ON THE STRUCTURE OF CONFORMALLY COMPACT EINSTEIN METRICS

MICHAEL T. ANDERSON

Abstract. Let M be an $(n+1)$-dimensional manifold with non-empty boundary, satisfying $\pi_1(M, \partial M) = 0$. The main result of this paper is that the space of conformally compact Einstein metrics on M is a smooth, infinite dimensional Banach manifold, provided it is non-empty. We also prove full boundary regularity for such metrics in dimension 4 and a local existence and uniqueness theorem for such metrics with prescribed metric and stress-energy tensor at conformal infinity, again in dimension 4. This result also holds for Lorentzian-Einstein metrics with a positive cosmological constant.

1. Introduction.

Let M be the interior of a compact $(n+1)$-dimensional manifold $\bar{M}$ with non-empty boundary ∂M. A complete metric g on M is $C^{m,\alpha}$ conformally compact if there is a defining function ρ on $\bar{M}$ such that the conformally equivalent metric

$$\widetilde{g} = \rho^2 g \tag{1.1}$$

extends to a $C^{m,\alpha}$ Riemannian metric on the compactification $\bar{M}$. A defining function ρ is a smooth, non-negative function on $\bar{M}$ with $\rho^{-1}(0) = \partial M$ and $d\rho \neq 0$ on ∂M.

The induced Riemannian metric $\gamma = \widetilde{g}|_{\partial M}$ is called the boundary metric associated to the compactification $\widetilde{g}$. Since there are many possible defining functions, and hence many conformal compactifications of a given metric g, only the conformal class $[\gamma]$ of γ on ∂M is uniquely determined by (M, g); the class $[\gamma]$ is called the conformal infinity of g. Any manifold M carries many conformally compact metrics and in this paper we are interested in Einstein metrics g, for which

$$Ric_g = -ng. \tag{1.2}$$

Conformally compact Einstein metrics are also called asymptotically hyperbolic (AH), in that $|K_g + 1| = O(\rho^2)$, where K_g denotes any sectional curvature of (M, g), at least when g has a C^2 conformal compactification.

In this paper, we prove several distinct results on conformally compact Einstein metrics. First, we prove boundary regularity for such metrics in dimension $n + 1 = 4$. Thus, suppose g is an Einstein metric on a 4-manifold M, which admits an $L^{2,p}$ conformal compactification, for some $p > 4$. If the resulting boundary metric is $C^{m,\alpha}$, or C^∞, or C^ω (real-analytic), then g is $C^{m,\alpha}$, or C^∞, or C^ω conformally compact respectively; see Theorem 2.3.

The proof of boundary regularity uses the fact that Einstein metrics on 4-manifolds satisfy a conformally invariant 4^{th} order equation, the Bach equation, given by

$$\delta d(Ric - \tfrac{s}{6}g) + W(Ric) = 0. \tag{1.3}$$

Here Ric and g are viewed as 1-forms with values in TM, $s = trRic$ is the scalar curvature, W is the Weyl curvature and $d = d^\nabla$ is the exterior derivative with $\delta = \delta^\nabla$ its L^2 adjoint. The equations (1.3) are the Euler-Lagrange equations for the L^2 norm of the Weyl curvature W, as a functional on the space of metrics on M. Since (1.3) is conformally invariant in dimension 4, any conformal compactification $\widetilde{g}$ of g satisfies (1.3) and boundary regularity is established by studying

Partially supported by NSF Grant DMS 0305865 and 0604735.

the boundary regularity of solutions of the non-degenerate equation (1.3) on $\bar{M}$. Theorem 2.3 corrects a small gap in the proof of boundary regularity in [2, Thm.2.4], cf. Remark 2.4.

As an application of these techniques, we also prove a local existence and uniqueness result. Thus, recall the Fefferman-Graham expansion of an AH Einstein metric [12]; in dimension 4, this is given by

$$\bar{g} = t^2 g \sim dt^2 + g_{(0)} + t^2 g_{(2)} + t^3 g_{(3)} + \cdots + t^k g_{(k)} + ..., \tag{1.4}$$

where t is a geodesic defining function, i.e. $t(x) = dist_{\bar{g}}(x, \partial M)$. The boundary metric γ is given by $\gamma = g_{(0)}$ and the term $g_{(2)}$ is intrinsically determined by γ. The Einstein constraint equations at conformal infinity ∂M are equivalent to the statement that the term $g_{(3)}$ is transverse-traceless on $(\partial M, \gamma)$, i.e. $\delta_\gamma g_{(3)} = tr_\gamma g_{(3)} = 0$, see for instance [11]. However, beyond this, the $g_{(3)}$ term is not determined by the boundary metric γ. All higher order terms in the expansion (1.4) are determined by $g_{(0)}$ and $g_{(3)}$ via the Einstein equations. It is also worth noting that from a physics perspective, the term $g_{(3)}$ is identified with the stress-energy tensor of the conformal infinity, cf. again [11] for instance.

