nu_g) - g_R(F_*(T_g + \nu_g), \nu_{g_R}) \cdot \nu_{g_R}, \tag{1.20}$$

where ν_{g_R} is the outward unit normal to $(\mathcal{C} \cap \mathcal{T}) \subset (\mathcal{T}, g_R)$.

We note Theorems 1.1 and 1.2 hold with $T_g + \nu_g$ replaced by $\lambda T_g + \mu \nu_g$ for any smooth functions $0 < \lambda, \mu < \infty$, cf. Remark 3.5. The allowable choices for H_{g_F} however depend on λ, μ. Also, in Theorem 1.2, T_g may be replaced by the time-like unit normal to the foliation $(\Sigma_G)_\tau$ within $T\mathcal{C}$, cf. the discussion preceding Proposition 2.7. As above, this IBVP is proved to be locally well-posed in Sobolev spaces H^s in Theorem 4.3.

The proofs of Theorems 1.1 and 1.2 rely on the existence of strong or boundary stable energy estimates for the localized or frozen coefficient system which are derived in §3. These together with basically standard methods from the theory of quasi-linear hyperbolic systems of IBVP's are used to establish local versions of Theorems 1.1 and 1.2, i.e. existence (and gauged-uniqueness) in sufficiently small neighborhoods of a corner point $p \in \Sigma$, cf. Theorems 4.1 and 4.3 respectively. It is then shown in §5 that such local solutions can be patched or glued together to obtain the full solutions (g, F) on a solid cylinder $\mathcal{T}$ containing S. For this gluing process, the role of the wave map F is crucial.

It is worth pointing out that the local existence statement in Theorems 1.1 and 1.2 holds without introduction of the wave map F. Thus, the proof of Theorem 1.1 gives also local existence of solutions of the IBVP

$$\begin{aligned} &Ric_g = 0 \;\text{ in }\; U \\ &(g|_S, K_g|_S) = (g_S, K) \;\text{ on }\; S \cap U \\ &[g^t] = [\gamma], \; T_g + \nu_g = \Theta \;\text{ on }\; \mathcal{C} \cap U, \end{aligned}$$

for U a sufficiently small (depending on the initial-boundary data $(g_S, K, [\gamma], \Theta)$) neighborhood of a boundary point $p \in \Sigma$. Here one chooses a *fixed* foliation S_τ near $\mathcal{C}$. The same result holds with respect to the $\mathbf{B}_{\mathcal{C}}$ boundary data. The construction of such solutions is discussed in Remark 4.4. With very special choices of an atlas, it is also possible to glue together such local solutions to larger domains, cf. Remark 5.3. However, it does not seem at all possible to extend this to more general gluings. More importantly, it does not seem at all possible to prove that such a construction gives rise to all possible local vacuum Einstein metrics satisfying the initial-boundary conditions, i.e. to prove that a general solution can be brought into such a form via diffeomorphisms in $\mathrm{Diff}_0(U)$. This is essentially the problem of geometric uniqueness mentioned above.

One of the main reasons for introduction of the wave map F, and one of the main consequences of the results above, is that it is possible to resolve, in a relatively simple way, the geometric uniqueness problem, (and the related gluing problem), for both sets of data $(\mathbf{I}, \mathbf{B})$ and $(\mathbf{I}, \mathbf{B}_{\mathcal{C}})$.

First we discuss $\mathbf{B}$ boundary data. Define two collections of initial data $\mathbf{I}_i = \{((g_S)_i, K_i, (E_0)_i, (E_1)_i\}$ and boundary data $\mathbf{B}_i = \{G_i, [\gamma_i], \Theta_i\}$, $(i = 1, 2)$ to be *equivalent* if there is a diffeomorphism $\psi \in \mathrm{Diff}(T)$, with $\psi(S) = S$ and $\psi(\mathcal{C}) = \mathcal{C}$, such that

$$\psi^*(\mathbf{I}_2, \mathbf{B}_2) = (\mathbf{I}_1, \mathbf{B}_1), \tag{1.21}$$

where the action $\psi^*(\mathbf{I}_2, \mathbf{B}_2)$ is as given in (1.12).

Theorem 1.3. *(Geometric Uniqueness I) Let* $(\mathcal{T}_1, g_1, F_1)$ *and* $(\mathcal{T}_2, g_2, F_2)$ *be two solutions of the system* (1.8)-(1.10) *with respect to initial and boundary data* $(\mathbf{I}_1, \mathbf{B}_1)$ *and* $(\mathbf{I}_2, \mathbf{B}_2)$ *respectively. If* $(\mathbf{I}_1, \mathbf{B}_1)$ *and* $(\mathbf{I}_2, \mathbf{B}_2)$ *are equivalent in the sense of* (1.21)*, then there are domains* $\mathcal{M}_1 \subset \mathcal{T}_1$ *and* $\mathcal{M}_2 \subset \mathcal{T}_2$*, diffeomorphic to solid cylinders and with* $S \subset \mathcal{M}_1$ *and* $S \subset \mathcal{M}_2$*, such that* $\Psi^* g_2 = g_1$ *and* $\Psi^* F_2 = F_1$ *for some diffeomorphism* $\Psi : \mathcal{M}_1 \to \mathcal{M}_2$ *with* $\Psi|_{T \cap \mathcal{M}_1} = \psi|_{T \cap \mathcal{M}_1}$.

In particular, $(\mathcal{T}_1, g_1)$ and $(\mathcal{T}_2, g_2)$ have a common isometric domain. We note that this is an exact analog of the uniqueness for solutions of the Cauchy problem. Note also that the converse of Theorem 1.3 is obvious.

A similar equivalence relation is defined for initial and boundary data $(\mathbf{I}, \mathbf{B}_{\mathcal{C}})$ as above, and the same geometric uniqueness result holds:

Theorem 1.4. *(Geometric Uniqueness II) Let* $(\mathcal{T}_1, g_1, F_1)$ *and* $(\mathcal{T}_2, g_2, F_2)$ *be two solutions of the system* (1.17)-(1.19) *with respect to initial and boundary data* $(\mathbf{I}_1, (\mathbf{B}_{\mathcal{C}})_1)$ *and* $(\mathbf{I}_2, (\mathbf{B}_{\mathcal{C}})_2)$ *respectively. If* $(\mathbf{I}_1, (\mathbf{B}_{\mathcal{C}})_1)$ *and* $(\mathbf{I}_2, (\mathbf{B}_{\mathcal{C}})_2)$ *are equivalent in the sense of* (1.21)*, where* $\psi^* \mathbf{B}_{\mathcal{C}} = \{G \circ \psi, [\gamma], H, \Theta_{\mathcal{C}}\}$*, then there are domains* $\mathcal{M}_1 \subset \mathcal{T}_1$ *and* $\mathcal{M}_2 \subset \mathcal{T}_2$*, diffeomorphic to solid cylinders and with* $S \subset \mathcal{M}_1$ *and* $S \subset \mathcal{M}_2$*, such that* $\Psi^* g_2 = g_1$ *and* $\Psi^* F_2 = F_1$ *for some diffeomorphism* $\Psi : \mathcal{M}_1 \to \mathcal{M}_2$ *with* $\Psi|_{T \cap \mathcal{M}_1} = \psi|_{T \cap \mathcal{M}_1}$.

Although phrased independently, the proofs of Theorems 1.1 and 1.3, and those of Theorems 1.2 and 1.4, given in §4 and §5, are not independent of each other but are closely interrelated.

We note that the results above for the $\mathbf{B}$ and $\mathbf{B}_{\mathcal{C}}$ boundary conditions hold also for other choices of (related) boundary data. The issue of identifying or classifying all possible such boundary data for which these results hold (with for instance Dirichlet boundary data for F), is beyond the scope of this work.

As pointed out to us recently by H. Friedrich and J. Smulevici, Theorems 1.3 and 1.4 do not establish the geometric uniqueness property for the vacuum metric g itself. For instance two solutions (g, F_1) and (g, F_2) may be metrically the same (or isometric), and yet may not be equivalent in the sense above; thus the equivalence relation does not capture the relation of isometry of vacuum solutions on suitable subdomains.

To address this issue, it is useful to change the perspective somewhat. Dirichlet boundary data for F (i.e. gauge source functions in the terminology of [9]) will no longer be freely specified in advance, but instead will be (indirectly) coupled to g. In addition, the initial data for F will be also be coupled equivariantly to g; here equivariance is with respect to the action of diffeomorphisms on the pair (g, F) as in (1.11).

We first discuss the choice of initial data for F. For this, we use a slice to the action of diffeomorphisms on the space of metrics on a surface Σ given by the uniformization theorem. For $\Sigma = S^2$, fix three distinct points $p_i \in S^2$ (to break the action of the conformal group of S^2). For

$\Sigma = T^2$, fix one point $p \in T^2$ (to break the action of translations). For Σ of higher genus, such base points are not needed. Let $\mathrm{Diff}'(\Sigma)$ be the group of diffeomorphisms of Σ, homotopic to the identity, fixing $\{p_i\}$ in the case of S^2 and fixing p in the case of T^2. By the uniformization theorem for surfaces, for any metric γ on Σ, there is a unique diffeomorphism $E_\gamma \in \mathrm{Diff}'(\Sigma)$ such that the pullback metric $(E_\gamma^{-1})^*\gamma$ is pointwise conformal to a space-form metric γ_0 on Σ; $\gamma = E_\gamma^*(\lambda^2\gamma_0)$. Here γ_0 is the round metric on $S^2(1)$ for $\Sigma = S^2$, while γ_0 is a quotient of the Euclidean plane $\mathbb{R}^2$ or hyperbolic plane $\mathbb{H}^2$ by a lattice in case Σ has genus at least zero. In suitable function space topologies, the mapping $\gamma \to E_\gamma$ is smooth.

Next, given an arbitrary Riemannian metric g_S on S, let ν_{g_S} be the inward pointing unit vector field normal to the equidistant foliation from $\partial S = \Sigma$ in S, defined in a neighborhood $V \subset S$ of Σ. Define $E_{g_S} : V \to S$ by

$$(E_{g_S})_*(\nu_{g_S}) = \partial_{r_0}, \tag{1.22}$$

with $E_{g_S}|_\Sigma = E_\gamma$, where $\gamma = g_S|_\Sigma$; as noted above, r_0 is fixed defining function for $\partial\Sigma$ in S. Clearly E_{g_S} is a diffeomorphism onto its image in S and note that E_{g_S} depends smoothly on g_S. This gives the coupling of the (previously free) initial data for F to the initial data g_S of the metric g.

The choice of the coupling (1.22) is not unique. The main reason for the choice (1.22) is that it satisfies the following equivariance property: for any $\psi \in \mathrm{Diff}'(S) := \{\psi \in \mathrm{Diff}(S) : \psi|_\Sigma \in \mathrm{Diff}'(\Sigma)\}$

$$E_{\psi^* g_S} = \psi^* E_{g_S} = E_{g_S} \circ \psi. \tag{1.23}$$

Next we turn to the boundary data. The $\mathbf{B}_{\mathcal{C}}$ boundary data is more natural or geometric in this context, so we focus on this case. Thus, let $\mathcal{U}$ be an open neighborhood of $\mathcal{C}$ in M which is a thickening of the boundary, i.e. $\mathcal{U} \cong [0, r_0) \times \mathcal{C}_\tau$. Consider the following system for the pair (g, φ_g):

$$\begin{cases} Ric_g = 0 \\ \Box_g \varphi_g + \Gamma(\varphi_g) g(\nabla\varphi_g, \nabla\varphi_g) = 0 \end{cases} \quad \text{in } \mathcal{U} \tag{1.24}$$

$$\begin{cases} g|_S = g_S,\ K_g|_S = K \\ \varphi_g = E_{g_S},\ (\varphi_g)_*(T_g) = T_{g_R} \end{cases} \quad \text{on } S \cap \mathcal{U} \tag{1.25}$$

and

$$\begin{cases} [(\varphi_g^{-1})^* g^t] = [\gamma] \\ H_{(\varphi_g^{-1})^* g} = H \\ (\varphi_g)_*(T_g + \nu_g)^T = T^c_{g_R} \end{cases} \quad \text{on } \mathcal{C} \cap \mathcal{U}. \tag{1.26}$$

Here the notation is the same as discussed above; the field T_{g_R} is the unit normal to S with respect to the background metric g_R while $T^c_{g_R}$ is the future-pointing unit normal to $\Sigma_\tau \subset (\mathcal{C}, g_R)$. The initial data E_{g_S} for the wave map φ_g (replacing F from before) is constructed as above with respect to the given initial data (S, g_S).

Observe there is no Dirichlet boundary data for φ_g. As mentioned following (1.12), Dirichlet boundary data is regarded as "pure-gauge" boundary data. It will be shown in §5 that when $([\gamma], H)$ is fixed, different choices of Dirichlet boundary data for φ_g only result in different solutions (g, φ_g) in the same equivalence (and so isometry) class.

Given the fixed choice $T^c_{g_R}$ (representing a fixed choice for $\Theta_{\mathcal{C}}$ in (1.19)), as shown below the subsystem of (1.24)-(1.26) obtained by fixing an arbitrary g has a unique solution φ_g. The equivariance property (1.23) implies that φ_g also transforms equivariantly, i.e.

$$\varphi_{\Psi^* g} = \varphi_g \circ \Psi, \tag{1.27}$$

for any $\Psi \in \mathrm{Diff}'(\mathcal{U}) = \{\Psi \in \mathrm{Diff}(\mathcal{U}) : \Psi|_\Sigma \in \mathrm{Diff}'(\Sigma)\}$. The particular choice $T^c_{g_R}$ is mainly for convenience; any other choice for the value of $\Theta_{\mathcal{C}}$ satisfying the compatibility conditions would also give rise to a unique φ_g. Moreover, the results to follow show that for any such choice of $\Theta_{\mathcal{C}}$, any vacuum metric g (together with its unique φ_g) can be constructed from the system above by assigning the appropriate boundary data $([\gamma], H)$. We thus view the pair $([\gamma], H)$ as the free geometric boundary data in (1.24)-(1.26). In comparison with the discussion following (1.12), note that *all* the boundary conditions in (1.26) are invariant under the action of diffeomorphisms of $\mathcal{C}$ mapping Σ to Σ.

In the following and throughout the paper, we use $(\mathbb{I}, \mathbb{B})$ to denote the free geometric initial and boundary data in (1.25)-(1.26),

$$\mathbb{I} = (g_S, K),\ \mathbb{B} = ([\gamma], H). \tag{1.28}$$

Theorem 1.5. *(Well-Posedness III) The IBVP for the coupled system* (1.24)-(1.26) *with initial data* $\mathbb{I} = (g_S, K)$ *and boundary data* $\mathbb{B} = ([\gamma], H)$ *is well-posed in* C^∞. *Thus, given smooth initial data* (g_S, K) *satisfying the constraint equations* (1.6), *and smooth boundary data* $([\gamma], H)$, *both defined on* T *and satisfying smooth compatibility conditions at the corner* Σ, *there exists a smooth pair* $(\mathcal{T}, g)$, *where* g *is a spacetime metric on a solid cylinder* $\mathcal{T} \subset M$, *such that*

$$Ric_g = 0 \quad \text{in } \mathcal{T} \tag{1.29}$$

$$g|_S = g_S,\ K_g|_S = K \quad \text{on } S \tag{1.30}$$

$$\begin{cases} [(\varphi_g^{-1})^* g^t] = [\gamma] \\ H_{(\varphi_g^{-1})^* g} = H \end{cases} \quad \text{on } \mathcal{C} \cap \mathcal{T} \tag{1.31}$$

where φ_g *is the unique wave map associated to* g *on some neighborhood* $\mathcal{U} \cong [0, r_0) \times (\mathcal{C} \cap \mathcal{T})$ *of the boundary* $\mathcal{C}$ *in* $\mathcal{T}$ *such that*

$$\Box_g \varphi_g + \Gamma(\varphi_g) g(\nabla\varphi_g, \nabla\varphi_g) = 0 \quad \text{in } \mathcal{U} \tag{1.32}$$

$$\varphi|_S = E_{g_S},\ \varphi_*(T_g) = T_{g_R} \quad \text{on } S \cap \mathcal{U} \tag{1.33}$$

$$\begin{cases} (\varphi_g)_*(T_g + \nu_g)^T = T^c_{g_R} \\ r_0 \circ \varphi_g = 0 \end{cases} \quad \text{on } \mathcal{C} \cap \mathcal{U}. \tag{1.34}$$

Further, the constructed solution $(\mathcal{T}, g)$ *depends continuously on the initial and boundary data.*

Note that φ_g is only defined in a neighborhood $\mathcal{U}$ of the boundary $\mathcal{C} \cap T$. This is sufficient to provide a foliation on $\mathcal{C}$ in which to describe the geometry of g. Thus φ_g can be understood as a preferred gauge near the boundary assigned in a unique way to each metric g.

Note also that the first boundary equation (1.34) has a dual interpretation. On the one hand, it provides well-posed Sommerfeld-type boundary data for the map φ_g. Allowing the datum $T^c_{g_R}$ to vary over all choices of $\Theta_{\mathcal{C}}$ as in (1.19), one sees that Dirichlet data for φ_g are determined as the value of the corresponding Sommerfeld-to-Dirichlet map for solutions φ_g of (1.32)-(1.34). (Recall that the condition $r_0 \circ F = 0$ on $\mathcal{C}$ is kept the same here). On the other hand, given the map φ_g, the first boundary condition (1.34) can be viewed as boundary data for the metric g; this gives 3 degrees of freedom when varying $T^c_{g_R}$ which, when combined with the 3 degrees of freedom in $([\gamma], H)$, gives the (expected) 6 degrees of freedom for the boundary data of g on $\mathcal{C}$.

The 3 degrees of freedom in the free boundary data (1.31) corresponds roughly to the degeneracy due to the action of boundary diffeomorphisms $\mathrm{Diff}(\mathcal{C})$ on the space of vacuum metrics. Thus the Dirichlet boundary values of the diffeomorphisms φ_g roughly parametrize $\mathrm{Diff}(\mathcal{C})$; the remaining "free" boundary data $([\gamma].H)$ then roughly parametrize the space of vacuum solutions modulo isometry, i.e. modulo the action of $\mathrm{Diff}(\mathcal{C} \cap \mathcal{T}) \subset \mathrm{Diff}(\mathcal{T})$; this is analogous to the discussion following (1.13).

The uniqueness statement for Theorem 1.5, i.e. the analog of Theorem 1.4 also holds here; this addresses the geometric uniqueness issue of Friedrich [11].

Theorem 1.6. *(Geometric Uniqueness III) Let $(\mathcal{T}_1, g_1)$ and $(\mathcal{T}_2, g_2)$ be two solutions of the system (1.29)-(1.31) with respect to initial and boundary data $((\mathbb{I}_1, \mathbb{B}_1)$ and $(\mathbb{I}_2, \mathbb{B}_2)$ respectively. If $(\mathbb{I}_1, \mathbb{B}_1)$ and $(\mathbb{I}_2, \mathbb{B}_2)$ are equivalent in the sense that there exists a diffeomorphism $\psi \in \mathrm{Diff}'(S)$ such that*

$$\{\psi^*(g_S)_1, \psi^* K_1, [\gamma_1], H_1\} = \psi^*(\mathbb{I}_1, \mathbb{B}_1) = (\mathbb{I}_2, \mathbb{B}_2) = \{(g_S)_2, K_2, [\gamma_2], H_2\}, \tag{1.35}$$

then there are domains $\mathcal{M}_1 \subset \mathcal{T}_1$ and $\mathcal{M}_2 \subset \mathcal{T}_2$, diffeomorphic to solid cylinders and with $S \subset \mathcal{M}_1$ and $S \subset \mathcal{M}_2$, such that

$$\Psi^* g_2 = g_1$$

for some diffeomorphism $\Psi : \mathcal{M}_1 \to \mathcal{M}_2$. In addition, $\Psi|_S = \psi$ and $\Psi|_{\mathcal{U}} = \varphi_{g_2}^{-1} \circ \varphi_{g_1}|_{\mathcal{U}}$ where $\mathcal{U}$ is a neighborhood of $\mathcal{C} \cap \mathcal{M}_1$ and $\varphi_{g_1}, \varphi_{g_2}$ are the unique diffeomorphisms associated to g_1, g_2 via (1.32)-(1.34).

Here the diffeomorphism ψ must belong to $\mathrm{Diff}'(S)$. It remains open if this can be generalized to $\psi \in \mathrm{Diff}(S)$, since for general $\psi \in \mathrm{Diff}(S)$, one may lose track of the boundary data when transforming from g to $\psi^* g$. We refer to Remark 5.8 for more detail.

Analogs of Theorems 1.5 and 1.6, with the same proofs, also hold with respect to $\mathbf{B}$ boundary data, where one replaces the boundary data $([\gamma], H)$ by $([\gamma], \eta)$, where η is the normal component

$$\eta = g_R((\varphi_g)_*(T_g + \nu_g), \nu_{g_R}); \tag{1.36}$$

see Remark 5.10.

We point out that the proofs of Theorems 1.5-1.6 rely on the "free-gauge-source" results Theorems 1.1-1.4.

The results above on the structure of solutions $(\mathcal{T}, g)$ to the IBVP closely parallel the well-known results on the structure of solutions $(\mathcal{T}_S, g)$ to the Cauchy problem on S. In [5], Choquet-Bruhat and Geroch proved the existence of a unique maximal globally hyperbolic vacuum development $(\widetilde{\mathcal{M}}_S, \widetilde{g}_S)$ of the Cauchy problem on $(S, \mathbb{I})$. We show that their techniques extend to cover the analogous result for the IBVP.

Theorem 1.7. *Given an initial boundary data set $(T, \mathbb{I}, \mathbb{B})$, there exists a unique (up to isometry) maximal globally hyperbolic vacuum development $(\widetilde{\mathcal{M}}, \widetilde{g})$. The development $(\widetilde{\mathcal{M}}, \widetilde{g})$ is an extension of any other globally hyperbolic vacuum development of $(T, \mathbb{I}, \mathbb{B})$.*

We refer to the discussion in §5 regarding globally hyperbolic manifolds with boundary and related issues, cf. Theorem 5.14.

The contents of the paper are briefly as follows. In §2, we discuss in detail the gauge reduced Einstein equations and derive the corresponding frozen coefficient linear system, for both sets of initial boundary conditions $(\mathbf{I}, \mathbf{B})$ and $(\mathbf{I}, \mathbf{B}_{\mathcal{C}})$. In §3, we derive the requisite energy estimates for these linear systems, based on Sommerfeld and Dirichlet energy estimates. These are then used in §4 to prove local existence and well-posedness of the gauge reduced and ungauged IBVP's. We also prove local versions of Theorems 1.1-1.4 in §4. Building on these prior results, the main section of the paper, Section 5, then discusses the gluing of local solutions to obtain the global solutions of the IBVP's, both in the context of pairs (g, F) (Theorems 1.1-1.4) as well as in the context of vacuum solutions g (Theorems 1.5-1.7). Finally in the Appendix, §6, we collect and derive a number of results used in the main text.

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2. Initial Set-up

In this section, we discuss details of the formulations in Theorems 1.1 and 1.2. In particular following the standard localization or frozen coefficient method of proving well-posedness, we set up the linearizations of these problems at background flat solutions. The data $(\mathbf{I}, \mathbf{B})$ (Theorem 1.1) is discussed first, and then followed with a similar analysis for the data $(\mathbf{I}_{\mathcal{C}}, \mathbf{B}_{\mathcal{C}})$ (Theorem 1.2). The following conventional index notation will be used throughout: Greek letters $\alpha \in \{0, 1, 2, 3\}$, lower case Roman indices $i \in \{1, 2, 3\}$ while upper case Roman $A \in \{2, 3\}$. Similarly, the Einstein summation convention that repeated indices are summed will always be used.

As discussed in the Introduction, we first work locally and in a neighborhood $U \subset M$ of an arbitrary corner point $p \in \Sigma = S \cap \mathcal{C}$. Choose standard Cartesian coordinates $(x_0^\alpha = \{(t_0, x_0^i)\})$ on $\mathbb{R}^4$. The standard corner domain is given by $\mathbf{R} = \{t_0 \geq 0, x_0^1 \leq 0\}$, so that $\partial_1 = \partial_{x_0^1}$ is outward pointing and $\partial_0 = \partial_{t_0}$ is future pointing.

For U as above, assume that U is in the domain of a chart $\chi : U \to \mathbf{R}$ such that $\chi(p) = 0$. In addition, χ carries the boundary $\mathcal{C} \cap U$ to the locus $\{x_0^1 = 0\}$ and carries the initial surface $S \cap U$ to the locus $\{t_0 = 0\}$. The corner Σ is thus mapped to a flat domain in $\mathbb{R}^2$ with coordinates x_0^A. In addition, it will always be assumed that the time function t_0 in the chart χ equals the fixed time function t_0 choosen on M. In the following we call a local chart which satisfies the conditions above a standard corner chart at p.

To solve for a local solution of the system (1.8)-(1.10) on U, we will expand the coordinate-free system to a system of (nonlinear) hyperbolic equations with complete initial and boundary conditions in the chart χ. To begin, according to common practice we introduce a local gauge condition. Let V be the vector field on U given by

$$V = (\Box_g x_0^\alpha) \cdot \partial_{x_0^\alpha}. \tag{2.1}$$

(The field V may be viewed as the tension field of the identity map $Id : (U, g) \to (\mathbb{R}^4, g_R)$, cf. [12] for example). When $V = 0$, the coordinates x_0^α of χ are harmonic (wave) coordinates with respect to g. For the data $(\mathbf{I}, \mathbf{B})$ (Theorem 1.1), we then consider the following system of reduced Einstein equations coupled to the gauge-type field F:

$$\begin{cases} Ric_g + \delta_g^* V = 0 \\ \Box_g F + \Gamma(F) g(\nabla F, \nabla F) + F_*(V) = 0 \end{cases} \quad \text{in } U \tag{2.2}$$

with initial conditions:

$$\begin{cases} g = q, \quad \frac{1}{2}\mathcal{L}_{\partial_{t_0}} g = k, \quad V = 0 \\ F = E_0, \; F_*(T_g) = E_1 \end{cases} \quad \text{on } U \cap S \tag{2.3}$$

and boundary conditions:

$$\begin{cases} V = 0 \\ F = G \\ [g_F^t] = [\gamma] \\ F_*(T_g + \nu_g) = \Theta \end{cases} \quad \text{on } U \cap \mathcal{C}. \tag{2.4}$$

It is well-known that the first equation in (2.2) is of the form

$$\Box_g g_{\alpha\beta} + Q_{\alpha\beta}(g, \partial g) = 0, \tag{2.5}$$

where Q is quadratic in g and ∂g. When the equations for (g, F) are uncoupled by fixing either the coefficients g or F, one obtains a system of wave-type equations for the unknown F or g.

Here we regard F as a map from U to another open set $U' \subset M$, cf. also Remark 2.1(ii) below. When the initial and boundary Dirichlet data E_0, G are given, U' is understood as an open neighborhood of the target corner point $p' = E_0(p) = G(p) \in M$. Local representations F_{loc} of F are given by $F_{loc} = \chi' \circ F \circ \chi^{-1} : \chi(U) \subset \mathbf{R} \to \mathbf{R}$, where χ' is a standard corner chart at p'. The term $F_*(V)$ in the second equation in (2.2) is introduced to simplify the form of the linearization, cf. (2.9) below.

The initial and boundary conditions (2.3)-(2.4) are understood to be the restriction of equations (1.9)-(1.10) to U plus gauge choices on the (local) initial and boundary surface. In (2.3), q is a 4-dimensional Lorentz metric on M restricted to S while k is a 4-dimensional symmetric bilinear form on M restricted to S. The pair (q, k) is understood as an extension of the 3-dimensional geometric initial data (g_S, K) via a certain choice of gauge (lapse and shift). The initial condition $V = 0$ in (2.3) is an implicit restriction on the choice of the initial data (q, k). We write it explicitly to emphasize this gauge condition.

We recall that $(\mathbf{I}, \mathbf{B})$ denotes the geometric (or ungauged) initial boundary data as in (1.9)-(1.10). In contrast, we use (I, B) to denote the local gauged initial boundary data as in (2.3)-(2.4), i.e. $I = \{(q, k), (E_0, E_1)\}$, $B = \{G, [\gamma], \Theta\}$ on the initial and boundary surface of U.

Throughout the paper, we will make the following assumptions on the data (I, B). This in return yield restrictions on the choice of global initial and boundary data $(\mathbf{I}, \mathbf{B})$ in Theorem 1.1.

- The gauge independent Riemanian metric g_S and symmetric 2-tensor K in (1.9) satisfy the constraint equations (1.6);
- The gauge dependent Lorentzian metric q and symmetric 2-tensor k are chosen so that the induced metric on $S \cap U$ by q is equal to the restriction $(g_S)|_{S\cap U}$ on $S \cap U$, i.e. $q_{i_0 j_0} = (g_S)_{i_0 j_0}$ in the standard corner chart χ;
- The extrinsic curvature of the initial surface $(S \cap U) \subset (U, g)$ computed from (q, k) is equal to the restriction $K|_{S\cap U}$ of the geometric initial data K. In the chart χ, this can be expressed as
$$K_{i_0 j_0} = \nabla_{i_0}(T_g)_{j_0} = \partial_{i_0}\big((T_g)_{j_0}\big) - (\Gamma_g)^{\alpha}_{i_0 j_0}(T_g)_\alpha,$$
where the unit timelike normal $(T_g)_{\alpha_0}$ and the Christoffel symbol $(\Gamma_g)^{\gamma_0}_{\alpha_0\beta_0}$ are uniquely determined by $q_{\alpha_0\beta_0}, k_{\alpha_0\beta_0}$.
- Moreover, we assume $(q_{\alpha_0\beta_0}, k_{\alpha_0\beta_0})$ are chosen so that $\Box_g x_0^\alpha = 0$ on S, which is exactly the gauge constraint $V = 0$ listed explicitly in (2.3).
- The map $E_0|_U : S \cap U \to S \cap U'$ in (2.3) is an orientation preserving diffeomorphism onto its image in S. (Correspondingly, we assume the global initial data $E_0 : S \to S$ is diffeomorphism in a neighborhood of Σ).
- The vector field $E_1|_U : S \cap U \to E_0^*(TM)|_{S\cap U'}$ is transverse to S.
- The map $G|_U : \mathcal{C} \cap U \to \mathcal{C} \cap U'$ in (2.4) is the restriction of a diffeomorphism $\mathcal{C} \to \mathcal{C}$ which is both orientation and time-orientation preserving.
- The initial data I and boundary data B satisfy C^k compatibility assumptions at the corner Σ, for suitable $k \geq 1$.

The C^k compatibility conditions are the relations induced between the initial data I and boundary data B at the corner Σ by a solution (g, F) of the system (2.2)-(2.4) which is $C^k \times C^{k+1}$ up to the boundary $T = \partial M = S \cup \mathcal{C}$, (i.e. the data (g, F) extend as $C^k \times C^{k+1}$ data to an open neighborhood of the closed domain M).

For the C^0 compatibility conditions, we first require
$$E_0|_\Sigma = G|_\Sigma \text{ and } [\gamma] = [(q_F)_{AB}] \quad \text{at } \Sigma.$$
In addition, note that g_S and E_0 uniquely determine the pull-back metric $(E_0^{-1})^* g_S$ on S (near Σ) which is supposed to be the induced metric by g_F. Moreover, E_1 gives its time-like unit normal since $E_1 = F_*(T_g) = T_{g_F}$. Thus the 4-dimensional Lorentzian metric g_F on S near Σ is fully determined by the initial data g_S, E_0, E_1. Consequently, since $\Theta = F_*(T_g + \nu_g) = T_{g_F} + \nu_{g_F}$, Θ must satisfy a

C^0 compatibility condition $\Theta = \Theta(g_S, E_0, E_1)$ at Σ. Similarly, C^1 compatibility requires $\mathcal{L}_{\partial_t}\Theta$ is determined by an algebraic expression in (q, k).

The C^1 compatibility condition of G is related with the gauged initial data q. Because $q = F^*(g_F)$ where g_F on S is determined as above and the linearization DF at the edge Σ is uniquely determined by DE_0 and DG, so we must have $q = q(g, E_0, E_1, G)$ at Σ which can be understood as a C^0 compatibility condition on q. On the other hand, this can also be interpreted as $DG = DG(q, E_0, E_1)$, i.e. a C^1 condition on G.

For $k \geq 2$, C^k compatibility requires using the bulk equations (2.2) and (2.5) to replace ∂_t^2 and higher order t-derivatives by x^α derivatives of lower order in t. Since it will not be necessary, we do not explicitly express the (complicated) higher order compatibility relations. The compatibility assumptions for the linearized equations are discussed in more detail below.

Remark 2.1. **(i).** As stated in the Introduction, ν_g is the unit space-like outward unit normal to $\mathcal{C} \cap U \subset (U, g)$. In particular ν_g is never a null-vector and so never tangent to $\mathcal{C}$. It follows then from the B-boundary condition in (2.4), (as well as the boundary condtions in (1.10) or (1.19)), that $(\mathcal{C} \cap U, g)$ is Lorentzian (i.e. $\mathcal{C} \cap U$ is time-like with respect to g) on its full domain,

(ii). Note also that the assumptions above that $E_0 : S \to S$ and $G : \mathcal{C} \to \mathcal{C}$ are diffeomorphisms near Σ and that the vector field E_1 is transverse to S imply that on any solution $(\mathcal{T}, g, F)$, there are neighborhoods $\mathcal{N}$ and $\mathcal{N}'$ of Σ in M such that F induces a diffeomorphism $F|_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}' = F(\mathcal{N}) \subset M$.

The following result is well-known.

Lemma 2.2. *Under the initial and boundary conditions that $V = 0$ on $T \cap U = (S \cup \mathcal{C}) \cap U$, one has*

$$V = 0 \quad \text{on} \quad U. \tag{2.6}$$

Thus a solution of the system (2.2) is a Ricci flat metric g with harmonic (or wave) coordinate chart χ (as well as wave map chart F). Moreover, if (h, F'), $h = g'$ is a solution of the linearization of the system (2.2) at a given solution (g, F), with $V_h' = 0$ on $T \cap U$, then

$$V_h' = 0 \quad \text{on} \quad U.$$

Proof. Let $\beta = \beta_g = \delta_g + \frac{1}{2}tr_g$ be the Bianchi operator with respect to g on symmetric bilinear forms. The Bianchi identity applied to (2.2) gives

$$\beta\delta^* V = 0 \quad \text{on} \quad U.$$

By a standard Weitzenbock formula, $2\beta\delta^*V = D^*DV - 2Ric(V) = 0$. This is a linear system of wave equations, coupled only at first order. As is well-known, cf. [13], given $V = 0$ on B_0, the constraint equations (1.6) imply that $\partial_t V = 0$ on B_0, so the initial data for V vanish. Since $V = 0$ also on the boundary $\mathcal{C} \cap U$, standard results on uniqueness of solutions of such linear wave systems imply (2.6), cf. [3] for instance. The same argument applies to the linearization (h, F'). ■

Conversely, we will need to know that a general (ungauged) solution to the system (1.8)-(1.10) restricted on U can be brought into the background gauge $V = 0$ by suitable diffeomorphisms in $\mathrm{Diff}_1(U)$ – the group consisting of diffeomorphisms that are equal to the identity to the first order on S and equal to the identity to the zero order on $\mathcal{C}$. It is easy to check that diffeomorphisms in $\mathrm{Diff}_1(U)$ preserve the initial and boundary conditions in (1.9)-(1.10). (In fact, they also preserve the gauged initial and boundary data (I, B) in (2.3)-(2.4)).

Lemma 2.3. *Let (g, F) be a local solution to the IBVP (1.8)-(1.10) in U and let $\chi = \{x_0^\alpha\}$ be a standard corner chart on U. Then there is an open subset $U' \subset U$ covering $S \cap U$ and a* *diffeomorphism* $\varphi \in \mathrm{Diff}_1(U')$*, such that*

$$V(\varphi^* g) = \Box_{\varphi^* g} x_0^\alpha = 0 \quad \text{on} \quad U'. \tag{2.7}$$

In addition, the pair $(\varphi^* g, \varphi^* F)$ *satisfies the same IBVP (with same initial and boundary data) restricted in* U'*.*

Proof. Given the background coordinates x_0^α on U, define new coordinates $\widetilde{x}_0^\alpha$ by solving the wave equation

$$\Box_g \widetilde{x}_0^\alpha = 0,$$

with the same initial and boundary conditions as that formed by x_0^α, i.e. $\widetilde{x}_0^a = x_0^a, \partial_{t_0}\widetilde{x}_0^\alpha = \partial_{t_0} x_0^\alpha$ on $U \cap S$ and $\widetilde{x}_0^\alpha = x_0^a$ on $\mathcal{C} \cap U$. This a well-posed IBVP for the simple linear wave equation, and so there is a unique solution. Define then a map $\varphi : U \to U$ by $\tilde{x}_0^\alpha|_{\varphi(q)} = x_0^\alpha|_q \quad \forall q \in U$. Based on the initial and boundary conditions of $\widetilde{x}_0^\alpha$, φ is diffeomorphism in a neighborhood U' of $S \cap U$. Clearly $\varphi \in \mathrm{Diff}_1(U')$. Then the pull-back $\widetilde{g} = \varphi^* g$ (and $\widetilde{F} = \varphi^* F$) satisfies (2.7). Moreover, it is proved in [19], see also [18], that for $g \in H^s(S_t)$, the new background coordinates $\widetilde{x}_0^\alpha \in H^{s+1}(S_t)$. ■

Remark 2.4. In the notation above, we have $\mathrm{Diff}_1(M) \subset \mathrm{Diff}_0(M) \subset \mathrm{Diff}(M)$. The quotient $\mathrm{Diff}_0(M)/\mathrm{Diff}_1(M)$ corresponds to diffeomorphisms which shift the time direction ∂_t transverse to S while keeping S fixed. Such diffeomorphisms shift the components $g_{0\alpha}$ of the initial metric g at S. In U, such diffeomorphisms may be realized by changing the initial data E_1 for F, keeping E_0 fixed. Roughly speaking E_1 is isomorphic to the tangent space $T(\mathrm{Diff}_0(M)/\mathrm{Diff}_1(M))$.

The system (2.2)-(2.4) is a quasi-linear hyperbolic system with mixed boundary conditions. The well-posedness of such an IBVP rests upon analysing the behavior in small regions, linearized around a point $p \in \Sigma$, (the frozen coefficient method). We next discuss in detail how this localization is done in the current setting.

For convenience, consider first a special case. Let $(\bar{g}_0)_R = dt_0^2 + \sum (dx_0^i)^2$ be the standard Euclidean metric on $\mathbb{R}^4 \supset \mathbf{R}$. The pair $(\bar{g}_0, \bar{F}_0)$ consisting of the Minkowski metric $\bar{g}_0 = -dt_0^2 + \sum (dx_0^i)^2$ and the identity map $\bar{F}_0 = Id : \mathbf{R} \to \mathbf{R}$ is then the unique solution to (2.2)-(2.4) on $\mathbf{R}$ with Cartesian initial and boundary data $(\bar{I}_0, \bar{B}_0)$:

$$\begin{aligned}
&\bar{I}_0 = \{q_{\alpha\beta} = \eta_{\alpha\beta},\ k_{\alpha\beta} = 0,\ E_0 = Id_{\{t_0 = 0\}},\ E_1 = \partial_{t_0}\},\\
&\bar{B}_0 = \{G = Id_{\{x_0^1 = 0\}},\ [\gamma] = [\delta_{AB}],\ \Theta = \partial_{t_0} + \partial_{x_0^1}\}.
\end{aligned}$$

Here $\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$ and $\delta_{AB} = \mathrm{diag}(1, 1)$.

More generally, let $(g_0)_R = \sum ((g_0)_R)_{\alpha\beta} dx_0^\alpha \cdot dx_0^\beta$ be a complete flat Riemannian metric on $\mathbf{R}$, with $((g_0)_R)_{\alpha\beta}$ constant functions on $\mathbf{R}$. Then the pair (g_0, F_0) consisting of a flat Lorentz (Minkowski-type) metric $g_0 = \sum (g_0)_{\alpha\beta} dx_0^\alpha \cdot dx_0^\beta$ and $F_0 = L : \mathbf{R} \to \mathbf{R}$ a linear map ($F_0(e^\alpha) = \ell_{\alpha\beta} e^\beta$ in the standard basis) is the unique solution to (2.2)-(2.4) on $\mathbf{R}$ with flat initial and boundary data (I_0, B_0):

$$\begin{aligned}
&I_0 = \{q_{\alpha\beta} = (g_0)_{\alpha\beta},\ k_{\alpha\beta} = 0,\ E_0 = L|_{\{t_0 = 0\}},\ E_1 = L_*(T_{g_0})|_{\{t_0=0\}}\}\\
&B_0 = \{G = L|_{\{x_0^1 = 0\}},\ [\gamma] = [((L^{-1})^* g_0)_{AB}],\ \Theta = L_*(T_{g_0} + \nu_{g_0})|_{\{x_0^1 = 0\}}\}.
\end{aligned} \tag{2.8}$$

Since we always assume the wave map F sends the initial and boundary surfaces to themselves, here we require $L : \{t_0 = 0\} \to \{t_0 = 0\}$, $L : \{x_0^1 = 0\} \to \{x_0^1 = 0\}$. Then it follows that $L : \{t_0 = \text{constant}\} \to \{t_0 = \text{constant}\}$, since L is a constant linear map.

We now show that for a general choice of initial and boundary data (I, B) in (2.3)-(2.4), the IBVP can be reduced to a problem with initial and boundary data sufficiently close to the flat data (2.8) above.

Choose given local data $I=(q,k,E_0,E_1)$, $B=(G,[\gamma],\Theta)$ in a neighborhood of a corner point $p\in\Sigma$. The map E_0 maps p to $p'=E_0(p)\in\Sigma$. Choose a pair χ, χ' of standard corner charts at p,p' so that $\chi(p)=0$ and $\chi'(p')=0$. Let x_0^α denote the coordinates in χ and x^α the coordinates in χ'. Now choose new coordinates $\tilde{x}_0^\alpha=\lambda^{-1}x_0^\alpha$ near p and correspondingly $\widetilde{x}^\alpha=\lambda^{-1}x^\alpha$ near p'. Define new initial data for g in the following way

$$\tilde{q}_{\tilde{\alpha}_0\tilde{\beta}_0}(\tilde{x}_0)=q_{\alpha_0\beta_0}(\lambda\tilde{x}_0),\ \tilde{k}_{\tilde{\alpha}_0\tilde{\beta}_0}(\tilde{x}_0)=\lambda k_{\alpha_0\beta_0}(\lambda\tilde{x}_0).$$

Here $\tilde{q}_{\tilde{\alpha}_0\tilde{\beta}_0}(\tilde{x}_0)$ denotes the component of $\tilde{q}$ at the point $\tilde{x}_0$ in the new coordinates $(\tilde{x}_0^\alpha)$. It equals the corresponding component $q_{\alpha_0\beta_0}(x_0)$ of q at the point $x_0=\lambda\tilde{x}_0$ in the original coordinates (x_0^α). The same explanation applies to the defining equations of $\tilde{k}$. Notice that when λ is very small, $\tilde{q}$ is very close to the Minkowski-type metric $\tilde{g}_0=\sum q_{\alpha_0\beta_0}(0)d\tilde{x}_0^\alpha\cdot d\tilde{x}_0^\beta$ where the components of the metric are constants $(\tilde{g}_0)_{\tilde{\alpha}_0\tilde{\beta}_0}=q_{\alpha_0\beta_0}(0)$; similarly $\tilde{k}$ is very close to zero.