In [12], Fefferman-Graham proved that if $\gamma = g_{(0)}$ is any real-analytic metric on an arbitrary 3-manifold ∂M, and one sets $g_{(3)} = 0$, so that the formal expansion (1.4) is even in t, then there exists a real-analytic AH Einstein metric defined in a thickening $M = [0, \varepsilon) \times \partial M$, with boundary metric γ. Thus, the series (1.4) converges to $\bar{g}$. This was proved by using results of Baouendi-Goulaouic on the convergence of formal series solutions to nonlinear Fuchsian systems of PDE's. A result analogous to this was proved earlier by LeBrun [19] for self-dual Einstein metrics on thickenings of 3-manifold boundaries, using twistor methods.

The following result generalizes these results to allow for an arbitrary $g_{(3)}$ term.

Theorem 1.1. *Let N be a closed 3-manifold, and let (γ, σ) be a pair consisting of a real-analytic Riemannian metric γ on N, and a real-analytic symmetric bilinear form σ on N satisfying $\delta_\gamma \sigma = tr_\gamma \sigma = 0$. Then there exists a unique (up to isometry), C^ω conformally compact Einstein metric g, defined on a thickening $N \times I$ of N, for which the expansion* (1.4) *converges to $\bar{g}$ and is given by*

$$\bar{g} = dt^2 + \gamma + t^2 g_{(2)} + t^3 \sigma + ... + t^k g_{(k)} + ... \tag{1.5}$$

The proof is based again on the Bach equation, together with the Cauchy-Kovalewsky theorem. An analogous result also holds for Lorentzian-Einstein metrics, i.e. solutions to the vacuum Einstein equations in general relativity with a positive cosmological constant Λ, cf. Theorem 2.6. The result in this case is related to work of H. Friedrich [13].

Next, we turn to the structure of the moduli space of AH Einstein metrics on a given $(n+1)$-manifold M. Let $E_{AH} = E_{AH}^{m,\alpha}$ be the space of AH Einstein metrics g on M which admit a $C^{m,\alpha}$ compactification $\widetilde{g}$ as in (1.1). We require that $m \geq 2$, $\alpha \in (0,1)$ but otherwise allow any value of m, including $m = \infty$ or $m = \omega$. The space $E_{AH}^{m,\alpha}$ is given the $C^{m,\mu}$ topology on $\bar{M}$, for any fixed $\mu < \alpha$, via a fixed compactification as in (1.1). Let $\mathcal{E}_{AH} = E_{AH}/\mathrm{Diff}_1(\bar{M})$, where $\mathrm{Diff}_1(\bar{M}) = \mathrm{Diff}_1^{m+1,\alpha}(\bar{M})$ is the group of $C^{m+1,\alpha}$ diffeomorphisms of $\bar{M}$ inducing the identity on ∂M, acting on E_{AH} in the usual way by pullback.

Regarding the boundary data, let $Met(\partial M) = Met^{m,\alpha}(\partial M)$ be the space of $C^{m,\alpha}$ metrics on ∂M and $\mathcal{C} = \mathcal{C}(\partial M)$ the corresponding space of pointwise conformal classes, endowed with the $C^{m,\mu}$ topology as above. There is a natural boundary map, (for any fixed (m, α)),

$$\Pi : \mathcal{E}_{AH} \to \mathcal{C}, \quad \Pi[g] = [\gamma], \tag{1.6}$$

which takes an AH Einstein metric g on M to its conformal infinity on ∂M.

We then have the following result on the structure of $\mathcal{E}_{AH}$ and the map Π.

Theorem 1.2. *Let M be a compact, oriented 4-manifold with boundary ∂M satisfying $\pi_1(M, \partial M) = 0$. If, for a given (m, α), $m \geq 3$, $\mathcal{E}_{AH}$ is non-empty, then $\mathcal{E}_{AH}$ is a C^∞ smooth infinite dimensional separable Banach manifold. Further, the boundary map*

$$\Pi : \mathcal{E}_{AH} \to \mathcal{C} \tag{1.7}$$

is a C^∞ smooth Fredholm map of Fredholm index 0.