Define new initial data for F as

$$\tilde{E}_0^{\tilde{\alpha}}(\tilde{x}_0)=\lambda^{-1}E_0^\alpha(\lambda\tilde{x}_0),\ \tilde{E}_1^{\tilde{\alpha}}(\tilde{x}_0)=E_1^\alpha(\lambda\tilde{x}_0).$$

Here $\tilde{E}_0^{\tilde{\alpha}}(\tilde{x}_0)$ denotes the $\tilde{x}^\alpha$ component of the image $\tilde{E}_0(\tilde{x}_0)$ in the new chart $\tilde{x}$. It equals to the rescaled (λ^{-1}) component of the image $E_0(\lambda\tilde{x}_0)$ expressed in the chart χ'. The equation for $\tilde{E}_1$ has the same meaning. As $\lambda\to 0$, the map $\tilde{E}_0$ approaches the constant linear map L_0 given in the standard basis by $L_0(e^{\tilde{\alpha}_0})=l_{\tilde{\alpha}_0\tilde{\beta}}e^{\tilde{\beta}}$ $(\alpha_0,\beta=1,2,3)$ where $l_{\tilde{\alpha}_0\tilde{\beta}}$ equal the coefficients of E_0 linearized at p in the chart χ, i.e. $l_{\tilde{\alpha}_0\tilde{\beta}}=(DE_0|_p)_{\alpha_0\beta}$. Since $E_0:S\to S$ maps $\{x_0^1=0\}$ to $\{x^1=0\}$ (edge to edge), the map $L_0:\{\tilde{t}_0=0\}\times\mathbb{R}^3\to\{\tilde{t}=0\}\times\mathbb{R}^3$ also maps $\{\tilde{x}_0^1=0\}$ to $\{\tilde{x}^1=0\}$. At the same time $\tilde{E}_1$ approaches the constant vector field $E_1^\alpha(0)\partial_{\tilde{\alpha}}$.

Similarly define new boundary data as

$$\tilde{G}^{\tilde{\alpha}}(\tilde{x}_0)=\lambda^{-1}G^\alpha(\lambda\tilde{x}_0),\ [\tilde{\gamma}_{\tilde{A}\tilde{B}}](\tilde{x}_0)=[\gamma_{AB}](\lambda\tilde{x}_0),\ \tilde{\Theta}^{\tilde{\alpha}}(\tilde{x}_0)=\Theta^\alpha(\lambda\tilde{x}_0).$$

It is easy to check that when $\lambda\to 0$, one has $\tilde{G}\to L_1=DG|_p$ in the same sense as the limiting approach of $\tilde{E}_0$ above. In addition, since G maps $\{t_0=0\}$ to $\{t=0\}$ (edge to edge), the map $L_1:\mathbb{R}\times\{\tilde{x}_0^1=0\}\times\mathbb{R}^2\to\mathbb{R}\times\{\tilde{x}^1=0\}\times\mathbb{R}^2$ also maps $\{\tilde{t}_0=0\}$ to $\{\tilde{t}=0\}$. In addition, since $E_0=G$ on the edge Σ, $L_0=L_1$ on $\{\tilde{t}_0=0\}\cap\{\tilde{x}_0^1=0\}$. Thus there is a unique constant linear transformation $\tilde{L}:\mathbb{R}^4\to\mathbb{R}^4$ such that $\tilde{L}|_{\{\tilde{t}_0=0\}}=L_0$ and $\tilde{L}|_{\{\tilde{x}_0^1=0\}}=L_1$. At the same time, $\tilde{\Theta}\to\Theta^\alpha(p')\partial_{\tilde{\alpha}}$ and $[\tilde{\gamma}_{\tilde{A}\tilde{B}}]\to[\gamma_{AB}](p')$.

Based on the compatibility conditions of (I,B) at Σ, we must have $E_1(p)=DF|_p(T_q)$ where $DF|_p$ is uniquely determined by $DE_0|_p$ and $DG|_p$. Thus the limit (as $\lambda\to 0$) of $\tilde{E}_1$, which is given by $E_1^\alpha(0)\partial_{\tilde{\alpha}}$, is equal to $DF|_p(T_{\tilde{q}})=\tilde{L}_*(T_{\tilde{g}_0})$. For the same reason, the limit of $\tilde{\Theta}$ makes $\Theta^\alpha(p')\partial_{\tilde{\alpha}}=\tilde{L}_*(T_{\tilde{g}_0}+\nu_{\tilde{g}_0})$ and the limit of $\tilde{\gamma}_{\tilde{A}\tilde{B}}$ is $[\gamma_{AB}](p')=[(\tilde{L}^{-1})^*(q(0))^t]=[(\tilde{L}^{-1})^*(\tilde{g}_0)^t]$.

The analysis above shows that the rescaled initial and boundary data $(\tilde{I}=\{\tilde{q},\tilde{k},\tilde{E}_0,\tilde{E}_1\},\tilde{B}=\{\tilde{G},[\tilde{\gamma}],\tilde{\Theta}\})$ limits to a set of flat initial and boundary data which is exactly (2.8) with $g_0=\tilde{g}_0$ and $L=\tilde{L}$ in the chart $(\tilde{x}_0^\alpha)$. Meanwhile, the same rescaling process is also applied to the given background Riemannian metric g_R on $U\subset M$, i.e. $(\tilde{g}_R)_{\tilde{\alpha}\tilde{\beta}}(\tilde{x})=(g_R)_{\alpha\beta}(\lambda\tilde{x})$. It then limits to a flat metric $(\tilde{g}_0)_R=\Sigma(g_R)_{\alpha\beta}(p')d\tilde{x}^\alpha\cdot d\tilde{x}^\beta$ as $\lambda\to 0$.

One may now set up a hyperbolic system in the same way as (2.2)-(2.4) in the new chart $(\tilde{x}_0^\alpha)$ with initial and boundary data $(\tilde{I},\tilde{B})$ constructed above in a fixed size neighborhood $\tilde{U}$ of p and with the rescaled background metric $\tilde{g}_R$. Choose λ small enough so that the data $(\tilde{I},\tilde{B})$ is sufficiently close to flat-type data as described above, and the terms in the wave equation of F with coefficients contributed by $\Gamma_{\tilde{g}_R}$ are close to zero. If $(\tilde{g},\tilde{F})$ is a solution of this rescaled IBVP in a (possibly smaller) domain $\tilde{U}$, then the pair $g(x) = \lambda^2\tilde{g}(\lambda^{-1}x)$ and $F(x) = \tilde{F}(\lambda^{-1}x)$ solves the system (2.2)-(2.4) with the original initial and boundary data (I, B) and background metric g_R. Thus there is a one-to-one correspondence between such local solutions in sufficiently small neighborhoods of corner points $p \in \Sigma$.

For the rest of this section, we assume that the initial and boundary data (I, B) have been localized as above, so they are close to flat data with harmonic coordinates x_0^α. The general case will be discussed in detail in §5. To analyse the solvability of the system (2.2)-(2.4) near flat data (g_0, F_0), (with the background flat Riemannian metric $(g_0)_R$), we first consider its linearization at (g_0, F_0) with infinitesimal deformation denoted as (h, F'). Without loss of generality (by linear transformation of the chart), we can assume $g_0 = -(dt_0)^2 + (dx_0^1)^2 + (g_0)_{AB} dx_0^A \cdot dx_0^B$ in the standard corner chart χ at p. For convenience, we also make the assumption that $\mathcal{C} \cap U$ is orthogonal to $S \cap U$ with respect to the flat metric g_0. This assumption, although not necessary, simplifies some of the computations to follow in §3. As noted in [14], it can always be realized by choosing a new initial slice $S' \subset \mathcal{V} \subset M$, with $\partial S' = \partial S = \Sigma$; here $\mathcal{V}$ is the (maximal) Cauchy development of the initial data set (S, I). In particular, one has then $T_{g_0} = \partial_{t_0}$ and $\nu_{g_0} = \partial_{x_0^1}$.

It is well-known that the linearization of the first equation in (2.2) at g_0 is given by the standard system $\Box_{g_0} h = 0$ of wave equations. Here and in the following, we work in $\mathbf{R}$ with harmonic coordinates x_0^α. For the second equation in (2.2), observe that the linearization of the Γ term at $(g_0)_R$ vanishes and

$$
\begin{aligned}
(\Box_g F + F_*(V))' &= \Box_{g_0} F' + \Box'_h F_0 + (F_0)_*(V'_h) \\
&= \Box_{g_0} F' + g_0(D^2_{g_0} F_0, h) - g_0(dF_0, \beta_{g_0} h) + (F_0)_*(V'_h) = \Box_{g_0} F',
\end{aligned}
\tag{2.9}
$$

where we use the fact that $V_{g_0} = 0$, $V'_h = \beta_{g_0} h$ and $D^2_{g_0} F_0 = 0$. Thus the corresponding linear hyperbolic system is given by:

$$
\begin{cases}
\Box_{g_0} h = 0 \\
\Box_{g_0} F' = 0
\end{cases}
\quad \text{in } \mathbf{R}
\tag{2.10}
$$

with initial and boundary conditions

$$
\begin{cases}
h_{\alpha\beta} = q'_{\alpha\beta}, \ \partial_{t_0} h_{\alpha\beta} = k'_{\alpha\beta} \ \ V'_h = 0 \\
F' = E'_0, \quad \partial_{t_0} F' + (F_0)_*(T') = E'_1
\end{cases}
\quad \text{on } \{t_0 = 0\}
\tag{2.11}
$$

and

$$
\begin{cases}
V'_h = 0 \\
F' = G' \\
h_{22} - h_{33} = c_2 \\
h_{23} = c_3 \\
T' + \nu' = b
\end{cases}
\quad \text{on } \{x_0^1 = 0\}.
\tag{2.12}
$$

We discuss in more detail the linearization of the initial and boundary conditions. To begin, $T' = T'_h + T'_{F'}$, where $T'_h = \frac{1}{2} h_{00} \partial_0 - h_{0i} \partial_i$ (cf. §6.2) and $T'_{F'}$ denotes the variation of T caused by the variation of the level set $(S_F)_\tau = F^{-1}(S_\tau)$. In addition, since ν_g does not depend on the foliation, $\nu' = \nu'_h = h_{10}\partial_0 - \frac{1}{2} h_{11} \partial_1 - h_{1A} \partial_A$. All the subscript indices here are with respect to the standard coordinates x_0^α.

Note that since $F|_S : S \to S$, $T'_{F'} = 0$ on S. Thus the term $(F_0)_*(T')$ on S is determined by $q'_{\alpha\beta}$ and $k'_{\alpha\beta}$. The last equation in (2.11) can thus be rewritten as $\partial_{t_0} F' = E'$ where E' involves E'_1, q' and k'.

The third and fourth equations in (2.12) come from the linearization of the boundary data of conformal metric $[g_F^t] = [\gamma]$. Since the conformal class of a 2-dim metric γ uniquely determines $(\det\gamma)^{-1/2}\gamma$, the linearization of this boundary equation at (g_0, F_0) yields

$$(g_F^t)' - \tfrac{1}{2}(\mathrm{tr}_{(g_0)_{F_0}^t}(g_F^t)')(g_0)_{F_0} = \sqrt{\tfrac{\det(g_0)_{F_0}}{\det\gamma}}[\gamma' - \tfrac{1}{2}\mathrm{tr}_\gamma\gamma'].$$

Here $(g_F^t)' = ((F_0^{-1})^*h)^t + (L_{F'}g_0)^t$ with $L_{F'}$ denoting the Lie derivative. Since F_0 is a constant linear mapping sending $\{t_0 = \text{constant}\}$ to $\{t_0 = \text{constant}\}$ on $\mathcal{C}$, we have $((F_0^{-1})^*h)^t = (F_0^{-1})^*(h^t)$. Thus the linearized equation above yields $h^t - (\mathrm{tr}_{g_0}h^t)g_0 = c$, where c is a tensor involving only γ, g_0, F_0 and γ', F'.

The last equation in (2.12) comes from the linearization of the boundary equation $F_*(T_g + \nu_g) = \Theta$:

$$(F_0)_*(T_h' + \nu') + (F_0)_*(T_{F'}') + L_{F'}(T_0 + \nu_0) = \Theta'.$$

Here $L_{F'}$ denotes the Lie derivative with respect to the deformation F'. This equation further implies $T_h' + \nu' = (F_0^{-1})_*[-(F_0)_*(T_{F'}') - L_{F'}(T_0 + \nu_0) + \Theta']$. Denoting the right-side term as b we obtain the last boundary equation (2.12).

As noted above we view the flat data (g_0, F_0) to be defined on the standard corner $\mathbf{R}$ (via the chart χ^{-1}) with coordinates x_0^α. Moreover the linearization of the initial and boundary data $(I', B') = \{(q', k'), (E_0', E_1'), (\gamma', G', \Theta')\}$ are extended to be defined on $\mathbf{R}$ by composing with a compactly supported bump function on the initial and boundary surface which equals to 1 on a subset $V \subset \bar{V} \subset U \cap (S \cup \mathcal{C})$. We always assume that the linearized initial and boundary data (I', B') satisfies the linearized compatibility conditions and the pair (q', k') satisfy the linearized constraint equations. Note that, given fixed boundary data, the system for (h, F') decouples into two independent systems for h and F'.

This leads to the following result.

Proposition 2.5. *Let $g_0 = \eta$ be the standard Minkowski metric on $\mathbf{R}$. The frozen coefficient or blow-up linearization of the equations* (2.2)-(2.4) *at a solution (g, F) at a corner point $p \in \Sigma$ may be written in the form*

$$\begin{cases} \Box_{g_0} h = 0 \\ \Box_{g_0} F' = 0 \end{cases} \quad \text{in } \mathbf{R} \tag{2.13}$$

with initial and boundary conditions

$$\begin{cases} h_{\alpha\beta} = q'_{\alpha\beta}, \;\; \partial_{t_0} h_{\alpha\beta} = k'_{\alpha\beta} \\ F' = E_0', \quad \partial_{t_0} F' = E' \\ V_h' = 0 \end{cases} \quad \text{on } \{t_0 = 0\} \tag{2.14}$$

and

$$\begin{cases} V_h' = 0 \\ F' = G' \\ h_{22} - h_{33} = c_2 \\ h_{23} = c_3 \\ \frac{1}{2}h_{00} + h_{10} = b_0 \\ h_{01} + \frac{1}{2}h_{11} = b_1 \\ h_{0A} + h_{1A} = b_A \quad A = 2,3 \end{cases} \quad \text{on } \{x_0^1 = 0\}. \tag{2.15}$$

Remark 2.6. Based on this form of the linearized system, we note that one has to set T_g to be the unit normal to $(S_F)_\tau$ in M instead of the unit normal to $(\Sigma_G)_\tau$ tangent to $\mathcal{C}$. Namely, if T_g

is chosen to be the unit time-like normal to $(\Sigma_G)\tau$ then its linearization at (g_0, F_0) only involves $\frac{1}{2}h_{00}\partial_0 - h_{0A}\partial_A$, i.e. the term h_{01} does not appear in the linearization. This will be problematic for the energy estimates discussed in the next section.

The compatibility or corner conditions for (h, F') at Σ are somewhat simpler to express in the system (2.13)-(2.15). Thus, the C^0 and C^1 compatibility conditions for F' are

$$G' = E_0', \quad \partial_t G' = E' \ \text{ on } \ \Sigma. \tag{2.16}$$

At 2nd order and 3rd order, $\partial_t^2 G' = \Delta E_0'$ and $\partial_t^3 G' = \Delta E'$ on Σ respectively and thus similarly at higher order. The C^0 compatibility conditions for h are $b_0 = \frac{1}{2}q_{00}' + q_{10}'$,$b_1 = q_{01}' + \frac{1}{2}q_{11}'$ and $b_A = q_{0a}' + q_{1A}'$ $(A = 2,3)$. The terms c_A are determined by the trace-free part of q_{AB}'. The t-derivatives of b_α and c_A are similarly determined by $k_{\alpha\beta}'$. One may compute the higher order compatibility conditions in a simlar way. A detailed analysis of the linearized IBVP (2.13)-(2.15) is given in the next section.

Next we provide a similar discussion for the boundary data $B_{\mathcal{C}}$ (or $\mathbf{B}_{\mathcal{C}}$). To begin, (2.2)-(2.4) is replaced by

$$\begin{cases} Ric_g + \delta_g^* V = 0 \\ \Box_g F + \Gamma(F)g(\nabla F, \nabla F) + F_*(V) = 0 \end{cases} \quad \text{in } U \tag{2.17}$$

with initial conditions:

$$\begin{cases} g_{\alpha\beta} = q_{\alpha\beta}, \ \partial_{t_0} g_{\alpha\beta} = k_{\alpha\beta}, \ V = 0 \\ F = E_0, \ F_*(T_g) = E_1 \end{cases} \quad \text{on } S \cap U \tag{2.18}$$

and boundary conditions:

$$\begin{cases} V = 0 \\ F = G \\ [g_F^t] = [\gamma] \\ H_{g_F} = H \\ F_*(T_g + \nu_g)^T = \Theta_{\mathcal{C}} \end{cases} \quad \text{on } \mathcal{C} \cap U. \tag{2.19}$$

Recall here from (1.20) that $F_*(T_g + \nu_g)^T = F_*(T_g + \nu_g) - g_R(F_*(T_g + \nu_g), \nu_{g_R})\nu_{g_R}$, where ν_{g_R} denotes the unit outward normal to $(\mathcal{C} \cap U) \subset (U, g_R)$. For the $(I_{\mathcal{C}}, B_{\mathcal{C}})$ (and so $\{\mathbf{I}_{\mathcal{C}}, \mathbf{B}_{\mathcal{C}}\}$) data, we impose the extra corner condition that

$$F_*(\nu_g) \in \text{span}\{\nu_{g_R}\}, \tag{2.20}$$

on the edge Σ. As noted in the C^0 compatibility conditions preceding Remark 2.1, F_* and ν_g are determined at Σ by the initial and boundary data q, E_0, E_1, G at Σ. Thus, the extra condition (2.20) is understood as a restriction on the choice of the data q, E_0, E_1, G at Σ. We point out that both Lemma 2.1 and Lemma 2.2 remain valid.

As before in (2.8), the flat data (g_0, F_0) in the chart (x_0^α) with background flat Riemannian metric $(g_R)_0$ satisfy this system with initial and boundary conditions:

$$(I_{\mathcal{C}})_0 = \{q_{\alpha\beta} = (g_0)_{\alpha\beta}, \ k_{\alpha\beta} = 0, \ E_0 = L|_{\{t_0=0\}}, \ E_1 = L_*(T_{g_0})\},$$
$$(B_{\mathcal{C}})_0 = \{[\gamma] = [(g_0^t)_{AB}], \ G = L|_{\{x_0^1=0\}}, \ H = 0, \ \Theta_{\mathcal{C}} = L_*(T_{g_0} + \nu_{g_0})^T\}.$$

Since (I, B) and $(I_{\mathcal{C}}, B_{\mathcal{C}})$ differ only by the last two terms in the boundary data, the same discussion as following (2.8) holds here, except that in the rescaling process one defines

$$\tilde{H}(\tilde{x}_0) = \lambda H(\lambda \tilde{x}_0), \quad \tilde{\Theta}_{\mathcal{C}}^{\tilde{\alpha}}(\tilde{x}_0) = \Theta_{\mathcal{C}}^{\alpha}(\lambda \tilde{x}_0).$$

Making λ sufficiently small, $\tilde{H}$ is close to zero and $\tilde{\Theta}_{\mathcal{C}}$ is close to $\tilde{L}(N_{\tilde{g}_0}+\nu_{\tilde{g}_0})^T$, where $\tilde{g}_0$, $\tilde{L}$ are the same as in the rescaling discussion for the data (I,B), and the projection operator ^{T} is respect to the rescaled Riemannian metric $\tilde{g}_R$. Given these minor modifications, the proof of the validity of the blow-up or rescaling process is exactly the same as in the (I,B) conditions above.

Passing then to the linearization at the flat data (g_0,F_0) as before, the linearization of $\Theta_{\mathcal{C}}$ condition at (g_0,F_0) is given by

$$(F_0)_*(T'_h+\nu')-(g_0)_R((F_0)_*(T'+\nu'),\nu_{(g_0)_R})\nu_{(g_0)_R}=b,$$

where b, as before, only involves F' in addition to $g_0,F_0,(g_0)_R$. As before, the conditions on E_0 and G imply that $(F_0)_*:\mathrm{span}\{\partial_{x_0^2},\partial_{x_0^3}\}\to\mathrm{span}\{\partial_{x_0^2},\partial_{x_0^3}\}$, and $(F_0)_*(\partial_{t_0})\to\mathrm{span}\{\partial_{t_0},\partial_{x_0^2},\partial_{x_0^3}\}$. Further, (2.20) implies that $(F_0)_*(\partial_1)\in\mathrm{span}\{\nu_{(g_R)_0}\}$. Now recall that $T'_h+\nu'=(\frac{1}{2}h_{00}+h_{10})\partial_0-(h_{01}+\frac{1}{2}h_{11})\partial_1-(h_{0A}+h_{1A})\partial_A$, cf. also §6.2. Therefore, the linearization of the last equation in (2.19) has the form

$$(\tfrac{1}{2}h_{00}+h_{10})\partial_0-(h_{0A}+h_{1A})\partial_A=(F_0)_*^{-1}(b).$$

Note that in contrast with the boundary condition (1.10) with data $\mathbf{B}$ (cf. Remark 2.6), for the $\mathbf{B_C}$ boundary conditions one may choose here T_g to be the unit timelike normal to $(\Sigma_G)_\tau\subset(\mathcal{C},g)$. This is because by taking the tangential projection as above, one obtains the same linearized term for these different choices of the timelike normal.

The analysis above leads to the analog of Proposition 2.5.

Proposition 2.7. *Let $g_0=\eta$ be the standard Minkowski metric on $\mathbf{R}$. The frozen coefficient or blow-up linearization of the equations* (2.17)-(2.19) *at a solution (g,F) at a corner point $p\in\Sigma$ may be written in the form*

$$\begin{cases}\Box_{g_0}h=0\\ \Box_{g_0}F'=0\end{cases}\quad \text{in } \mathbf{R}\tag{2.21}$$

with initial and boundary conditions

$$\begin{cases}h_{\alpha\beta}=q'_{\alpha\beta},\ \ \partial_{t_0}h_{\alpha\beta}=k'_{\alpha\beta}\\ F'=E'_0,\quad \partial_{t_0}F'=E'\\ V'_h=0\end{cases}\quad \text{on } \{t_0=0\}\tag{2.22}$$

and

$$\begin{cases}V'_h=0\\ F'=G'\\ h_{22}-h_{33}=c_2\\ h_{23}=c_3\\ (H_{g_F})'_{h,F'}=H'\\ \frac{1}{2}h_{00}+h_{10}=b_0\\ h_{0A}+h_{1A}=b_A\quad A=2,3\end{cases}\quad \text{on } \{x_0^1=0\}.\tag{2.23}$$

The compatibility conditions are essentially the same as those in and following (2.16). In the next section we discuss the precise definition of H_{g_F} and its linearization.

3. Analysis of Linearized Equations

In this section, we derive the main H^1 and H^s energy estimates for the linearized systems (2.13)-(2.15) and (2.21)-(2.23) at the core of the existence and well-posedness results, Theorems 1.1 and 1.2.

As in §2, we work in the domain $\mathbf{R} = \mathbb{R}^+ \times (\mathbb{R}^{\mathbf{3}})^+$ and set $S_t = \{t = const\}$, $\Sigma_t = \{t = const\} \cap \{x^1 = 0\}$ and let $\mathcal{C}_t = \cup\{\Sigma_s : 0 \leq s \leq t\}$, $\mathcal{T}_t = \cup\{S_s : 0 \leq s \leq t\}$. As in §2, we assume the initial and boundary data in (2.13)-(2.15) and (2.21)-(2.23) are of compact support in $\mathbf{R}$.

As a simple model for the systems (2.13)-(2.15) and (2.21)-(2.23) consider a scalar function u on $\mathbf{R}$ satisfying the inhomogeneous wave equation

$$\Box_{g_0} u = \varphi, \tag{3.1}$$

with initial conditions $u(0,\cdot) = u_0$, $\partial_t(0,\cdot) = u_1$ and boundary data $B(u) = b$ on $\mathcal{C}$, all of compact support. The boundary operator B, specified further below, is assumed to contain derivative operators of order at most j, with $j = 0$ or $j = 1$. As usual, define the bulk and boundary energies by

$$\mathcal{E}_{S_t}(u) = \int_{S_t} u_t^2 + |du|^2 + u^2 \ \text{ and } \ \mathcal{E}_{\mathcal{C}_t}(u) = \int_0^t \int_{\Sigma_t} u_t^2 + |du|^2 + u^2,$$

where du is the full collection of spatial derivatives $\partial_i u$, $i = 1,2,3$ and the integration is with respect to the volume forms induced on S_t, Σ_t by g_0.

For $\Omega = S_t, \Sigma_t$ or $\mathcal{C}_t$, let $H^s(\Omega)$ denote the Sobolev space of functions with weak derivatives up to order s in $L^2(\Omega)$, $s \in \mathbb{R}^+$. For notational convenience, we let $\bar{H}^s$ denote the analogous norm consisting of all space-time derivatives, (not just those tangent to Ω). In this notation,

$$\mathcal{E}_{S_t}(u) = ||u||^2_{\bar{H}^1(S_t)}, \ \text{ and } \ \mathcal{E}_{\mathcal{C}_t}(u) = ||u||^2_{\bar{H}^1(\mathcal{C}_t)}.$$

As is well-known, the well-posedness of (3.1) and similar more complicated systems of wave equations rests on the existence of the main H^1 energy estimate

$$\mathcal{E}_{S_t}(u) + c\mathcal{E}_{\mathcal{C}_t}(u) \leq C[\mathcal{E}_{S_0}(u) + ||\varphi||^2_{L^2(M_t)} + ||b||^2_{H^{1-j}(\mathcal{C}_t)}], \tag{3.2}$$

for constants $c, C > 0$ independent of u and b. Similarly, one requires higher order energy estimates of the form

$$||u||^2_{\bar{H}^s(S_t)} + c||u||^2_{\bar{H}^s(\mathcal{C}_t)} \leq C[||u||^2_{\bar{H}^s(S_0)} + ||\varphi||^2_{\bar{H}^{s-1}(\mathcal{T}_t)} + ||b||^2_{H^{s-j}(\mathcal{C}_t)}]. \tag{3.3}$$

These estimates require that u is a smooth, (or at least sufficiently smooth) solution of (3.1). It will always be assumed $s > \frac{n}{2} + 1 = \frac{5}{2}$, so that by Sobolev embedding $C^{1,\alpha} \subset H^s$ in dimension 3. It is important to observe that the last term in (3.2) or (3.3) involves only derivatives of b tangent to $\mathcal{C}$.

For completeness, these energy estimates for solutions u of (3.1) are derived in Appendix §6.1 for Sommerfeld ($b = \partial_{x^0} + \partial_{x^1}$) and Dirichlet boundary conditions, which suffice for our purposes. It is well-known, cf. [3] for example, that the IBVP for (3.1) is well-posed with respect to either of these boundary conditions.

Returning to the linearized systems (2.13)-(2.15) and (2.21)-(2.23), throughout the following, we assume the initial data (g'_S, K') for g' are in $H^s(S) \times H^{s-1}(S)$ and the boundary data (b_α, c_A) are in $H^s(\mathcal{C})$. Similarly, we assume the initial data (E'_0, E'_1) for F' are in $H^{s+1}(S) \times H^s(S)$ while the boundary data G' for F' are in $H^{s+1}(\mathcal{C})$. In addition, we assume the $C^{s-1} \times C^s$ compatibility conditions hold for (g', F') at the corner Σ.

We first prove the energy estimates for F'.

Proposition 3.1. *Under the assumptions on the initial and boundary data above, one has an H^{s+1} energy estimate for F'. Thus*

$$||F'||^2_{\bar{H}^{s+1}(S_t)} + c||F'||^2_{\bar{H}^{s+1}(\mathcal{C}_t)} \leq C[||F'||^2_{\bar{H}^{s+1}(S_0)} + ||G'||^2_{H^{s+1}(\mathcal{C}_t)}].$$

Proof. As already noted in §2, the system for F' in (2.13)-(2.15) and (2.21)-(2.23) decouples from the h-system. In both cases of initial-boundary data one has the system

$$\begin{aligned} &\Box_{g_0} F' = 0 \text{ in } \mathbf{R} \\ &F' = E_0', \quad \partial_{t_0} F' = E', \text{ on } \{t=0\} \\ &F' = G' \text{ on } \{x_0^1 = 0\}. \end{aligned} \tag{3.4}$$

The system (3.4) is an uncoupled system of wave equations for F' with (inhomogeneous) Dirichlet boundary data. It is well-known that such systems admit H^1 energy estimates as in (3.2) and higher order energy estimates (3.3) given the C^s compatability conditions; cf. again [3] or the Appendix §6.1.

■

Next we turn to energy estimates for the blow-up linearization (2.13)-(2.15) of the system (2.2)-(2.4) with boundary data B.

Proposition 3.2. *For the linear system* (2.13)-(2.15) *, one has an H^s energy estimate*

$$|||h||^2_{\bar{H}^s(S_t)} + c||h||^2_{\bar{H}^s(\mathcal{C}_t)} \leq C[|||h||^2_{\bar{H}^s(S_0)} + ||\underline{b}||^2_{\bar{H}^s(\mathcal{C}_t)} + ||\underline{c}||^2_{H^s(\mathcal{C}_t)}],$$

where $\underline{b} = \{b_\alpha\}$, $\alpha = 0,1,2,3$, $\underline{c} = \{c_A\}$, $A = 2,3$.

Proof. We begin by analysing the gauge boundary conditions $V_h' = 0$ on the boundary $\mathcal{C} = \{x_0^1 = 0\}$, which have the form:

$$\begin{aligned} (V_h')_0 &= -\partial_0 h_{00} + \partial_1 h_{01} + \partial_A h_{0A} - \tfrac{1}{2}\partial_0(trh) = 0 \\ (V_h')_1 &= -\partial_0 h_{01} + \partial_1 h_{11} + \partial_A h_{1A} - \tfrac{1}{2}\partial_1(trh) = 0 \\ (V_h')_A &= -\partial_0 h_{0A} + \partial_1 h_{1A} + \partial_B h_{BA} - \tfrac{1}{2}\partial_A(trh) = 0. \end{aligned} \tag{3.5}$$

We recall that $A = 2,3$, $B = 2,3$ and the Einstein summation convention is used. Let $\tau = \frac{1}{2}(h_{22} + h_{33})$. Since $trh = -h_{00} + h_{11} + 2\tau$, this gives

$$\begin{aligned} &-\tfrac{1}{2}\partial_0(h_{00} + h_{11}) + \partial_1 h_{01} - \partial_0 \tau + \partial_A h_{0A} = 0 \\ &\tfrac{1}{2}\partial_1(h_{00} + h_{11}) - \partial_0 h_{01} - \partial_1 \tau + \partial_A h_{1A} = 0 \\ &-\partial_0 h_{0A} + \partial_1 h_{1A} + \tfrac{1}{2}\partial_A(h_{00} - h_{11} - 2\tau) + \partial_B h_{AB} = 0. \end{aligned} \tag{3.6}$$

Simple modification of the equations gives

$$\begin{aligned} &-(\partial_0 + \tfrac{1}{2}\partial_1)h_{00} - \partial_0\tau - \tfrac{1}{2}\partial_0(h_{11} - h_{00}) + \partial_1(h_{01} + \tfrac{1}{2}h_{00}) + \partial_A h_{0A} = 0 \\ &(\tfrac{1}{2}\partial_0 + \partial_1)h_{00} - \partial_1\tau + \tfrac{1}{2}\partial_1(h_{11} - h_{00}) - \partial_0(h_{01} + \tfrac{1}{2}h_{00}) + \partial_A h_{1A} = 0 \\ &-\partial_0 h_{0A} + \partial_1 h_{1A} + \tfrac{1}{2}\partial_A(h_{00} - h_{11} - 2\tau) + \partial_B h_{AB} = 0. \end{aligned} \tag{3.7}$$

which leads easily to the following system:

$$-(\partial_0 + \tfrac{1}{2}\partial_1)h_{00} - \partial_0\tau - \tfrac{1}{2}\partial_0(h_{11} - h_{00}) + \partial_1(h_{01} + \tfrac{1}{2}h_{00}) + \partial_A h_{0A} = 0 \tag{3.8}$$

$$(\tfrac{1}{2}\partial_0 + \partial_1)h_{00} - \partial_1\tau + \tfrac{1}{2}\partial_1(h_{11} - h_{00}) - \partial_0(h_{01} + \tfrac{1}{2}h_{00}) + \partial_A h_{1A} = 0 \tag{3.9}$$

$$-(\partial_0 + \partial_1)(h_{0A} - h_{1A}) - (\partial_0 - \partial_1)(h_{0A} + h_{1A}) + \partial_A(h_{00} - h_{11}) - 2\partial_A\tau + 2\partial_B h_{AB} = 0. \tag{3.10}$$

In light of (3.8)-(3.10) and the well-known existence of energy estimates for Sommerfeld and Dirichlet boundary data discussed in §6.1, it is then natural to impose Dirichlet boundary data on the terms $h_{11} - h_{00}$, $\frac{1}{2}h_{00} + h_{01}$ and $h_{0A} + h_{1A}$. Note that these terms are all included in the Dirichlet boundary conditions in (2.15). Namely, the full set of boundary data in (2.15) are

$$h_{22} - h_{33} = c_2 \tag{3.11}$$

$$h_{23} = c_3 \tag{3.12}$$

$$\tfrac{1}{2}h_{00} + h_{01} = b_0 \tag{3.13}$$

$$\tfrac{1}{2}h_{11} + h_{01} = b_1 \tag{3.14}$$

$$h_{0A} + h_{1A} = b_A. \tag{3.15}$$

Observe that (3.13)$-$(3.14) gives $h_{00} - h_{11} = 2(b_0 - b_1)$. All of these linearly combined components of h i.e. $h_{22} - h_{33}$, h_{23}, $h_{00} - h_{11}$, $\frac{1}{2}h_{00} + h_{01}$ and $h_{0A} + h_{1A}$, satisfy the wave equation (3.1) with Dirichlet boundary conditions. Hence the H^s energy estimate (3.3) holds for them.

Next in (3.10), fix the index A, say $A = 2$. Then the last two terms are $-2\partial_2\tau + 2\partial_2 h_{22} + 2\partial_3 h_{23} = -\partial_2(h_{22} + h_{33}) + 2\partial_2 h_{22} + 2\partial_3 h_{23} = \partial_2(h_{22} - h_{33}) + 2\partial_3 h_{23}$. Thus by (3.11)-(3.12) and (3.13)-(3.14), one has H^{s-1} control of the last 3 terms in (3.10). Further, as discussed in the Appendix §6.1, control of the Dirichlet boundary value gives control of the Neumann (normal derivative) boundary value; this is the boundedness of the Dirichlet-to-Neumann map. Thus (3.15) gives H^{s-1} control of the second term in (3.10). It follows that $(\partial_0 + \partial_1)(h_{02} - h_{12})$ is controlled in H^{s-1} on $\mathcal{C}$. This is a Sommerfeld boundary operator and since $(h_{02} - h_{12})$ is a solution of the wave equation, $\Box_{g_0}(h_{02} - h_{12}) = 0$ (with $\bar{H}^s$ initial conditions), this gives H^s control on $h_{02} - h_{12}$. In addition we already have H^s energy control of $h_{02} + h_{12}$. Thus we obtain H^s energy estimates for h_{02} and h_{12}. The same argument applies to the case $A = 3$.

Now according to (3.8), (3.9), the H^s Dirichlet control on h_{0A}, h_{1A}, $\frac{1}{2}h_{00} + h_{01}$ and $h_{00} - h_{11}$ at the boundary $\mathcal{C}$ gives H^{s-1} control of $u = (\partial_0 + \frac{1}{2}\partial_1)h_{00} + \partial_0\tau$ and $v = (\frac{1}{2}\partial_0 + \partial_1)h_{00} - \partial_1\tau$. Consider the combination $u - (2+\sqrt{3})v$, which is then also controlled in H^{s-1}. Simple computation gives

$$u - (2 + \sqrt{3})v = (\partial_0 + (2 + \sqrt{3})\partial_1)(-\tfrac{\sqrt{3}}{2}h_{00} + \tau),$$

which is a Sommerfeld boundary operator on $(-\frac{\sqrt{3}}{2}h_{00} + \tau)$. Thus the H^s energy of $w = (-\frac{\sqrt{3}}{2}h_{00} + \tau)$ is controlled. Furthermore $u = (\partial_0 + \frac{1}{2}\partial_1)h_{00} + \partial_0(\frac{\sqrt{3}}{2}h_{00} - \frac{\sqrt{3}}{2}h_{00} + \tau) = [(1 + \frac{\sqrt{3}}{2})\partial_0 + \frac{1}{2}\partial_1]h_{00} + \partial_0 w$ which thus gives a bound on the Sommerfeld operator $[(1 + \frac{\sqrt{3}}{2})\partial_0 + \frac{1}{2}\partial_1]h_{00}$ on h_{00}. Thus we obtain H^s energy estimates for h_{00}, which further yields H^s energy estimates for τ via (3.8). Combined with (3.11), (3.13), (3.14), this also gives energy estimates for h_{22}, h_{33}, h_{11} and h_{01}.

■

Remark 3.3. The method of proof of Proposition 3.2 shows that the harmonic gauge condition $V = 0$ on $\mathcal{C}$ determines a natural choice of Dirichlet-type boundary data (3.13)-(3.15), at least given the choice of the conformal class $[\gamma]$ as boundary data. The method of proof also has an upper-triangular character, similar to the upper-triangular form or bootstrap method introduced and employed in [15], [16], cf. also [14].

Next we consider boundary conditions more intrinsic to the boundary $\mathcal{C}$. We will keep the three boundary conditions for the projection $\Theta_{\mathcal{C}} = \{\theta_\alpha\}$, $\alpha = 0, 2, 3$ of Θ tangential to $\mathcal{C}$, but drop the condition for the normal or transverse component θ_1; instead we seek a replacement for θ_1 with data more intrinsic or geometric to the boundary $\mathcal{C}$. We first present below a general discussion of this situation. The result of this analysis is then summarized in Proposition 3.4 below.

In the following, we denote by O_j a boundary term which has been controlled in $H^{s-j}(\mathcal{C})$ by preceding arguments and let $O = O_0$. Thus from (3.11) or (2.23), we have $h_{22} = h_{33} = \tau + O$, while from (2.23), $h_{01} = -\frac{1}{2}h_{00} + O$ and $h_{1A} = -h_{0A} + O$. Applying these replacements in (3.5), we obtain

$$-\tfrac{1}{2}\partial_0 h_{11} - \tfrac{1}{2}(\partial_0 + \partial_1)h_{00} - \partial_0\tau + X = O \tag{3.16}$$

$$\tfrac{1}{2}\partial_1 h_{11} + \tfrac{1}{2}(\partial_0 + \partial_1)h_{00} - \partial_1\tau - X = O \tag{3.17}$$

$$-(\partial_0 + \partial_1)h_{0A} + \tfrac{1}{2}\partial_A(h_{00} - h_{11}) = O, \tag{3.18}$$

where X denotes $\partial_A h_{0A}$.

We first seek an equation involving only the terms h_{00} and h_{11}. To do this, we use Hamiltonian constraint (6.13), (6.16) on both the timelike boundary $\mathcal{C} = \{x^1 = 0\}$ as well as the spacelike hypersurfaces $\{t_0 = \text{constant}\}$. The linearization of these equations is given in the Appendix §6.2. From (6.15), on $\mathcal{C}$ one has

$$(\partial_0\partial_0 - \partial_1\partial_1)h_{00} + (\partial_0\partial_0 + \partial_1\partial_1)\tau - 2\partial_0 X = O_2, \tag{3.19}$$

while on the hypersurfaces $\{t_0 = \text{constant}\}$, from (6.17) one has

$$(\partial_0\partial_0 - \partial_1\partial_1)h_{11} + (\partial_1\partial_1 + \partial_0\partial_0)\tau + 2\partial_1 X = O_2. \tag{3.20}$$

Taking the difference (3.19)−(3.20) gives

$$(\partial_0\partial_0 - \partial_1\partial_1)(h_{00} - h_{11}) - 2(\partial_0 + \partial_1)X = O_2, \tag{3.21}$$

which is equivalent to

$$(\partial_0 + \partial_1)[(\partial_0 - \partial_1)(h_{00} - h_{11}) - 2X] = O_2. \tag{3.22}$$

This is a Sommerfeld boundary condition on $[(\partial_0 - \partial_1)(h_{00} - h_{11}) - 2X]$ and thus as in Proposition 3.3, we obtain:

$$(\partial_0 - \partial_1)(h_{00} - h_{11}) - 2X = O_1. \tag{3.23}$$

Now taking $\partial_1(3.16) - \partial_0(3.17)$ yields:

$$-\partial_0\partial_1 h_{11} - \tfrac{1}{2}(\partial_0 + \partial_1)^2 h_{00} + (\partial_1 + \partial_0)X = O_2. \tag{3.24}$$

Taking then 2×(3.24)+(3.21) gives

$$-2\partial_0\partial_1 h_{11} - (\partial_0 + \partial_1)^2 h_{00} + (\partial_0\partial_0 - \partial_1\partial_1)(h_{00} - h_{11}) = O_2.$$

which can be simplified as:

$$(-\partial_0\partial_0 + \partial_1\partial_1 - 2\partial_0\partial_1)h_{11} - 2\partial_1(\partial_0 + \partial_1)h_{00} = O_2. \tag{3.25}$$

Factorizing gives $(-\partial_0\partial_0 + \partial_1\partial_1 - 2\partial_0\partial_1)h_{11} = (\partial_1 + (\sqrt{2} - 1)\partial_0)(\partial_1 - (\sqrt{2} + 1)\partial_0)h_{11}$. The first factor, again of Sommerfeld type, leads to suitable energy estimates as in Proposition 3.2; the second factor however does not. Thus we seek a remaining boundary operator of the form

$$(\partial_0 + \partial_1)h_{00} - \alpha\partial_0 h_{11} - \beta\partial_1 h_{11} = O_1. \tag{3.26}$$

Based on the Dirichlet-to-Neumann estimate as discussed in the proof of Proposition 3.2, taking ∂_1 of (3.26) yields

$$\partial_1(\partial_0 + \partial_1)h_{00} - \alpha\partial_0\partial_1 h_{11} - \beta\partial_1\partial_1 h_{11} = O_2, \tag{3.27}$$

and adding this to (3.25) gives $(-\partial_0\partial_0 + \partial_1\partial_1 - 2\partial_0\partial_1)h_{11} - 2\alpha\partial_0\partial_1 h_{11} - 2\beta\partial_1\partial_1 h_{11} = O_2$, i.e.

$$[\partial_0\partial_0 + 2(\alpha + 1)\partial_0\partial_1 + (2\beta - 1)\partial_1\partial_1]h_{11} = O_2. \tag{3.28}$$

For this to be a well-posed (Sommerfeld-type) boundary condition, one must have

$$\alpha + 1 \geq 0,\ 2\beta - 1 \geq 0,\ (\alpha + 1)^2 \geq 2\beta - 1. \tag{3.29}$$

If the inequalities above are satisfied, (3.28) can be taken as a"double Sommerfeld" type boundary condition on h_{11}.