Implicit in Theorem 1.2 is the boundary regularity statement that an AH Einstein metric with $C^{m,\alpha}$ conformal infinity has a $C^{m,\alpha}$ compactification. Versions of Theorem 1.2 also hold in arbitrary dimensions $n > 4$; see Theorems 5.5 and 5.6 for the precise statements.

The condition $\pi_1(M, \partial M) = 0$ is equivalent to the statements that ∂M is connected and the inclusion map $\iota : \partial M \to M$ induces a surjection $\pi_1(\partial M) \to \pi_1(M) \to 0$. It is not clear whether Theorem 1.2 holds globally without this assumption, although generic metrics in $\mathcal{E}_{AH}$ always have smooth manifold neighborhoods, cf. Remark 3.2 and the discussion following Theorem 4.1.

Theorem 1.2 is a generalization of previous results of Graham-Lee [15] and Biquard [8], who proved a local analogue of this result, (without full boundary regularity), in neighborhoods of metrics $g \in \mathcal{E}_{AH}$ which are regular points of the map Π. The proof of Theorem 1.2 uses methods introduced in [15] and [8]. Further, Theorem 1.2 is formally analogous to results on the space of minimal surfaces, cf. [9] and especially [26], [27] and we have also been influenced by this work.

The proof of Theorem 1.2 requires a rather subtle understanding of the behavior of infinitesimal AH Einstein deformations which are in $L^2(M, g)$; one needs to know that such deformations satisfy a suitable unique continuation property at infinity. This was proved in [6] and is presented here in §3, cf. Proposition 3.1, after some preliminary introductory material. The main work in the proof of Theorem 1.2 is then given in §4, with the final proof given in §5. We also point out that it is proved in Theorem 5.7 that the spaces $\mathcal{E}_{AH}^{m,\alpha}$ are stable in (m, α); they are all diffeomorphic and the inclusion of $\mathcal{E}_{AH}^{m',\alpha'}$ into $\mathcal{E}_{AH}^{m,\alpha}$ for $m' + \alpha' > m + \alpha$ is dense, including the case $m' = \infty$ or $m' = \omega$. Thus, the structure of the spaces $\mathcal{E}_{AH}^{m,\alpha}$ is essentially independent of (m, α).

The results in this paper are also used in [5], which studies the existence problem for conformally compact Einstein metrics with prescribed conformal infinity on 4-manifolds.

2. The Bach equation and AH Einstein metrics.

In this section, we prove boundary regularity for AH Einstein metrics in dimension 4, together with Theorem 1.1 and various applications.

We begin with the study of boundary regularity. Let g be an AH Einstein metric on a 4-manifold M. Then g satisfies the conformally invariant Bach equation (1.3). Hence, any conformal compactification $\widetilde{g}$ of g also satisfies (1.3). In the following, to simplify notation, we work with a given conformal compactification $\widetilde{g}$ of an AH Einstein metric g and drop the tilde from the notation; thus, from now on until Corollary 2.2, g denotes $\widetilde{g}$.

By a standard Weitzenbock formula, (1.3) may be rewritten in the form

$$D^*DRic = -\tfrac{1}{3}D^2 s - \tfrac{1}{6}\Delta s + \mathcal{R}, \tag{2.1}$$

where $\mathcal{R}$ is a term quadratic in the curvature of g.

As it stands, the equation (2.1) (or (1.3)) does not form an elliptic system, due to its invariance under diffeomorphisms and conformal deformations. Since we wish to cast (2.1) in the form of an elliptic boundary value problem, two choices of gauge are needed to break these symmetries.

First, with regard to the diffeomorphism invariance, we use, as is now common, harmonic coordinates. Thus let x^i, $i = 1, 2, 3$ be local harmonic coordinates on $(\partial M, \gamma)$ and extend x^i locally into M by requiring that x^i is harmonic with respect to g (i.e. $\widetilde{g}$):

$$\Delta x^i = 0, \tag{2.2}$$

where the Laplacian is with respect to (M,g). Also, let x^0 be a local "harmonic defining function", satisfying

$$\Delta x^0 = 0, \quad x^0|_{\partial M} = 0. \tag{2.3}$$

Thus, the functions x^i, $i = 0,1,2,3$, form a local coordinate system for M up to its boundary, harmonic with respect to g.

Suppose in a given fixed (or background) C^∞ atlas for M near ∂M, the compactified metric g is in $L^{k,p}$, for some $k \geq 2$, $p > 4$, or in $C^{m,\alpha}$, $m \geq 2$. Then it is well-known, and will be frequently be used below, that g is $L^{k,p}$ or $C^{m,\alpha}$ respectively in local boundary g-harmonic coordinates. Further, these harmonic coordinates are $L^{k+1,p}$ or $C^{m+1,\alpha}$ functions of the background local coordinates respectively. The same remarks pertain in the real-analytic case. Thus, harmonic coordinates give optimal regularity properties.

With regard to the conformal invariance, it is natural to specify the scalar curvature s to determine the conformal gauge. At a later point, we will choose a Yamabe gauge, where $s = const$. However, for the moment, we assume that s, the scalar curvature of g, is a given function with a given degree of smoothness. In particular, the two scalar curvature terms on the right in (2.1) are thus "determined". In the following, Greek indices α,β run over $0,1,2,3$ while Latin indices i, j run over $1,2,3$.

In local harmonic coordinates, $-2D^*DRic = \Delta\Delta g_{\alpha\beta}$ + (3^{rd} order terms). Thus, the system (2.1) may be rewritten in local harmonic coordinates as

$$\Delta\Delta g_{\alpha\beta} = F_{\alpha\beta}(g,s), \tag{2.4}$$

where F is of order 3 in g, order 2 in s, with real-analytic coefficients; here $\Delta = g^{\alpha\beta}\partial_\alpha\partial_\beta$ and s is treated as given.

This is a 4^{th} order elliptic system, with leading order term in diagonal form.

We now set up the boundary conditions for this system. On the surface, it would be simplest to just choose Dirichlet and Neumann boundary conditions on the set of all $g_{\alpha\beta}$. However, via the map Π in (1.6), at the boundary we only have information on the intrinsic metric $\gamma_{ij} = g_{ij}$; the $g_{0\alpha}$ terms are gauge dependent, and have no apriori prescribed form at ∂M. Thus, it is not clear if $\{g_{0\alpha}\}$ satisfy any particular boundary conditions apriori. Moreover, in general Bach-flat metrics are not conformally Einstein; conformally Einstein metrics thus necessarily induce certain special boundary conditions. In the case of geodesic gauge, this is discussed in the proof of Theorem 1.1 below, and is closely related to the Fefferman-Graham expansion. However, such a gauge is badly behaved for elliptic boundary value problems. While it is an interesting open question to characterize the boundary conditions for Bach-flat metrics to be conformally Einstein (in arbitrary gauges), this issue will not be addressed here; instead we will derive certain boundary conditions for the Bach-flat equations as a consequence of the metric being conformally Einstein.

To begin, we divide the collection $\{g_{\alpha\beta}\}$ into two parts. First g_{ij} are the tangential components of g, with $1 \leq i,j \leq 3$. The remaining terms $g_{0\alpha}$ are the mixed and normal terms. The basic idea is then to impose Dirichlet boundary conditions on g_{ij}, of 0^{th} and 2^{nd} order, while imposing Neumann-type boundary conditions on $g_{0\alpha}$, of 1^{st} and 3^{rd} order. These latter two conditions come from the gauge choice of harmonic coordinates.

In more detail:

$\{1\}$. Dirichlet boundary conditions on g_{ij}:

$$g_{ij} = \gamma_{ij} \;\text{ on }\; \partial M \;\text{ (locally)}, \tag{2.5}$$

where γ is the given boundary metric on ∂M.

$\{2\}$. Neumann-type boundary conditions on $g_{0\alpha}$. It is convenient to set these up for the inverse variables $g^{0\alpha}$. These are of the form

$$N(g^{00}) = -2Hg^{00}, \tag{2.6}$$

$$N(g^{0i}) = \tfrac{1}{2}q^{i\beta}\partial_\beta g^{00} - Hg^{0i}. \tag{2.7}$$

Here $N = (g^{00})^{-1/2}g^{0\beta}\partial_\beta = q^{0\beta}\partial_\beta$ is the unit normal, and $q^{\alpha\beta} = (g^{00})^{-1/2}g^{\alpha\beta}$. The term H is the mean curvature, $H = g^{ij}A_{ij}$, with $A_{ij} = \frac{1}{2}q^{0\alpha}[\partial_\alpha g_{ij} - (\partial_i g_{\alpha j} + \partial_j g_{\alpha i})]$ the 2^{nd} fundamental form of the boundary. The fact that the components $\{g^{0\alpha}\}$ satisfy (2.6)-(2.7) in boundary harmonic coordinates $\{x_\alpha\}$ was derived in [2], cf. also [3]. This does not require the metric to be conformally Einstein or Bach-flat; it holds in general.