It remains to check what mean curvature condition H_{g_F} can lead to (3.26). Note that since we already have energy estimate for F' by Proposition 3.1, in the following we only consider variation of various mean curvature terms with respect to the deformation h. Let K be the second fundamental form of t_0-hypersurface in $\mathbf{R}$, and A be the second fundamental form of the time-like boundary $\{x_0^1 = 0\} \subset \mathbf{R}$. Let $tr_{S_t}K$ denote the full trace of K on the t-hypersurface S_t and $tr_{\Sigma_t}K$ the restricted trace of K on Σ_t. Similarly, let $tr_{\mathcal{C}}A$ be the full trace of A on the time-like boundary, with $tr_{\Sigma_t}A$ the restricted trace. The linearizations of these terms are given by (cf. Appendix §6.2):

$$2(tr_{S_t}K)'_h = \partial_0(h_{11} + h_{AA}) - 2\partial_1 h_{01} - 2\partial_A h_{0A} = \partial_0 h_{11} + \partial_1 h_{00} + 2\partial_0\tau - 2X + O_1 \tag{3.30}$$

$$2(tr_{\Sigma_t}K)'_h = \partial_0 h_{AA} - 2\partial_A h_{0A} = 2(\partial_0\tau - X) + O_1 \tag{3.31}$$

$$2(tr_{\mathcal{C}}A)'_h = \partial_1(-h_{00} + h_{AA}) + 2\partial_0 h_{10} - 2\partial_A h_{1A} = -(\partial_1 + \partial_0)h_{00} + 2\partial_1\tau + 2X + O_1 \tag{3.32}$$

$$2(tr_{\Sigma_t}A)'_h = \partial_1 h_{AA} - 2\partial_A h_{1A} = 2(\partial_1\tau + X) + O_1. \tag{3.33}$$

Here in (3.30) and (3.32) we have used the control on $\frac{1}{2}h_{00} + h_{01}$ as well as the control given by the Dirichlet-to-Neumann map, as in the proof of Proposition 3.2 above. Substituting the relations (3.16)-(3.18) into these equations one easily obtains

$$\begin{aligned} \partial_0 h_{00} &= -2(tr_{S_t}K)'_h, \\ \partial_1 h_{00} &= 2(tr_{\Sigma_t}A)'_h - 2(tr_{\mathcal{C}}A)'_h + 2(tr_{S_t}K)'_h, \\ \partial_0 h_{11} &= -2(tr_{\Sigma_t}K)'_h - 2(tr_{\Sigma_t}A)'_h + 2(tr_{\mathcal{C}}A)'_h, \\ \partial_1 h_{11} &= 2(tr_{\mathcal{C}}A)'_h. \end{aligned} \tag{3.34}$$

Substituting these into (3.26) transforms (3.26), after simple manipulations, into

$$\alpha(tr_{\Sigma_t}K)'_h - (\alpha+\beta+1)(tr_{\mathcal{C}}A)'_h + (\alpha+1)(tr_{\Sigma_t}A)'_h = O_1.$$

Thus we set

$$H_{g_F} = \alpha tr_{\Sigma_t}K_{g_F} - (\alpha+\beta+1)tr_{\mathcal{C}}A_{g_F} + (\alpha+1)tr_{\Sigma_t}A_{g_F}, \tag{3.35}$$

and require that α, β satisfy (3.29).

This leads to the following analog of Proposition 3.2.

Proposition 3.4. *For the gauged system* (2.17)-(2.19) *with $B_{\mathcal{C}}$ boundary data where $H_{g_F} = H$ and H_{g_F} is given by* (3.35) *satisfying* (3.29)*, its blow-up linearization* (2.21)-(2.23) *admits an H^s energy estimate*

$$|||h||^2_{\bar{H}^s(B_t)} + c||h||^2_{\bar{H}^s(\mathcal{C}_t)} \leq C[||h||^2_{\bar{H}^s(B_0)} + ||\underline{b}||^2_{\bar{H}^s(\mathcal{C}_t)} + ||\underline{c}||^2_{H^s(\mathcal{C}_t)}],$$

where $\underline{b} = \{b_\alpha\}$, $\alpha = 0, 2, 3$, $\underline{c} = \{c_A\}$, $A = 2, 3$.

Proof. The proof is the same as that of Proposition 3.2. Namely, if α, β satisfy (3.29), then by (3.28) one obtains an H^s energy estimate for h_{11}. Via (3.26) and the Dirichlet-to-Neumann estimate in §6.1, this gives an H^s energy estimate for h_{00}. Since the Dirichlet condition on $\frac{1}{2}h_{00} + h_{01}$ gives an H^s energy estimate for this term, one has the H^s energy estimate for h_{01}. Now equation (3.18) yields H^s control of h_{0A} for $A = 2, 3$ and equation (3.16) yields H^s control of τ. The Dirichlet boundary conditions give H^s control on the remaining components of $h = h_{\alpha\beta}$.

■

Remark 3.5. There are many other expressions for H_{g_F} besides (3.35) for which Proposition 3.4 remains valid; this arises from the fact that there are numerous other variants of the algebraic manipulations in (3.25)-(3.26). Similarly, other expressions for H_{g_F} preserving the validity of

Proposition 3.4 may be obtained by changing the boundary condition $\frac{1}{2}h_{00} + h_{01}$ to $\lambda h_{00} + \mu h_{01}$, for arbitrary smooth $\lambda, \mu > 0$. We will not pursue this further in general here however.

It is worth noting that when $\alpha = 0$ and $\beta = 1$, H_{g_F} is the mean curvature boundary condition L in [14]. However, the method of proof of [14], relying on estimates with pure Neumann boundary data, is rather different than the proof above.

Finally, it would be interesting to know if (with a suitable choice of λ, μ for instance), one can choose $H_{g_F} = tr_{\mathcal{C}} A$, the mean curvature of the boundary $\mathcal{C}$, as in [9].

4. Local well-posedness and geometric uniqueness for the IBVP

In this section, we use the results above to prove local versions of Theorems 1.1-1.4. We first prove well-posedness of the gauged IBVP's in (2.2)-(2.4) and (2.17)-(2.19). Following this, we turn to ungauged systems (1.8)-(1.10) and (1.17)-(1.19) and the issue of local uniqueness.

For the following results, let V be a neighborhood of $p \in \Sigma$ in M and (given a metric g) let $D^+(T \cap V)$ denote the future domain of dependence of the initial boundary data set $T \cap V$ in (V, g).

Theorem 4.1. *The IBVP for the gauged system* (2.2)-(2.4) *with initial data I as in* (2.3) *and boundary data B as in* (2.4) *is locally well-posed in*

$$H^s \times H^{s+1},$$

for $s \geq 4$, $s \in \mathbb{N}^+$. More precisely, suppose in a neighborhood V of a corner point p, equipped with a standard corner chart χ, one is given gauged g-initial data $(q, k) \in H^{s+\frac{1}{2}}(S \cap V) \times H^{s-\frac{1}{2}}(S \cap V)$ satisfying the constraint equations (1.6)*, and F-initial data $(E_0, E_1) \in H^{s+\frac{3}{2}}(S \cap V) \times H^{s+\frac{1}{2}}(S \cap V)$ together with boundary data $(G, [\gamma], \Theta) \in H^{s+\frac{3}{2}}(\mathcal{C} \cap V) \times H^{s+\frac{1}{2}}(\mathcal{C} \cap V) \times H^{s+\frac{1}{2}}(\mathcal{C} \cap V)$ as in* (2.3)-(2.4) *and satisfying the C^{s-1} compatibility conditions at $\Sigma \cap V$.*

Then there exists a triple (U, g, F) with $U \subset V \subset M$, $p \in U$, satisfying the following properties:

- *The pair (g, F) is a solution of the system* (2.2) *with*

$$(g, F) \in H^s(U) \times H^{s+1}(U).$$

  The trace of (g, F) on $T \cap U$ is in $H^s(T \cap U) \times H^{s+1}(T \cap U)$ and realizes the initial and boundary conditions (2.3)-(2.4)*.*
- *$U = D^+(T \cap U)$.*
- *On the domain U, the solution (g, F) is unique.*
- *The solution (U, g, F) on the fixed domain depends continuously on the initial and boundary data.*

Proof. Proposition 3.2 gives the existence of strong or boundary stable energy estimates for the frozen coefficient system, i.e. the linearization of the system at a standard flat configuration. The proof of well-posedness then follows from the general theory of quasi-linear initial-boundary value problems.

In more detail, consider the linearization of the system (2.2)-(2.4) at any smooth background configuration (g, F). The bulk equations are then a system of linear wave equations, coupled only at lower order. As in Proposition 3.2, the boundary conditions are the 4 Dirichlet boundary conditions for F, 6 Dirichlet boundary conditions for g and 4 gauge boundary conditions $V'_h = 0$, all satisfying the compatibility conditions. Given the existence of energy estimates for the frozen (constant) coefficient system, one obtains existence of energy estimates for the general linearized system by localization in a sufficiently small neighborhood of any corner point $p \in \Sigma$. This uses a partition of unity, giving local data of compact support, and rescaling, as discussed in §2. We refer for example to [3, Theorem 9.1], for details of this extension of energy estimates for the constant coefficient system to the general linear system. It follows that the general linearization of the system (2.2)-(2.4) at any given background has boundary stable energy estimates.

The frozen coefficient system admits a reduction to a first order symmetric hyperbolic system, (i.e. there exists a Friedrichs symmetrizer) with non-characteristic boundary. The strong or boundary stable H^s energy estimates are equivalent to the statement that the boundary conditions (2.4) are strictly maximally dissipative, cf. [3], [22]. It then follows from [3, Theorem 9.16], that the system (2.2)-(2.4), linearized at any smooth background (g, F), is well-posed in $C^r([0,t], H^{s-r}(S)) \times C^r([0,t], H^{s+1-r}(S))$, $0 \leq r \leq s$.

Finally, by a technically involved argument, the quasi-linear system is proved to be well-posed by a standard iteration or contraction mapping principle applied to a sequence of solutions of the linearized system, cf. [20], [17], [3]; the particular formulation given in Theorem 4.1 is an application of [3, Theorem 11.1].

■

We note that of course the 'size' of the domain U depends on the initial boundary data (I, B). However, the domain U on which a solution (g, F) exists is not unique; for example one may consider solutions on domains $U' \subset U$. It is only claimed that on the fixed point-set $U \subset M$, the solution (g, F) is unique. It is well-known that such uniqueness fails on domains $\widehat{U}$ which strictly contain the domain of dependence of their initial boundary data, i.e. for $\widehat{U}$ such that $D^+(\widehat{U} \cap T) \subset\subset \widehat{U}$.

Remark 4.2. The regularity stated in Theorem 4.1 is likely not optimal in that there is a loss of half of derivative in the statement. This will not be pursued further here, cf. also [3, Ch. 11]. Note that Theorem 4.1 also proves well-posedness in the space

$$(g, F) \in C^r(I, H^{s-1-r}(S)) \times C^r(I, H^{s-r}(S)),$$

for $0 \leq r \leq s-1$.

The same result and proof holds for $(I_{\mathcal{C}}, B_{\mathcal{C}})$ initial and boundary data.

Theorem 4.3. *The gauged IBVP for the system* (2.17)-(2.19) *with initial data* $I_{\mathcal{C}}$ *as in* (2.18) *and boundary data* $B_{\mathcal{C}}$ *as in* (2.19) *is locally well-posed in*

$$H^s \times H^{s+1},$$

for $s \geq 4$, $s \in \mathbb{N}^+$. *More precisely, suppose in a neighborhood* V *of a corner point* p, *equipped with a standard corner chart* χ, *one is given gauged* g*-initial data* $(q, k) \in H^{s+\frac{1}{2}}(S \cap V) \times H^{s-\frac{1}{2}}(S \cap V)$ *satisfying the constraint equations* (1.6), *and* F*-initial data* $(E_0, E_1) \in H^{s+\frac{3}{2}}(S \cap V) \times H^{s+\frac{1}{2}}(S \cap V)$ *as in* (2.18) *together with boundary data* $(G, [\gamma], \Theta_{\mathcal{C}}, H) \in H^{s+\frac{3}{2}}(\mathcal{C} \cap U) \times H^{s+\frac{1}{2}}(\mathcal{C} \cap U) \times H^{s+\frac{1}{2}}(\mathcal{C} \cap U) \times H^{s-\frac{1}{2}}(\mathcal{C} \cap U)$ *as in* (2.19), *and satisfying the* C^{s-1} *compatibility conditions at* $\Sigma \cap V$.

Then there exists a triple (U, g, F) *with* $U \subset V \subset M$, $p \in U$, *satisfying the following properties:*

- *The pair* (g, F) *is a solution of the system* (2.2) *with*

$$(g, F) \in H^s(U) \times H^{s+1}(U).$$

  The trace of (g, F) *on* $T \cap U$ *is in* $H^s(T \cap U) \times H^{s+1}(T \cap U)$ *and realizes the initial and boundary data* (2.18)-(2.19).
- $U = D^+(T \cap U)$.
- *On the domain* U, *the solution* (g, F) *is unique.*
- *The solution* (U, g, F) *on the fixed domain depends continuously on the initial and boundary data.*

Proof. The proof is the same as that of Theorem 4.1, using Proposition 3.4 in place of Proposition 3.2. ■

Remark 4.4. As mentioned in the Introduction, a simple inspection of the proofs shows that Theorems 4.1 and 4.3 remain valid when the wave map variable F is dropped, specifying then initial-boundary data solely for the gauge-reduced Einstein equations for g. Moreover, this may be done with respect to an arbitrary fixed (smooth) foliation $\mathcal{F} = \{\tau = const\}$ of U (not necessarily the standard $\{t_0 = const\}$ foliation. However, as discussed further below, it does not appear possible to glue such local solutions together to obtain solutions on larger domains in general.

The results above are for the reduced Einstein equations (2.2), so in the gauge $V_g = 0$ and with gauged initial boundary data (I, B) or $(I_{\mathcal{C}}, B_{\mathcal{C}})$. While solutions of the reduced Einstein equations give solutions of the Einstein equations (1.4), (by Lemma 2.1), with data $(\mathbf{I}, \mathbf{B})$ or $(\mathbf{I}_{\mathcal{C}}, \mathbf{B}_{\mathcal{C}})$ the uniqueness statement no longer holds. The next result removes this gauge condition and, when used in conjunction with Theorem 4.1, gives a local version of Theorem 1.1 and Theorem 1.3.

Proposition 4.5. *(Local Geometric Uniqueness I) A local solution (U, g, F) to the IBVP (1.8)-(1.10) with $U = D^+(T \cap U)$ is locally unique up to the action of $\mathrm{Diff}_0(U)$, i.e. if (g, F) and $(\widetilde{g}, \widetilde{F})$ are solutions in U with the same $(\mathbf{I}, \mathbf{B})$ data, then there exists an open subset $U' \subset U$ covering $S \cap U$ and a diffeomorphism $\varphi \in \mathrm{Diff}_0(U')$ such that*

$$\varphi^*(\widetilde{g}, \widetilde{F}) = (g, F). \tag{4.1}$$

In particular $\widetilde{g}$ is isometric to g in U'.

Proof. Fix a standard corner chart $\chi = (x^\alpha)$ on U. Since g and $\widetilde{g}$ induce the same Riemannian metric g_S and second fundamental form K on $S \cap U$, there exists a diffeomorphism ψ fixing the initial surface and boundary such that in χ the coordinate components of $\psi^*\widetilde{g}$ and its time derivative on $S \cap U$ agree with the gauged initial data ($q = g_{\alpha\beta}, k = \partial_t g_{\alpha\beta}$) of g. Meanwhile, ψ preserves the gauge independent initial and boundary data $(\mathbf{I}, \mathbf{B})$. By Lemma 2.3, there is an open subset $U' \subset U$ covering $S \cap U$ and a diffeomorphism $\varphi_1 \in \mathrm{Diff}_1(U')$ such that $(g_1, F_1) = (\varphi_1^* g, F \circ \varphi_1)$ is a solution to (2.2)-(2.4) with the $(\mathbf{I}, \mathbf{B})$ data restricted to U'. Similarly there is a diffeomorphism $\varphi_2 \in \mathrm{Diff}_1(U')$ (shrink U' if necessary) such that $(g_2, F_2) = (\varphi_2^* \psi^* \widetilde{g}, \widetilde{F} \circ \psi \circ \varphi_2)$ is also a solution to (2.2)-(2.4) with the same geometric initial and boundary data. Moreover, because the diffeomorphisms φ_1, φ_2 are equal to identity to the first order at $S \cap U'$, they preserve the gauged initial data of the metric. Hence in the chart $\chi|_{U'}$, (g_1, F_1) and (g_2, F_2) have the same gauged data (I, B). Therefore, $(g_2, F_2) = (g_1, F_1)$ by the uniqueness in Theorem 4.1 above. This proves the result. ∎

Of course the domain U in Proposition 4.5 is not unique. For instance if (U, g, F) is a solution, then so is (U', g, F) for any open subset $U' \subset U$ with $T \cap U' = T \cap U$ and $U' = D^+(T \cap U')$. Nevertheless, the same proof as above shows that if (U_1, g_1, F_1) and (U_2, g_2, F_2) are two such solutions with the same initial and boundary data on $T \cap U_1 = T \cap U_2$, then there are subdomains $V_1 \subset U_1$, $V_2 \subset U_2$ with $S \cap V_i = S \cap U_i$, $V_i = D^+(T \cap V_i)$ and a diffeomorphism $\varphi : V_1 \to V_2$, equal to the identity on $T \cap V_i$, such that $\varphi^*(g_2, F_2) = (g_1, F_1)$.

The same discussion and result holds for $(\mathbf{I}_{\mathcal{C}}, \mathbf{B}_{\mathcal{C}})$ data, leading to the local version of Theorem 1.2.

Proposition 4.6. *(Local Geometric Uniqueness II) A local solution (U, g, F) to the IBVP in (1.17)-(1.19) with $U = D^+(T \cap U)$ is locally unique up to the action of $\mathrm{Diff}_0(U)$, i.e. if (g, F) and $(\widetilde{g}, \widetilde{F})$ are solutions in U with the same $(\mathbf{I}_{\mathcal{C}}, \mathbf{B}_{\mathcal{C}})$ data, then there exists an open subset $U' \subset U$ covering $S \cap U$ and a diffeomorphism $\varphi \in \mathrm{Diff}_0(U')$ such that*

$$\varphi^*(\widetilde{g}, \widetilde{F}) = (g, F). \tag{4.2}$$

In particular $\widetilde{g}$ is isometric to g in U'.

Proof. The proof is the same as the proof of Proposition 4.5. ∎

Remark 4.7. These uniqueness results in Propositions 4.5 and 4.6 no longer hold when the wave map F is dropped from the data as in Remark 4.4. The reason is that boundary data consisting of g alone are not invariant under $\mathrm{Diff}_1(U)$ or $\mathrm{Diff}_0(U)$. In particular, the normal component ν_g is not invariant under $\mathrm{Diff}_1(U)$.

Also, as noted in (1.14), there are a number of alternate boundary conditions one may impose in place of those for Θ or $\Theta_{\mathcal{C}}$ in (1.10) or (1.19) to obtain existence results analogous to Theorems 4.1 and 4.3. For this, one only requires the linearization of the Θ-term at flat data (g_0, F_0) to have the same basic form as that analysed in §3. However, to preserve the uniqueness results above requires significant restrictions on the choice of boundary conditions for Θ; in particular the boundary data must be invariant under $\mathrm{Diff}_0(U)$.

Now recall the equivalence relation (1.21) on the initial boundary data $(\mathbf{I}, \mathbf{B})$, applied here locally in U. The same relation holds for the $(\mathbf{I}_{\mathcal{C}}, \mathbf{B}_{\mathcal{C}})$ initial boundary data, i.e. $((\mathbf{I}_{\mathcal{C}})_1, (\mathbf{B}_{\mathcal{C}})_1)) \sim ((\mathbf{I}_{\mathcal{C}})_2, (\mathbf{B}_{\mathcal{C}})_2)$ in U if and only if there is a diffeomorphism $\psi \in \mathrm{Diff}(T \cap U)$ such that

$$\psi^*((\mathbf{I}_{\mathcal{C}})_2, (\mathbf{B}_{\mathcal{C}})_2) = ((\mathbf{I}_{\mathcal{C}})_1, (\mathbf{B}_{\mathcal{C}})_1). \tag{4.3}$$

Corollary 4.8. *(Local Geometric Uniqueness) If the initial and boundary data of two pairs (U, g_1, F_1) and (U, g_2, F_2) are equivalent in the sense of either (1.21) or (4.3), then there exists an open subset $U' \subset U$ covering $S \cap U$ and a diffeomorphism $\Psi \in \mathrm{Diff}(U')$ such that*

$$\Psi^*(g_2) = g_1 \text{ and } \Psi^* F_2 = F_1 \quad \textit{in } U'. \tag{4.4}$$

Proof. The proof is the same for both sets of boundary data, so we work with $(\mathbf{I}, \mathbf{B})$ boundary data. Let (g_1, F_1) and (g_2, F_2) be solutions in U satisfying (1.17)-(1.19) with initial and boundary data $(\mathbf{I}_1, \mathbf{B}_1)$ and $(\mathbf{I}_2, \mathbf{B}_2)$ respectively. Suppose $(\mathbf{I}_1, \mathbf{B}_1)$ and $(\mathbf{I}_2, \mathbf{B}_2)$ are related as in (1.21), so that there is a diffeomorphism $\psi \in \mathrm{Diff}(T \cap U)$ such that $\psi^*(\mathbf{I}_2, \mathbf{B}_2) = (\mathbf{I}_1, \mathbf{B}_1)$. Extend ψ to a diffeomorphism $\psi \in \mathrm{Diff}(U)$. Then $\psi^*(g_2, F_2)$ is a solution with initial and boundary data $(\mathbf{I}_1, \mathbf{B}_1)$. By Proposition 4.5, there is a diffeomorphism $\varphi \in \mathrm{Diff}_0(U')$ for some $U' \subset U$ covering $S \cap U$ such that $\varphi^* \psi^*(g_2, F_2) = (g_1, F_1)$ in U'. Thus (4.4) holds with $\Psi = \varphi \circ \psi$. In particular we have $\Psi|_{T \cap U'} = \psi$.

■

Remark 4.9. The local geometric uniqueness above for the IBVP is the same as that for the Cauchy problem. Namely, for the Cauchy problem recall that if two solutions have equivalent geometric initial data (S, g, K), then there exists a local 4-diffeomorphism, i.e. isometry, relating the two solutions, restricting to suitable domains if necessary. In the terminology used in §5, any two vacuum developments of (S, g, K) have a common sub-development. Similarly here, to demonstrate that equivalent boundary data generate isometric solutions, one does not need to solve the IBVP.

However, while for the Cauchy problem the choice of initial data is geometric and independent of any gauge choice, this is not the case for the IBVP. The choice of possible boundary conditions giving rise to a well-posed IBVP is strongly dependent on the choice of gauge $V = 0$.

Remark 4.10. The existence and uniqueness results above show that local solutions (U, g, F) of the system (1.4) are (bijectively) parametrized by their initial-boundary data $(\mathbf{I}, \mathbf{B})$ or $(\mathbf{I}_{\mathcal{C}}, \mathbf{B}_{\mathcal{C}})$ on $T \cap U$. We note that this parametrization depends on the choice of time function t_0 and background metric g_R. However, this choice is not essential; different choices of (t_0, g_R) give different bijective parametrizations of the space of solutions on U.

5. Gluing and geometric uniqueness for the IBVP.

Up until this point, all the discussion has been local, in a sufficiently small neighborhood U of a corner point $p \in \Sigma$ and for the pair (g, F) consisting of a vacuum solution together with a (free) wave map gauge. We now turn to the global (in space) issue of well-posedness in a full neighborhood of the initial surface S in M. This is obtained by gluing local solutions together, using local geometric uniqueness. This gluing is first carried out in a full neighborhood $\mathcal{U}$ of the corner Σ for the pair (g, F) in Theorem 5.2 below and then followed by a similar analysis for the vacuum metric g alone in Theorems 5.5.

To pass to the interior, away from the corner Σ, recall that given initial data (g_S, K) on S, it is proved in [5] that there is a maximal solution of the Einstein vacuum equations, i.e. a maximal globally hyperbolic vacuum spacetime (M_S, g) where $S \setminus \partial S$ is embedded in M_S as a Cauchy surface with induced metric and second fundamental form equal to $\mathbb{I} = (g_S, K)$. (It is easy to generalize this result to the Cauchy problem for (M_S, g, F), with F being the wave map satisfying (1.3) and initial conditions as in (1.9)). To obtain vacuum developments of the full initial boundary data $(T, \mathbb{I}, \mathbb{B})$, we will show that the solution $(\mathcal{U}, g)$ above may be smoothly patched with the interior solution (M_S, g), giving then a global development. (The same may done when including the free wave map F).

Note that (up to isometry) solutions g of the Cauchy problem for the vacuum Einstein equations do not depend on any choice of gauge or local coordinates while the definition of boundary data is gauge (i.e. φ_g or F) dependent. This is another reason that the analysis needs to be separated into the (pure) Cauchy problem (without boundary) and the IBVP in a neighborhood $\mathcal{U}$ of the boundary $\mathcal{C}$.

We first make a couple of definitions. The arguments to follow regarding pairs (g, F) do not depend on the choice of $\mathbf{B}$ or $\mathbf{B}_{\mathcal{C}}$ boundary data, so we will not distinguish $\mathbf{B}$ and $\mathbf{B}_{\mathcal{C}}$ and use $\mathcal{B}$ to denote either one of them. An *initial boundary data set* $(T, \mathbf{I}, \mathcal{B})$ is an initial boundary set $T = S \cup \mathcal{C} \subset M$ together with initial and boundary data $(\mathbf{I}, \mathbf{B})$ as in (1.5)-(1.7) or $(\mathbf{I}_{\mathcal{C}}, \mathbf{B}_{\mathcal{C}})$ as in (1.15)-(1.16) on T satisfying the C^{s-1} compatibility conditions on Σ. As previously, we fix a time function t_0 on M and a complete Riemannian metric g_R on the target space $\widetilde{M} \supset M \supset T$.

Let r_0 denote the distance function to Σ on S with respect to the background Riemannian metric g_R. A partial initial boundary set $P \subset T$ is an initial boundary set of the form

$$P = P_{r_0} \cup (\mathcal{C}_G)_\tau$$

where $P_{r_0} \subset S$ is the r_0-tubular neighborhood of $\Sigma = \partial S$ with respect to g_R and $(\mathcal{C}_G)_\tau = \{p \in \mathcal{C} : t_0(G(p)) \in [0, \tau)\}$ with G the Dirichlet boundary data of F given in $\mathcal{B}$. We will allow $\tau = \infty$ but assume r_0 is small, so that $P_{r_0} \cong I \times \Sigma$. A partial initial boundary data set $(P, \mathbf{I}, \mathcal{B})$ is defined as the restriction of $(T, \mathbf{I}, \mathcal{B})$ to the subset P.

Definition 5.1. A *boundary vacuum development with gauge* for the partial data set $(P, \mathbf{I}, \mathcal{B})$ is a manifold-with-corner $\mathcal{U}$, equipped with a pair (g, F) solving (1.8) in $\mathcal{U}$ such that:

(1) $\mathcal{U} \cong [0, r_0) \times (\mathcal{C}_G)_\tau \subset M$ is diffeomorphic to a product neighborhood of $(\mathcal{C}_G)_\tau$ with its initial and boundary surface identified with P in a natural way.
(2) $P_{r_0} \cap \mathcal{U}$ is spacelike and $(\mathcal{C}_G)_\tau \cap \mathcal{U}$ is timelike in $(\mathcal{U}, g)$.
(3) F is a diffeomorphism in $\mathcal{U}$ and $(S_F)_t = F^{-1}(\{t_0 = t\})$ is spacelike in $(\mathcal{U}, g)$ for all $t \in [0, \tau)$.
(4) $(\mathcal{U}, g, F)$ satisfies the conditions (1.9)-(1.10) or (1.18)-(1.19) with the given initial and boundary data $\{\mathbf{I}, \mathcal{B}\}$ on P.
(5) By choosing a smaller neighborhood if necessary, we require that $\mathcal{U} = D^+(P)$, i.e. $\mathcal{U}$ is the future domain of dependence of P in $(\mathcal{U}, g)$.

We also recall from Remark 2.1(ii) that for any local solution (U, g, F) near Σ, the map F is a diffeomorphism from its domain onto its image $F(U) \subset M$ in a neighborhood of Σ.

The semi-global analog of Theorems 4.1 and 4.3 is:

Theorem 5.2. *Let $(P, \mathbf{I}, \mathcal{B})$ be a partial initial boundary data set on $P = P_{r_0} \cup (\mathcal{C}_G)_\tau$, with g-initial data $(g_S, K) \in H^{s+\frac{1}{2}}(P_{r_0}) \times H^{s-\frac{1}{2}}(P_{r_0})$ satisfying the constraint equations* (1.6)*, and F-initial* *data* $(E_0, E_1) \in H^{s+\frac{3}{2}}(P_{r_0}) \times H^{s+\frac{1}{2}}(P_{r_0})$ *as in* (1.5)*, together with boundary data* $(G, [\gamma], \Theta) \in H^{s+\frac{3}{2}}((\mathcal{C}_G)_\tau) \times H^{s+\frac{1}{2}}((\mathcal{C}_G)_\tau) \times H^{s+\frac{1}{2}}((\mathcal{C}_G)_\tau)$ *as in* (1.7) *(or* $(G, [\gamma], H, \Theta_{\mathcal{C}}) \in H^{s+\frac{3}{2}}((\mathcal{C}_G)_\tau) \times H^{s+\frac{1}{2}}((\mathcal{C}_G)_\tau) \times H^{s-\frac{1}{2}}((\mathcal{C}_G)_\tau) \times H^{s+\frac{1}{2}}((\mathcal{C}_G)_\tau)$ *as in* (2.19)*), satisfying the* C^{s-1} *compatibility conditions. Then there exists* $\tau' > 0$*,* $r_0' > 0$ *so that the subset* $P' = P_{r_0'} \cup (\mathcal{C}_G)_{\tau'}$ *admits a boundary vacuum development with gauge, i.e.* $(\mathcal{U}, g, F)$ *with*

$$(g, F) \in H^s(\mathcal{U}) \times H^{s+1}(\mathcal{U}), \tag{5.1}$$

and (g, F) *has trace on* P' *in* $H^s(P') \times H^{s+1}(P')$ *realizing the conditions in Definition 5.1.*

Two boundary vacuum developments of the same partial initial boundary data $(P, \mathbf{I}, \mathcal{B})$ *are equivalent in a neighborhood of* $P' \subset P$*.*

Proof. Here we give the proof in the case $\mathcal{B} = \mathbf{B}$. The same proof works for $\mathcal{B} = \mathbf{B}_{\mathcal{C}}$. By the local existence theorem, Theorem 4.1, for any point $p \in \Sigma$, there exists an open neighborhood V_p in T admitting a vacuum development. Choose then a finite collection of open subsets $\{V_n\}_{n=1}^m$ of T covering the corner Σ. Each V_n is equipped with initial and boundary data $(\mathbf{I}_n, \mathbf{B}_n)$ obtained by restricting $(\mathbf{I}, \mathbf{B})$ to V_n and each $(V_n, \mathbf{I}_n, \mathbf{B}_n)$ admits a vacuum development (U_n, g_n, F_n). When two subsets V_n and V_m overlap, their vacuum developments can be patched together in the following way.

Let U_n' denote the image of U_n under F_n i.e. $U_n' = F_n(U_n)$ and let $(g_F)_n$ denote the pull-back metric $(F_n^{-1})^* g_n$ on U_n'. Then $((g_F)_n, Id_{U_n'})$ is a solution to (1.8)-(1.10) on U_n' with the initial and boundary data $(\mathbf{I}_n', \mathbf{B}_n')$ on $T \cap U_n'$ given by

$$\mathbf{I}_n' = \{\big((E_0^{-1})^* g_S, (E_0^{-1})^* K\big), (Id_{S \cap U_n'}, E_1)\}, \quad \mathbf{B}_n' = \{Id_{\mathcal{C} \cap U_n'}, [\gamma], \Theta\}.$$

The same applies to (U_m, g_m, F_m), so we obtain $(U_m', (g_F)_m, Id_{U_m'})$. Observe that on the common overlapping initial boundary surface $T \cap U_n' \cap U_m'$, $((g_F)_n, Id_{U_n'})$ and $((g_F)_m, Id_{U_m'})$ have the same initial and boundary data. By geometric uniqueness Theorem 4.5, there is a subdomain U_{nm}' covering $P' \cap U_n' \cap U_m'$ for some $P' = P_{r_0'} \cup (\mathcal{C}_G)_{\tau'}$ with τ' small and a diffeomorphism $\varphi \in \mathrm{Diff}_0(U_{mn}')$ such that $(g_F)_n = \varphi^*(g_F)_m$ and $Id_{U_{nm}'} = Id_{U_{nm}'} \circ \varphi$. Obviously from the latter equation, $\varphi = Id_{U_{nm}'}$. Hence

$$(g_F)_n = (g_F)_m$$

in the overlap U_{mn}'. It follows by induction that the local metrics g_n' can be trivially glued together to obtain a solution $(g_F, Id_{\mathcal{U}'})$ on some neighborhood $\mathcal{U}'$ of Σ in M satisfying (1.8)-(1.10) with initial and boundary data given by

$$\mathbf{I}' = \{\big((E_0^{-1})^* g_S, (E_0^{-1})^* K\big), (Id_{S \cap \mathcal{U}_n'}, E_1)\}, \quad \mathbf{B}' = \{Id_{\mathcal{C} \cap \mathcal{U}_n'}, [\gamma], \Theta\}.$$

Since $\mathcal{U}'$ is patched up by finite local solutions, it is easy to adjust the domain so that $\mathcal{C} \cap \mathcal{U}' = \{t_0 \in [0, \tau)\}$ for some $\tau > 0$ and $S \cap \mathcal{U}' = E_0(P_{r_0'})$ for some $r_0' > 0$. Now construct a diffeomorphism $F : M \to M$ such that $F|_S = E_0$ and $F|_{\mathcal{C}} = G$. Let $g = F^* g_F$ and $\mathcal{U} = F^{-1}(\mathcal{U}')$. Then it is easy to check that $(\mathcal{U}, g, F)$ is a boundary vacuum development with gauge of some sub-data $(P', \mathbf{I}, \mathbf{B})$ of $(P, \mathbf{I}, \mathcal{B})$.

Next let $(\mathcal{U}_1, g_1, F_1)$ and $(\mathcal{U}_2, g_2, F_2)$ be a pair of boundary vacuum developments with gauge of the same $(P, \mathbf{I}, \mathcal{B})$. By local uniqueness, at every corner point $p \in \Sigma$ there is an open neighborhood U and a diffeomorphism $\varphi \in \mathrm{Diff}_0(U)$ such that $\varphi^* g_1 = g_2$ and $F_1 \circ \varphi = F_2$ on U. Since F_i $(i = 1, 2)$ is a local diffeomorphism, the second equation uniquely determines $\varphi = F_1^{-1} \circ F_2$. Patching up naturally such local neighborhoods, we obtain an open neighborhood $\mathcal{U}$ covering a subset $P' \subset P$ in which $\Phi^* g_1 = g_2$ and $F_1 \circ \Phi = F_2$ for the unique $\Phi \in \mathrm{Diff}_0(\mathcal{U})$ determined by $\Phi = F_1^{-1} \circ F_2$. ∎

Remark 5.3. As noted in Remark 4.4, with regard to local existence one may drop the wave map F and locally solve the IBVP for the metric g with $\mathbf{B}$ or $\mathbf{B}_{\mathcal{C}}$ boundary conditions. This is done with respect to a local chart $\chi : U \to \mathbf{R}$ in which the coordinate functions are g-harmonic. Suppose $\chi' : U' \to \mathbf{R}$ is another local chart with $U \cap U' \neq \emptyset$ giving rise to a solution g' in U'. If the chart χ is affinely related to the χ chart on $U \cap U'$, then the coordinates of χ' are also harmonic with respect to g, and so the uniqueness in Theorem 4.1 implies that $g' = g$ on $U \cap U'$.

In this very special case, where the domain of g has an atlas of affinely related charts preserving the manifold-with-corner structure, (so the domain has an affine-flat structure) with corresponding affine-related initial and boundary data, one may patch together local solutions to obtain a larger solution g. However, there appears to be no method to prove such solutions are unique.

Theorem 5.2 contains the main part of the proofs of Theorems 1.1 and 1.2, as well as the main parts of the uniqueness results in Theorems 1.3-1.4; it remains only to show that solutions $(\mathcal{U}, g, F)$ may be consistently glued with the maximal solution to the Cauchy problem with initial data $\mathbf{I}$ on S. This is now quite straightforward; see Remark 5.9. Instead, we proceed to discuss the analog of Theorem 5.2 for the coupled system (g, φ_g) with preferred gauge φ_g as in Theorem 1.5.

Note first that given a fixed metric g, there is a unique solution φ_g of the system (1.32)-(1.34) in a domain $\mathcal{U}$ containing Σ. Since the boundary conditions for φ_g are a simple combination of Sommerfeld and Dirichlet boundary conditions, this existence and uniqueness follows by standard results for IBVP's of systems of semi-linear wave equations. The conditions on the initial data (E_{g_S}, T_{g_R}) imply that $\varphi_g : \mathcal{U} \to \mathcal{U}' \subset M$ is a diffeomorphism onto its image $\mathcal{U}'$. The uniqueness also gives the equivariance property (1.27).

Recall that $\mathbb{I} = (g_S, K)$, $\mathbb{B} = ([\gamma], H)$ denote the initial and boundary data in the system (1.29)-(1.31). In the following the initial boundary data set $(T, \mathbb{I}, \mathbb{B})$ and partial initial boundary data set $(P, \mathbb{I}, \mathbb{B})$ are naturally defined as above.

Definition 5.4. A *boundary vacuum development* for the partial data set $(P, \mathbb{I}, \mathbb{B})$ is a manifold-with-corner $\mathcal{U}$, equipped a Ricci-flat Lorentz metric g in $\mathcal{U}$ such that:

(1) $\mathcal{U} \cong [0, r_0) \times (\mathcal{C}_{\varphi_g})_\tau \subset M$ is diffeomorphic to a product neighborhood of $(\mathcal{C}_{\varphi_g})_\tau$ with its initial and boundary surface identified with P naturally.
(2) The unique wave map φ_g associated to g via (1.32)-(1.34) is a diffeomorphism in $\mathcal{U}$; and the partial level set $(S_{\varphi_g})_t = \varphi_g^{-1}(S_t) \cap \mathcal{U}$ is space-like in $(\mathcal{U}, g)$ for all $t \in [0, \tau)$.
(3) $P_{r_0} \cap \mathcal{U}$ is spacelike and $(\mathcal{C}_{\varphi_g})_\tau \cap \mathcal{U}$ is timelike in $(\mathcal{U}, g)$.
(4) $(\mathcal{U}, g)$ satisfies the conditions (1.30)-(1.31) with the given initial and boundary data $(\mathbb{I}, \mathbb{B})$ on P.
(5) By choosing a smaller neighborhood if necessary, we require that $\mathcal{U} = D^+(P)$, i.e. $\mathcal{U}$ is the future domain of dependence of P in $(\mathcal{U}, g)$.

The semi-global analog of Theorem 4.3 is:

Theorem 5.5. *Let $(P, \mathbb{I}, \mathbb{B})$ be a partial initial boundary data set on $P = P_{r_0} \cup \mathcal{C}_\tau$, with g-initial data $(g_S, K) \in H^{s+\frac{1}{2}}(P_{r_0}) \times H^{s-\frac{1}{2}}(P_{r_0})$ satisfying the constraint equations (1.6), together with boundary data $([\gamma], H) \in H^{s+\frac{1}{2}}((\mathcal{C}_G)_\tau) \times H^{s-\frac{1}{2}}((\mathcal{C}_G)_\tau)$ as in (1.28) satisfying the C^{s-1} compatibility conditions. Then there exists $\tau' > 0$, $r_0' > 0$ so that there is a boundary vacuum development $(\mathcal{U}, g)$ for the subset $P' = P_{r_0'} \cup (\mathcal{C}_{\varphi_g})_{\tau'} \subset P$ with*

$$g \in H^s(\mathcal{U}), \tag{5.2}$$

and g has trace on P' in $H^s(P')$ realizing the conditions in Definition 5.1.

Two boundary vacuum developments of the same partial initial boundary data $(P, \mathbb{I}, \mathbb{B})$ are equivalent, and so in particular isometric, in a neighborhood of $P' \subset P$.

Proof. To show existence of a vacuum development $(\mathcal{U}, g)$, we expand the initial and boundary data $(\mathbb{I}, \mathbb{B})$ to

$$\mathbf{I} = \{g_S, K, E_0 = E_{g_S}, E_1 = T_{g_R}\}, \ \mathcal{B} = \{G, [\gamma], H, \Theta_{\mathcal{C}} = T^c_{g_R}\},$$

where the notation is the same as in the coupled system (1.24)-(1.26). Here G is an arbitrary diffeomorphism $G : \mathcal{C} \to \mathcal{C}$ satisfying the compatibility conditions. Then Theorem 5.2 shows that there is a boundary vacuum development with gauge (g, F) for some subset P' of the partial initial boundary data $(P, \mathbf{I}, \mathcal{B})$. Observe here that the wave map F must be equal to the unique diffeomorphism φ_g determined by (1.32)-(1.34). It then follows that the so obtained $(\mathcal{U}, g)$ is a boundary vacuum development of P'.

Suppose $(\mathcal{U}_1, g_1)$ and $(\mathcal{U}_2, g_2)$ are two boundary vacuum developments of the same partial initial boundary data $(P, \mathbb{I}, \mathbb{B})$. We can set $g = g_1$ (and $g = g_2$) in the system (1.32)-(1.34) and then solve for the unique φ_{g_1} in $\mathcal{U}_1$ (and φ_{g_2} in $\mathcal{U}_2$). Next consider the pull-back metrics $g_{\varphi_1} = (\varphi_{g_1}^{-1})^* g_1$ and $g_{\varphi_2} = (\varphi_{g_2}^{-1})^* g_2$. Let $\mathcal{U} = \varphi_{g_1}(\mathcal{U}_1) \cap \varphi_{g_2}(\mathcal{U}_2)$. Then the triples $(\mathcal{U}, g_{\varphi_1}, Id_{\mathcal{U}})$ and $(\mathcal{U}, g_{\varphi_2}, Id_{\mathcal{U}})$ are both vacuum developments with gauge of some common subset $P' \subset (P, \mathbf{I}, \mathcal{B})$ where

$$\mathbf{I} = \{g_S, K, E_0 = Id_S, E_1 = T_{g_R}\}, \ \mathcal{B} = \{G = Id_{\mathcal{C}}, [\gamma], H, \Theta_{\mathcal{C}} = T^c_{g_R}\}.$$

By the uniqueness result in Theorem 5.2, there exists a diffeomorphism $\Phi \in \mathrm{Diff}_0(\mathcal{U})$ such that $\Phi^* g_{\varphi_1} = g_{\varphi_2}$ and $\varphi_{g_1} \circ \Phi = \varphi_{g_2}$. Therefore, g_1 and g_2 are equivalent – in fact they are related by the unique diffeomorphisms determined by (1.32)-(1.34), i.e. $(\varphi_{g_1}^{-1} \circ \varphi_{g_2})^* g_{\varphi_1} = g_{\varphi_2}$. ∎

Next we define the vacuum development of global initial and boundary data $(T, \mathbb{I}, \mathbb{B})$ on M.

Definition 5.6. A *vacuum development* of the initial boundary data $(T, \mathbb{I}, \mathbb{B})$ is an open subset $\mathcal{M} \subset M$ such that $\{p \in M : t_0(p) < \tau\} \subset \mathcal{M}$ for some $\tau > 0$, equipped with a Ricci-flat Lorentz metric g such that:

(1) S is spacelike and $\mathcal{C} \cap \mathcal{M}$ is timelike with respect to g. In addition, $\mathcal{M} = D^+(T \cap \mathcal{M})$ i.e. $\mathcal{M}$ is the future domain of dependence of $T \cap \mathcal{M}$ in $(\mathcal{M}, g)$.
(2) The unique wave map φ_g associated to g via (1.32)-(1.34) is a diffeomorphism in a neighborhood $\mathcal{U}$ of $\mathcal{C} \cap \mathcal{M}$; and the partial level set $(S_{\varphi_g})_t = \varphi_g^{-1}(S_t) \cap \mathcal{U}$ is space-like in $(\mathcal{M}, g)$ for all $t \in [0, \tau)$.
(3) $(\mathcal{M}, g)$ satisfies the initial and boundary conditions in (1.30)-(1.31) with the given initial and boundary data $(\mathbb{I}, \mathbb{B})$ restricted to $T \cap \mathcal{M}$.
(4) The boundary $\mathcal{M}_{\mathcal{C}} = \mathcal{M} \cap \mathcal{C}$ has the form $\mathcal{M}_{\mathcal{C}} = (\mathcal{C}_g)_\tau = \varphi_g^{-1}\{t_0 \in [0, \tau)\}$ where φ_g is the unique diffeomorphism determined by g via (1.32)-(1.34).

Regarding condition (4) above, because the boundary conditions include the specification of a conformal class of metrics on the surfaces $(\Sigma_{\varphi_g})_\tau = {\varphi_g}^{-1}\{t_0 = \tau\}$, the domain of $\mathcal{M}_{\mathcal{C}}$ must be a union of such surfaces. Since $\mathcal{M}_{\mathcal{C}}$ is path connected, one has either $\mathcal{M}_{\mathcal{C}} = {\varphi_g}^{-1}\{t_0 \in [0, \tau)\}$ or $\mathcal{M}_{\mathcal{C}} = {\varphi_g}^{-1}\{t_0 \in [0, \tau]\}$, for some $\tau > 0$. Note that, without loss of generality, one may choose $\mathcal{M}_{\mathcal{C}}$ to be open in $\mathcal{C}$. Namely, any slice $(\Sigma_{\varphi_g})_{\tau'} \subset \mathcal{M}_{\mathcal{C}}$ extends to a partial initial data set $P_{r_0} \subset (S_{\varphi_g})_{\tau'}$ which is spacelike for r_0 small, giving then a local partial initial boundary data set $(P, \mathbf{I}, \mathcal{B})$ at $(\Sigma_{\varphi_g})_{\tau'}$. By Theorem 5.2, one may then extend the given vacuum development for a small time to the future of $(\Sigma_{\varphi_g})_{\tau'}$.

Combining Theorem 5.5 with the solution of the Cauchy problem gives the following result, which is a more precise version of Theorem 1.5 (and part of Theorem 1.6). Let $T_\tau = \mathcal{C}_\tau \cup S = \{p \in T : t_0(p) \in [0, \tau)\}$.

Theorem 5.7. *Let $(T, \mathbb{I}, \mathbb{B})$ be an initial boundary data set satisfying the assumptions of Theorem 5.5. Then $(T, \mathbb{I}, \mathbb{B})$ admits a vacuum development, i.e. there exists a pair $(\mathcal{M}, g)$ such that*

$$g \in H^s(\mathcal{M}), \tag{5.3}$$

and with trace on $T \cap \mathcal{M}$ in $H^s(T \cap \mathcal{M})$ realizing the conditions in Definition 5.6.

Two vacuum developments of the same initial boundary data $(T, \mathbb{I}, \mathbb{B})$ are equivalent, and so in particular isometric, in a neighborhood of T_τ, for some $\tau > 0$.

Proof. By Theorem 5.5, there is a partial initial boundary data $(P, \mathbb{I}, \mathbb{B})$ of $(T, \mathbb{I}, \mathbb{B})$ admitting a boundary vacuum development $(\mathcal{U}, g)$ defined in a neighborhood $\mathcal{U}$ of the corner Σ and unique up to diffeomorphisms equal to the identity on $S \cap P$. On the other hand, by the solution to the Cauchy problem for the vacuum Einstein equations, the interior initial data $(S, \mathbb{I}) = (S, (g_S, K))$ also admits a vacuum development $(\mathcal{U}_{\rm int}, g_{\rm int})$, unique up to diffeomorphism in $\mathrm{Diff}_0(\mathcal{U}_{\rm int})$. For convenience, we choose $\mathcal{U}_{\rm int}$ to be the maximal Cauchy development of the initial data and view $\mathcal{U}_{\rm int} \subset M$.

By construction $\mathcal{U}_{\rm int} \cap \mathcal{U}$ is an open neighborhood of $\mathcal{U} \cap S$ in M. Since g and $g_{\rm int}$ both solve the Ricci-flat equation (1.8) in $\mathcal{U}_{\rm int} \cap \mathcal{U}$ and satisfy the same geometric initial condition $\mathbb{I}$ on $S \cap \mathcal{U}$, there is an open $\mathcal{V} \subset \mathcal{U}_{\rm int} \cap \mathcal{U}$ covering $S \cap \mathcal{U}$ and a diffeomorphism $\varphi : \mathcal{V} \to \mathcal{V}$ fixing $S \cap \mathcal{U}$ such that $g = \varphi^* g_{\rm int}$ on $\mathcal{V}$. By shrinking the open sets $\mathcal{U}$ and $\mathcal{U}_{\rm int}$, we can assume $\mathcal{U} \cap \mathcal{U}_{\rm int} = \mathcal{V}$ and then extend φ to be a diffeomorphism $\mathcal{U}_{\rm int} \to \mathcal{U}_{\rm int}$ which fixes S. We can then glue $(\mathcal{U}, g)$ with $\varphi^*(\mathcal{U}_{\rm int}, g_{\rm int})$ naturally to obtain $(\mathcal{M}, g)$ which solves (1.29) and satisfies condition (2),(3) in Definition 5.6. It is easy to adjust the spacetime $(\mathcal{M}, g)$ so that it satisfies the other two conditions (1),(4) in the definition.

Next suppose $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$ are both vacuum developments of the same initial boundary data $(T, \mathbb{I}, \mathbb{B})$. Then by Theorem 5.5 there is a neighborhood $\mathcal{U}$ of the partial boundary $P = P_{r_0} \cup \mathcal{C}_\tau$ for some $r_0 > 0$ and $\tau > 0$ so that $(\varphi_{g_1}^{-1} \circ \varphi_{g_2})^* g_1 = g_2$ in $\mathcal{U}$. By standard uniqueness results in the solution of the Cauchy problem on $(S, \mathbb{I})$, there is a neighborhood $\mathcal{U}_S$ of the initial surface S and a diffeomorphism φ fixing S such that $\varphi^* g_1 = g_2$ on $\mathcal{U}_S$. Observe that in the overlap $\mathcal{U} \cap \mathcal{U}_S$, the maps $(\varphi_{g_1}^{-1} \circ \varphi_{g_2})$ and φ both equal the identity on S and push forward T_{g_1} to T_{g_2}. In addition they both pull back the metric g_1 to g_2. It follows that $\varphi_{g_1}^{-1} \circ \varphi_{g_2} = \varphi$ in $\mathcal{U} \cap \mathcal{U}_S$. The map φ may thus be naturally extended to a map $\varphi : \mathcal{M}' = \mathcal{U} \cup \mathcal{U}_S \to M$ which is a diffeomorphism onto its image and which fixes the initial surface S. Hence g_1 and g_2 are related by φ in $\mathcal{M}'$ and in particular related by $(\varphi_{g_1}^{-1} \circ \varphi_{g_2})$ in a neighborhood of the time-like boundary. ■

At this point, we make several remarks.

Remark 5.8. The proof of Theorem 1.6 (Geometric Uniqueness III) now basically follows as in the proof of Corollary 4.8. Suppose g_1, g_2 are two vacuum solutions in $\mathcal{M}$ with $\psi^*(\mathbb{I}_2, \mathbb{B}_2) = (\mathbb{I}_1, \mathbb{B}_1)$ as in (1.35) for some $\psi \in \mathrm{Diff}'(S)$. First extend ψ to a diffeomorphism of $\mathcal{M}$ and set $\widetilde{g}_2 = \psi^* g_2$. Their initial data is then related by $(\widetilde{g}_2)_S = (\psi|_S)^*(g_2)_S$, so it follows that the initial data constructed for the system (1.32)-(1.34) are related by $E_{(\widetilde{g}_2)_S} = E_{(g_2)_S} \circ \psi|_S$ in a neighborhood of Σ. Thus the unique wave map associated to $\widetilde{g}_2$ is given by $\varphi_{\widetilde{g}_2} = \varphi_{g_2} \circ \psi$ in some neighborhood $\mathcal{U}$ of the boundary $\mathcal{C} \cap \mathcal{M}$. Consequently, $\widetilde{g}_2$ and g_2 have the same boundary data $([(\varphi_g^{-1})^* g^t], H_{(\varphi_g^{-1})^* g})$ when expressed in their preferred gauge. Now initial and boundary data of g_1 and $\widetilde{g}_2$ are equal $(\widetilde{\mathbb{I}}_2, \widetilde{\mathbb{B}}_2) = (\mathbb{I}_1, \mathbb{B}_1)$. It follows from the theorem above that $\widetilde{g}_2$ and g_1 are isometric in a common subdomain of $\mathcal{M}$ and hence so are g_2 and g_1.

Remark 5.9. The proof of Theorem 5.7 also holds for the systems (g, F) in (1.8)-(1.10) and (1.17)-(1.19). To see this, note that it is straightforward to extend the existence of a maximal Cauchy development (M_S, g) of initial data $(S, \mathbb{I})$ to existence of a maximal Cauchy development (M_S, g, F)

where F is a wave map as in (3.1) satisfying initial conditions $\mathbf{I}$ as in (1.5). The proof of Theorem 5.7 for triples $(\mathcal{M}, g, F)$ then proceeds in the same way. This completes the proofs of Theorems 1.1 and 1.2. The proofs of Theorems 1.3 and 1.4 then follow again in the same way as in Corollary 4.8.

Remark 5.10. Finally, we note that Theorem 5.7, and hence Theorems 1.5 and 1.6, also holds with boundary data $\mathbb{B} = ([\gamma], H)$ replaced by the data $([\gamma], \eta)$ for η as in (1.36). The proof is the same.

Theorem 5.7 is an exact analog of the situation for vacuum developments of Cauchy data $(S, \mathbb{I})$ on an initial time surface and so it is natural to consider the existence and uniqueness of a maximal vacuum development of an initial boundary data set $(T, \mathbb{I}, \mathbb{B})$. To do this, we proceed along the same lines as in [5], following the exposition in [13], cf. also especially [21], (as well as [23], [25] for related but distinct approaches).

To begin, as with the Cauchy problem, we pass to the abstract setting and will include both future and past developments. Fix $M \cong \mathbb{R} \times S$ with a time function t_0 on it. Let $\widetilde{M}$ be a thickening of M equipped with a Riemmanian metric g_R. The boundary data $\mathbb{B}$ are now defined on $\mathcal{C} \simeq \mathbb{R} \times \Sigma$ (and not $[0, \infty) \times \Sigma$ as before). As previously, we use S_τ and Σ_τ to denote the level sets of t_0 on M and $\mathcal{C}$. The initial boundary set T is now given by $S \cup \mathcal{C}$, with ∂S identified with $\{0\} \times \Sigma$ in the natural way. For simplicity, we work in the C^∞ setting in the analysis below.

Definition 5.11. An *(abstract) vacuum development* for the initial boundary data $(T, \mathbb{I}, \mathbb{B})$ is a manifold-with-boundary $\mathcal{M}$ with $\partial \mathcal{M} \simeq \mathcal{C}$ equipped with a Ricci-flat Lorentz metric g on $\mathcal{M}$ such that:

(1) $(\mathcal{M}, g)$ is a globally hyperbolic spacetime with timelike boundary admitting a Cauchy hypersurface-with-boundary $\mathcal{S}$.
(2) The unique wave map φ_g associated to g via (1.32)-(1.34) is a diffeomorphism in a neighborhood $\mathcal{U}$ of $\partial \mathcal{M}$; and the partial level set $(S_{\varphi_g})_t = \varphi_g^{-1}(S_t) \cap \mathcal{U}$ is space-like in $(\mathcal{M}, g)$ for all $t \in (\tau_1, \tau_2)$ with some $\tau_1 < 0, \tau_2 > 0$.
(3) There is an embedding $\iota : T' \to \mathcal{M}$ of some $T' \subset T$ such that $\iota(S) = \mathcal{S}$, $\iota(\mathcal{C} \cap T') = \partial \mathcal{M}$. In addition g induces the data $\iota^*(\mathbb{I}, \mathbb{B})$ on T' and $\iota(\mathcal{C} \cap T') = \varphi_g^{-1}(\{p \in \mathcal{C} : t_0(p) \in (\tau_1, \tau_2)\})$.

Here we regard φ_g as a map from a neighborhood of $\partial \mathcal{M}$ in the (abstract) manifold $\mathcal{M}$ to a neighborhood of $\mathcal{C}$ in the fixed manifold $(\widetilde{M}, t_0, g_R)$, determined by g via (1.32)-(1.34). In the following we use $T \cap \mathcal{M}$ to denote both the subset $T' \subset T$ and the image $\iota(T') \subset \mathcal{M}$ which can be identified via the embedding ι.

Globally hyperbolic manifolds with timelike boundary are defined in the same way as globally hyperbolic manifolds (without boundary) and have the same essential properties, cf. [1] for a recent analysis. In particular, inextendible time-like curves intersect the Cauchy surface $\mathcal{S}$ exactly once and $\mathcal{M}$ is diffeomorphic to $M = \mathbb{R} \times S$. Note that by definition $\mathcal{M}$ includes its boundary $\partial \mathcal{M} \subset \mathcal{M}$. Further, one has

$$\mathcal{M} = D(\mathcal{M} \cap T), \tag{5.4}$$

where D is the full (future and past) domain of dependence. In particular, any (abstract) vacuum development can be realized as a solution $(\mathcal{M}, g)$ with $\mathcal{M} \subset M$, as in Theorem 5.7.

Conversely, by taking the union of both future and past vacuum developments in M, Theorem 5.7 shows that any initial boundary data set $(T, \mathbb{I}, \mathbb{B})$ admits an (abstract) vacuum development.

We now turn to the existence of maximal developments. First we give a precise definition of extension.

Definition 5.12. An *extension* of the vacuum development $(\mathcal{M}, g)$ of the data set $(T, \mathbb{I}, \mathbb{B})$ is a development $(\mathcal{M}', g')$ of the same initial and boundary data such that there exists an isometric embedding $\psi : (\mathcal{M}, g) \to (\mathcal{M}', g')$ with $\psi|_{S \cap \mathcal{M}} = Id_{S \cap \mathcal{M}}$, $\psi|_{\partial \mathcal{M}} = \varphi_{g'}^{-1} \circ \varphi_g|_{\partial \mathcal{M}}$.

Lemma 5.13. *The isometric embedding from a vacuum development to its extension is unique.*

Proof. The proof is the standard one from [5]. Suppose $(\mathcal{M}', g')$ is an extension of $(\mathcal{M}, g)$ with embedding $\psi : (\mathcal{M}, g) \to (\mathcal{M}', g')$. Take any point $p \in \mathcal{M}$ and let σ be an inextendible timelike geodesic in $\mathcal{M}$ starting from p. By (5.4), σ must hit the boundary $T \cap \mathcal{M}$ at a unique point q, for which the image $\psi(q)$ is uniquely determined by $\psi|_{T\cap\mathcal{M}}$. The length or proper time $\ell(\sigma)$ and angle α between σ' and the tangent space of T uniquely determine σ. This data is preserved under an isometric embedding. Since the point $\psi(p)$ is uniquely determined by this data and $\psi(q)$, the embedding ψ is unique.

■

Theorem 5.14. *Given an initial boundary data set* $(T, \mathbb{I}, \mathbb{B})$*, up to isometry there exists a unique maximal development* $(\widetilde{\mathcal{M}}, \widetilde{g})$*. The vacuum development* $(\widetilde{\mathcal{M}}, \widetilde{g})$ *is an extension of any other vacuum development of* $(T, \mathbb{I}, \mathbb{B})$*.*

Proof. The proof follows closely that in [5], cf. also [13].

Let $\mathcal{D} = \mathcal{D}(T, \mathbb{I}, \mathbb{B})$ be the set of all vacuum developments of a given initial boundary data set $(T, \mathbb{I}, \mathbb{B})$. By Theorem 5.7, $\mathcal{D}$ is nonempty. This set is partially ordered by the extension relation; $\mathcal{M}_1 \leq \mathcal{M}_2$ in $\mathcal{D}$ if $\mathcal{M}_2$ is an extension of $\mathcal{M}_1$. If $\{M_\alpha\}$ is a totally ordered subset, then the uniqueness from Lemma 5.13 implies that the union $\cup\mathcal{M}_\alpha$ is also a vacuum development which is clearly an upper bound for $\{\mathcal{M}_\alpha\}$. It follows from Zorn's Lemma that $\mathcal{D}$ has a maximal element $(\widetilde{\mathcal{M}}, \widetilde{g})$. Any extension of $\widetilde{\mathcal{M}}$ thus equals $\widetilde{\mathcal{M}}$.

The main issue is to prove uniqueness. Suppose $(\mathcal{M}', g')$ is another vacuum development of $(T, \mathbb{I}, \mathbb{B})$; we need to prove $\widetilde{\mathcal{M}}$ is an extension of $\mathcal{M}'$.

By Theorem 5.7, any two vacuum developments of $(T, \mathbb{I}, \mathbb{B})$ are extensions of a common sub-development. Given $\widetilde{\mathcal{M}}$ and $\mathcal{M}'$, consider then the set $C(\widetilde{\mathcal{M}}, \mathcal{M}')$ of all common sub-developments of $\widetilde{\mathcal{M}}$ and $\mathcal{M}'$. This set is again partially ordered by extension and hence *it* has a maximal element $(\widehat{\mathcal{M}}, \widehat{g}, \widehat{F})$, with isometric embeddings $\psi_1 : (\widehat{\mathcal{M}}, \widehat{g}) \to (\widetilde{\mathcal{M}}, \widetilde{g})$ and $\psi_2 : (\widehat{\mathcal{M}}, \widehat{g}) \to (\mathcal{M}', g')$.

One then forms the union $\mathcal{M}' \cup \widehat{\mathcal{M}} \cup \widetilde{\mathcal{M}'}$ and divides by the equivalence relation

$$\overline{\mathcal{M}} = \widetilde{M} \cup \widehat{\mathcal{M}} \cup \mathcal{M}'/\sim,$$

where for $p \in \widehat{\mathcal{M}}$, $p \sim \psi_1(p) \in \widetilde{\mathcal{M}}$ and $p \sim \psi_2(p) \in \mathcal{M}'$. Thus one is gluing the spaces $\widetilde{\mathcal{M}}$ and $\mathcal{M}'$ together along their common isometrically embedded subspace $\widehat{\mathcal{M}}$. The data $\bar{g}$ on $\overline{\mathcal{M}}$ is well-defined.

The main claim is that $\overline{\mathcal{M}}$ is Hausdorff. Given this, the space $\overline{\mathcal{M}}$ is then a vacuum development of $(T, \mathbb{I}, \mathbb{B})$ which is an extension of both $\widetilde{\mathcal{M}}$ and $\mathcal{M}'$. Since the only extension of $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{M}}$ itself, it follows that $\widetilde{\mathcal{M}}$ is an extension of $\mathcal{M}'$. This proves the uniqueness and the fact that any vacuum development has an extension to the maximal development $\widetilde{\mathcal{M}}$.

The proof of the Hausdorff property is by contradiction. If $\overline{\mathcal{M}}$ is not Hausdorff, then there are points $\widetilde{p} \in \partial(\psi_1(\widehat{M})) \in \widetilde{M}$ and $p' \in \partial(\psi_2(\widehat{M})) \in M'$ such that every neighborhood $U \subset \widetilde{\mathcal{M}}$ of $\widetilde{p}$ has the property that the closure $\overline{\psi_2\psi_1^{-1}(U)} \subset \mathcal{M}'$ contains p'. Given $\widetilde{p}$, the associated point p' is unique and there exist neighborhoods $\widetilde{U}$ of $\widetilde{p}$ in $\widetilde{\mathcal{M}}$ and U' of p' in $\mathcal{M}'$ such that $\psi_2 \circ \psi_1^{-1}$ maps $\widetilde{U} \cap \psi_1(\widehat{M})$ to $U' \cap \psi_2(\widehat{M})$ and it can be extended to a diffeomorphism $\Psi : U \to U'$ (cf. [13]). It follows that the set $\mathcal{H}$ of all non-Hausdorff points of $\partial(\psi_1(\widehat{\mathcal{M}})) \subset \widetilde{\mathcal{M}}$ is open in $\partial(\psi_1(\widehat{\mathcal{M}}))$ and thus $\mathcal{H} \cap \widetilde{\mathcal{M}}_{\text{int}}$ is nonempty where $\widetilde{\mathcal{M}}_{\text{int}}$ denotes the interior of $\widetilde{\mathcal{M}}$.

Without loss of generality, assume a non-Hausdorff point $\widetilde{p} \in \mathcal{H} \cap \widetilde{\mathcal{M}}_{\text{int}}$ is in the future domain of dependence of $\mathcal{S}$ in $\widetilde{M}$. Let $\widetilde{\sigma}$ be an inextendible past-directed null geodesic in $\widetilde{\mathcal{M}}$ starting at $\widetilde{p}$. By the diffeomorphism Ψ above, $\sigma' = \Psi(\widetilde{\sigma}) \in \mathcal{M}'$ extends similarly to a past directed null geodesic starting at p' in $\mathcal{M}'_{\rm int}$. To each point of $\widetilde{\sigma} \cap \partial(\psi_1(\widehat{M}))$ there corresponds a point of $\sigma' \cap \partial(\psi_2(\widehat{M}))$ and so $\widetilde{\sigma} \cap \partial(\psi_1(\widehat{M})) \subset \mathcal{H}$.

Now the globally hyperbolic property (5.4) implies either

(i) $\widetilde{\sigma}$ leaves $\partial(\psi_1(\widehat{M}))$ and enters the interior of $\psi_1(\widehat{\mathcal{M}}) \subset \widetilde{\mathcal{M}}$ at some point $\widetilde{q} \in \mathcal{H} \cap \widetilde{M}_{\rm int}$,

or

(ii) for some time $s > 0$, $\widetilde{\sigma}([0, s)) \in \mathcal{H} \cap \widetilde{M}_{\rm int}$ and $\widetilde{\sigma}$ hits the timelike boundary $\partial\widetilde{M}$ at $\widetilde{q} = \widetilde{\sigma}(s)$.

In Case (i), one obtains the same contradiction as in [5], [13]. Thus we consider only Case (ii).

In Case (ii), at the point $\widetilde{q} \in \mathcal{H} \cap \mathcal{C} \subset \widetilde{\mathcal{M}}$, $\widetilde{\sigma}$ can be taken as a future-directed null geodesic starting from $\widetilde{q}$ pointing inwards $\widetilde{\mathcal{M}}$. By the definition of vacuum development, $\widetilde{q} \in (\Sigma_{\varphi_{\widetilde{g}}})_\tau$ for some $\tau > 0$ where $(\Sigma_{\varphi_{\widetilde{g}}})_\tau$ is the boundary of the spacelike hypersurface $(S_{\varphi_{\widetilde{g}}})_\tau = \varphi_{\widetilde{g}}^{-1}(S_\tau)$. Here S_τ denotes a collar neighborhood of Σ_τ in the t_0-level set S_τ, since $\varphi_{\widetilde{g}}$ is only defined near $\partial\widetilde{\mathcal{M}}$. It follows that there is an open neighborhood $S_{\widetilde{q}}$ of $\widetilde{q}$ in the partial level set $(S_{\varphi_{\widetilde{g}}})_\tau$ providing a local spacelike hypersurface such that $S_{\widetilde{q}} \setminus ((\Sigma_{\varphi_{\widetilde{g}}})_\tau \cap S_{\widetilde{q}}) \subset \psi_1(\widehat{\mathcal{M}})$.

Now the same process can be carried out for σ' hitting the timelike boundary $\mathcal{C} \cap \mathcal{M}'$ for the first time at $q' \in \Psi(\mathcal{H}) \cap \mathcal{M}'$. Then we must have $q' \in (\Sigma_{\varphi_{g'}})_\tau$ with the same τ as for $\widetilde{q}$. Setting $S_{q'} = \Psi(S_{\widetilde{q}})$ for Ψ as above based at $\widetilde{q}$ in place of $\widetilde{p}$ gives a local spacelike hypersurface containing q' with $S_{q'} \setminus ((\Sigma_{\varphi_{g'}})_\tau \cap S_{q'}) \subset \psi_2(\widehat{\mathcal{M}})$. In fact, $S_{q'}$ is a neighborhood of q' in the partial level set $(S_{\varphi_{g'}})_\tau$. The initial data $\mathbb{I}$ induced on $S_{\widetilde{q}}$ and $S_{q'}$ by $\widetilde{g}$ and g' respectively are equivalent, and the boundary data induced on $\mathcal{C}$ are the same. Neighborhoods of each of these local initial boundary data sets in $\widetilde{\mathcal{M}}$ and $\mathcal{M}'$ respectively are local vacuum developments and hence by the uniqueness of local solutions in Theorem 5.5, there is a common sub-development $\mathcal{M}^*$ of these neighborhoods.

Joining $\mathcal{M}^*$ with $\widehat{\mathcal{M}}$ gives an extension of $\widehat{\mathcal{M}}$ of which both $\widetilde{\mathcal{M}}$ and $\mathcal{M}'$ are extensions. This contradicts the maximality of $\widehat{\mathcal{M}} \in C(\widetilde{\mathcal{M}}, \mathcal{M}')$, which (together with Case (i)) proves the Hausdorff property.

■

We conclude the paper with a few final remarks.

Remark 5.15. (i). For initial data set $S \subset T$, let $\mathcal{M}_S$ be the unique maximal Cauchy development of S. Clearly $\mathcal{M}_S \subset \widetilde{\mathcal{M}}$. The existence of boundary vacuum developments $\mathcal{U} \subset \widetilde{\mathcal{M}}$ as in Theorem 5.2 implies that in small neighborhoods $\mathcal{U}$ of Σ, $\mathcal{M}_S \cap \mathcal{U}$ has a Cauchy horizon in $\mathcal{U}$. In general, the "boundary" of $\mathcal{M}_S$ may be very complicated, consisting of regions where the solution g has (curvature) singularities and is in general not well understood. Thus, the presence of boundary data near Σ has the effect of regulating the metric near the boundary Σ.

(ii). For a maximal solution $(\widetilde{\mathcal{M}}, \widetilde{g})$, let $\tau_0 = \sup\{\tau : \widetilde{\mathcal{M}} \cap (\mathcal{C}_{\varphi_{\widetilde{g}}})_\tau \neq \emptyset\}$; τ_0 is the maximal time of existence of the solution $\widetilde{M}$ at the boundary, measured in the time coordinate t_0. The solution $(\widetilde{\mathcal{M}}, \widetilde{g})$ may break down or degenerate at t_0 in two ways. One way is that the metric $\widetilde{g}$ becomes degenerate so we cannot extend the solution further. On the other hand, it may happen that the solution breaks down only because the wave map $\varphi_{\widetilde{g}}$ becomes degenerate (i.e. is no longer a diffeomorphism) at t_0. In the latter case it is possible to extend $(\widetilde{\mathcal{M}}, \widetilde{g})$ to a larger domain by defining the new initial data E_{g_S} on the partial level set $((S_{\varphi_{\widetilde{g}}})_{\tau_0}, \widetilde{g})$ for a new (or continuing) wave map and then solving for a continuing vacuum metric in this preferred gauge. In addition, it may be possible that such an extension could be done only near certain regions of $(\Sigma_{\varphi_{\widetilde{g}}})_{\tau_0}$.

(iii). We note finally that the analog of Theorem 1.7 holds with 'free' wave map gauge F, i.e. there exists a maximal vacuum development with gauge $(\widetilde{M}, \widetilde{g}, \widetilde{F})$ for a given initial boundary data $(T, \mathbf{I}, \mathcal{B})$, unique up to the action of $\mathrm{Diff}_0(\mathcal{M})$. In this case, as a vacuum solution $\widetilde{g}$ to the Einstein equations, the maximal development $(\widetilde{\mathcal{M}}, \widetilde{g}, \widetilde{F})$ depends on the choice of gauge $\widetilde{F}$, i.e. on the initial boundary data (E_0, E_1, G). Of course one may act by diffeomorphisms in $\mathrm{Diff}(\widetilde{\mathcal{M}})$ to relate the maximal developments of equivalent boundary data in the sense of (1.21) or (4.3).

6. Appendix

In this section we collect a number of results and formulas (mostly standard) used in the main text.

6.1. **Boundary Conditions and Energy Estimates.**

In this subsection, we summarize the energy estimates for the IBVP for a scalar wave equation on a Minkowski half-space with Sommerfeld, Dirichlet and also Neumann boundary data. These estimates are basically well-known and included for completeness.

Consider the scalar wave equation

$$\Box_{g_0} u = \varphi. \tag{6.1}$$

on the region $\mathbf{R} = [0, \infty) \times (\mathbb{R}^3)^+$ of Minkowski spacetime with standard coordinates (t, x^i). The stress-energy tensor S of u is given by

$$S = du^2 - \tfrac{1}{2}|du|^2 g,$$

As is well-known, the symmetric bilinear form S is conserved on-shell, i.e. if u solves the equation of motion (6.1), then

$$\delta S = -\Box_{g_0} u du = -\varphi du,$$

(cf. [13] for example). For any smooth vector field Z, one then has

$$\delta(S(Z)) = (\delta S)(Z) + \langle S, \delta^* Z\rangle = -\varphi Z(u) + \langle S, \delta^* Z\rangle.$$

Let U be any open domain in $\mathbf{R}$ with compact closure and piecewise smooth boundary ∂U. Applying the divergence theorem to the left side then gives

$$\int_{\partial U} S(Z, N) = \int_U \langle S, \delta^* Z\rangle - \varphi Z(u), \tag{6.2}$$

where N is the outward g-unit normal at the boundary. The equation (6.2) leads to the basic energy estimates.

Let

$$E_{S_t}(u) = \tfrac{1}{2}\int_{S_t} u_t^2 + |du|^2,$$

where du is the full spatial derivative. Here and below, integration is with respect to the standard measures. As in the main text, let S_t be the level set of t, $\mathcal{C}_s = \{x^1 = 0\} \cap \{t \in [0, s]\}$, $\Sigma_t = \mathcal{C} \cap S_t$ and $M_s = \{t \in [0, s]\}$. Also for this section, let $x = x^1$, and $(x^2, x^3) = (y, z)$.

Consider first $Z = \partial_t$. Then $\delta^* Z = 0$ and one obtains from (6.2)

$$\frac{d}{dt} E_{S_t}(u) + \int_{S_t} \varphi u_t = \int_{\Sigma_t} u_x u_t. \tag{6.3}$$

For $\varphi = 0$, this immediately gives the relation

$$E_{S_t}(u) = E_{S_0}(u) + \int_{\mathcal{C}_t} u_x u_t.$$

For general φ, since $|\varphi u_t| \le \frac{1}{2}(u_t^2 + \varphi^2)$, one has

$$E_{S_t}(u) \le E_{S_0}(u) + \int_0^t E_{S_s}(u) ds + \tfrac{1}{2}\int_{M_t} \varphi^2 + \int_{\mathcal{C}_t} u_x u_t.$$

The integral form of the standard Gronwall inequality then gives the bound

$$E_{S_t}(u) \leq E_{S_0}(u) + Ce^t[\int_{M_t} \varphi^2 + \int_{\mathcal{C}_t} u_x u_t]. \tag{6.4}$$

For data φ of compact support, the factor e^t may be absorbed into the constant C.

Next for $Z = \partial_x$ again first with $\varphi = 0$, (6.2) gives

$$\frac{d}{dt}\int_{S_t} u_x u_t = \int_{\Sigma_t} u_x^2 - \tfrac{1}{2}|du|^2 = \tfrac{1}{2}\int_{\Sigma_t} u_x^2 + u_t^2 - |d_A u|^2. \tag{6.5}$$

Thus

$$\tfrac{1}{2}\int_{\mathcal{C}_t} |d_A u|^2 = \tfrac{1}{2}\int_{\mathcal{C}_t} u_x^2 + u_t^2 - \int_{S_t} u_x u_t \leq \tfrac{1}{2}\int_{\mathcal{C}_t} u_x^2 + u_t^2 + E_{S_t}(u) + E_{S_0}(u). \tag{6.6}$$

For the inhomogeneous equation, using (6.4) one obtains in the same way that

$$\tfrac{1}{2}\int_{\mathcal{C}_t} |d_A u|^2 \leq \tfrac{1}{2}\int_{\mathcal{C}_t} u_x^2 + u_t^2 + C(E_{S_t}(u) + E_{S_0}(u) + \int_{M_t} \varphi^2), \tag{6.7}$$

with again C depending only on t.

Sommerfeld Boundary data: This is boundary data of the form

$$u_t + u_x = b, \tag{6.8}$$

where b is a given function on the boundary cylinder $\mathcal{C}$. Then $u_x = b - u_t$ so that

$$\int_{\Sigma_t} u_t u_x = -\int_{\Sigma_t} u_t^2 + \int_{\Sigma_t} b u_t.$$

Since $|bu_t| \leq \frac{1}{2}(u_t^2 + b^2)$, we obtain from (6.3)

$$\frac{d}{dt}\int_{S_t} u_t^2 + |du|^2 + \int_{\Sigma_t} u_t^2 \leq \int_{\Sigma_t} b^2, \tag{6.9}$$

giving the basic energy estimate

$$E_{S_t}(u) + \int_{\mathcal{C}_t} u_t^2 \leq E_{S_0}(u) + \int_{\mathcal{C}_t} b^2.$$

To extend this to a strong or boundary stable estimate, note that $u_x^2 \leq 2(u_t^2 + b^2)$, so that $u_t^2 + u_x^2 \leq 3u_t^2 + 2b^2$. Substituting this in (6.9) gives

$$\frac{d}{dt}\int_{S_t} u_t^2 + |du|^2 + \tfrac{1}{3}\int_{\Sigma_t} u_t^2 + u_x^2 \leq 2\int_{\Sigma_t} b^2.$$

Using the relation (6.6), one easily derives that

$$\int_{S_t} u_t^2 + |du|^2 + \tfrac{1}{4}\int_{\mathcal{C}_t} u_t^2 + u_x^2 + |d_A u|^2 \leq E_{S_0}(t) + 3\int_{\mathcal{C}_t} b^2,$$

for solutions u of (6.1) with $\varphi = 0$. When $\varphi \neq 0$, using (6.7), the same arguments give

$$\int_{S_t} u_t^2 + |du|^2 + \tfrac{1}{4}\int_{\mathcal{C}_t} u_t^2 + u_x^2 + |d_A u|^2 \leq E_{S_0}(u) + C[\int_{\mathcal{C}_t} b^2 + \int_{M_t} \varphi^2].$$

As is well-known, this estimate can be promoted to a full energy estimate, i.e. including the L^2 norm of u, by noting that if u satisfies (6.1), then $v = e^{ct}u$ satisfies

$$(\Box + c^2)v = \varphi v + 2cv_t.$$

The same arguments then give an energy estimate for v including the L^2 norm, which then translates to a similar energy estimate for u. In sum and in the notation of §3, this gives the strong or boundary stable H^1 estimate

$$\mathcal{E}_{S_t}(u) + \tfrac{1}{2}\mathcal{E}_{\mathcal{C}_t}(u) \leq E_{S_0}(u) + C[\int_{\mathcal{C}_t} b^2 + \int_{M_t} \varphi^2], \tag{6.10}$$

for solutions u of (6.1) with Sommerfeld boundary condition.

One obtains higher order H^s energy estimates by simple differentiation. Thus, for $i = 0, 2, 3$, so ∂_i is tangent to the boundary $\mathcal{C}$, one has, for $u_i = \partial_i u$,

$$\Box_{g_0} u_i = \partial_i \varphi,$$

and the boundary condition (6.8) becomes

$$(u_i)_t + (u_i)_x = \partial_i b.$$

It follows that one has the H^1 energy estimate for each u_i, given H^1 control on b and φ. For the normal derivative u_x, the bulk equation (6.1) gives $u_{xx} = \Box_{\mathcal{C}} u - \varphi$. The term $\Box_{\mathcal{C}} u$ is bounded in L^2 by the estimate above giving then an L^2 bound on u_{xx}, which gives then a full H^2 energy estimate. One continues in this way inductively for each s.

Dirichlet Boundary Data: Here

$$u = b$$

on $\mathcal{C}$. In this context one has

$$\int_{\Sigma_t} u_x u_t \leq \varepsilon \int_{\Sigma_t} u_x^2 + \varepsilon^{-1} \int_{\Sigma_t} u_t^2 = \varepsilon \int_{\Sigma_t} u_x^2 + \varepsilon^{-1} \int_{\Sigma_t} b_t^2,$$

so that to control $E_{S_t}(u)$, it suffices to control the Neumann derivative u_x. Also, as in (6.5), we have

$$\tfrac{1}{2} \int_{\mathcal{C}_t} u_x^2 + u_t^2 \leq E_{S_t}(u) + E_{S_0}(u) + \tfrac{1}{2} \int_{\mathcal{C}_t} |d_A u|^2, \tag{6.11}$$

so that

$$\tfrac{1}{2} \int_{\mathcal{C}_t} u_x^2 \leq E_{S_t}(u) + E_{S_0}(u) + \tfrac{1}{2} \int_{\mathcal{C}_t} |d_A b|^2. \tag{6.12}$$

The estimate (6.12) shows that one can control Neumann boundary data of u at $\mathcal{C}$ in terms of Dirichlet control of u on $\mathcal{C}$ (and the energy). In other words, consider the Dirichlet-to-Neumann map $\mathcal{N}(b) = u_x$, where u is the unique solution to the IBVP (6.1) with Dirichlet boundary data b and zero initial data. Then (6.12) gives an $L^2(\mathcal{C})$ bound for $\mathcal{N}$. This estimate is important for boundary stable energy estimates.

The same arguments as above then give the energy estimate (6.10) with Dirichlet boundary value b, with constants suitably adjusted. Similarly, in the same way as above, one obtains higher order H^s energy estimates.

Remark 6.1. An estimate of the form (6.12) with Dirichlet and Neumann data interchanged, i.e. an estimate of the form

$$\int_{\mathcal{C}_t} |du|^2 \leq C[E_{S_t}(u) + E_{S_0}(u) + \int_{\mathcal{C}_t} u_x^2]$$

does not hold, i.e. Dirichlet data cannot be controlled by Neumann data at the same level of differentiability. There is a definite loss of regularity or diffentiability, cf. [24] for a detailed analysis.

We note that in proving the well-posedness of the IBVP of quasi-linear systems such as those in (2.2)-(2.4), it is important to have boundary stable energy estimates as in (6.10).

6.2. **Linearization Formulas.**

In this subsection, for convenience we derive the formulas (3.30)-(3.33) and (3.19)-(3.20). Considering linearizations at the standard configuration $(\mathbf{R}, \eta)$, the linearization of the normal vectors T (normal to S) and ν (normal to $\mathcal{C}$) are

$$T'_h = \tfrac{1}{2}h_{00}\partial_0 - h_{0i}\partial_i, \quad \nu'_h = h_{10}\partial_0 - \tfrac{1}{2}h_{11}\partial_1 - h_{1A}\partial_A.$$

For the second fundamental form K, one has $2K = \mathcal{L}_T g$, so that $2K'_h = \mathcal{L}_T h + \mathcal{L}_{T'} g$. This gives

$$2K'_h = \nabla_T h + dh_{00} \cdot dt_0 - 2dh_{0i}dx^i_0.$$

Taking the trace with respect to η then gives (3.30) as well as (3.31). Replacing T by ν, similar computation gives (3.32)-(3.33).

Next, the Hamiltonian constraint (Gauss equation) for a vacuum solution $Ric_g = 0$ on the timelike boundary $\mathcal{C}$ is

$$R_{\mathcal{C}} - (tr_{\mathcal{C}} A)^2 + |A|^2 = 0, \tag{6.13}$$

where A is the second fundamental form of $\mathcal{C} \subset (U, g)$. For a linearization h at the flat metric η with $Ric'_h = 0$, it follows that

$$(R_{\mathcal{C}})'_h = 0. \tag{6.14}$$

It is standard that $(R_{\mathcal{C}})'_h = -\Box_{\mathcal{C}}(tr_{\mathcal{C}} h) + \delta_{\mathcal{C}}\delta_{\mathcal{C}} h_{\mathcal{C}} - g_{\mathcal{C}}(Ric_{\mathcal{C}}, h)$, which is computed as follows:

$$\begin{aligned}(R_{\mathcal{C}})'_h &= -\Box_{\mathcal{C}}(-h_{00} + 2\tau) + \partial_0\partial_0 h_{00} - 2\partial_0\partial_A h_{0A} + \partial_A\partial_B h_{AB}\\ &= \Delta_{\Sigma_t} h_{00} + \partial_0\partial_0(2\tau) - \Delta_{\Sigma_t} 2\tau - 2\partial_0\partial_A h_{0A} + \partial_A\partial_A h_{AA} + O_2\\ &= (\partial_0\partial_0 - \partial_1\partial_1)h_{00} + \partial_0\partial_0(2\tau) - \Delta_{\Sigma_t} 2\tau - 2\partial_0 X + \Delta_{\Sigma_t}\tau + O_2\\ &= (\partial_0\partial_0 - \partial_1\partial_1)h_{00} + (\partial_0\partial_0 + \partial_1\partial_1)\tau - 2\partial_0 X + O_2\end{aligned}$$

Here we have used the facts that $h_{23} = O$, $h_{22} = \tau + O$, $h_{33} = \tau + O$, $\Box h_{00} = 0$, $\Box\tau = 0$, so that for instance $\Delta_{\Sigma_t}\tau = \partial_0\partial_0\tau - \partial_1\partial_1\tau$. Thus from (6.14) we obtain

$$(\partial_0\partial_0 - \partial_1\partial_1)h_{00} + (\partial_0\partial_0 + \partial_1\partial_1)\tau - 2\partial_0 X = O_2 \tag{6.15}$$

Similarly, the Hamiltonian constraint or Gauss equation on the hypersurfaces $S_t = \{t = \text{constant}\}$ gives:

$$R_{S_t} + (tr_{S_t} K)^2 - |K_{S_t}|^2 = 0 \tag{6.16}$$

The same analysis as above then gives

$$-\partial_1\partial_1\tau - \partial_0\partial_0\tau - \Delta_{\Sigma_t} h_{11} - 2\partial_1\partial_A h_{0A} = O_2. \tag{6.17}$$

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