The equations (2.6)-(2.7) can be reexpressed as Neumann-type conditions on the coefficients $g_{0\alpha}$, since g_{ij} is given on ∂M by (2.5). However, these expressions will not be given explicitly, since only the linearized versions of (2.6)-(2.7) need to be actually computed.

$\{3\}$. Dirichlet boundary conditions on Ric_{ij}. One has, in harmonic coordinates in general, $-2Ric_{ij} = \Delta^M g_{ij} + Q(g, \partial g)$, where the Laplacian is with respect to (M, g) and Q is a 1^{st} order term in g. A simple and standard calculation shows that for conformally Einstein metrics g, Ric_{ij} is determined by $(Ric_\gamma)_{ij}$ at ∂M, modulo lower order terms. In fact, cf. [2,Lemma 1.3] for instance, the extrinsic and intrinsic Ricci curvatures are related by

$$Ric_{ij} = 2(Ric_\gamma)_{ij} + \tfrac{1}{6}(s - \tfrac{3}{2}s_\gamma)\gamma_{ij} - (\tfrac{H}{3})^2\gamma_{ij}. \tag{2.8}$$

Since also $-2(Ric_\gamma)_{ij} = \Delta^{\partial M}\gamma_{ij} + Q(\gamma, \partial\gamma)$, this gives

$$\Delta^M g_{ij} = F(\gamma, \partial\gamma, \partial^2\gamma, H, s) \ \text{ on } \ \partial M, \tag{2.9}$$

where F is real-analytic in its arguments; observe that H is real-analytic in g and its 1^{st} derivatives.

$\{4\}$. Neumann-type boundary conditions on $Ric_{0\alpha}$. These turn out to be

$$q^{0\beta}N(\Delta^M g_{0\beta}) + \tfrac{1}{3}\Delta^{\partial M}H = -\partial_0 s + Q_0, \tag{2.10}$$

$$q^{0\beta}N(\Delta^M g_{i\beta}) = -\tfrac{2}{3}\partial_i s + Q_i, \tag{2.11}$$

where $Q = (Q_0, Q_i)$ is an operator of order less than 3. Recall that the scalar curvature s is treated as given.

The equations (2.10)-(2.11) are basically a consequence of the Bianchi identity. To derive them, the contracted Bianchi identity gives $\delta Ric = -\frac{1}{2}ds$, so that for any α, $-\delta Ric(\partial_\alpha) = \frac{1}{2}\partial_\alpha s$. By definition, $-\delta Ric(\partial_\alpha) = (\nabla_N Ric)(N, \partial_\alpha) + (\nabla_{e_j} Ric)(e_j, \partial_\alpha)$, where e_j runs over an orthonormal basis tangent to ∂M. Also $(\nabla_N Ric)(N, \partial_\alpha) = N(Ric(N, \partial_\alpha)) - Ric(N, \nabla_N\partial_\alpha) - Ric(\nabla_N N, \partial_\alpha)$, and $N(Ric(N, \partial_\alpha)) = q^{0\beta}N(Ric_{\alpha\beta}) + N(q^{0\beta})Ric_{\alpha\beta}$. Putting these together, the Bianchi identity may be rewritten as

$$q^{0\beta}N(Ric_{\alpha\beta}) + e_j(Ric(e_j, \partial_\alpha)) = \tfrac{1}{2}\partial_\alpha s + Q_\alpha, \tag{2.12}$$

where Q_α involves the Ricci curvature of g and the 1^{st} derivatives of g at ∂M; thus Q_α is of order less than 3. Using again the fact that $-2Ric_{\alpha\beta} = \Delta^M g_{\alpha\beta} + Q(g, \partial g)$ in harmonic coordinates, the equation above may be rewritten as

$$q^{0\beta}N(\Delta^M g_{0\beta}) + div^{\partial M}(\Delta^M g_{0\cdot}) = -\partial_0 s + Q_0. \tag{2.13}$$

$$q^{0\beta}N(\Delta^M g_{i\beta}) + div^{\partial M}(\Delta^M g_{i\cdot}) = -\tfrac{2}{3}\partial_i s + Q_i, \tag{2.14}$$

where we have separated the cases $\alpha = 0$ and $\alpha = i > 0$.

These equations have to be modified somewhat, since, together with $\{1\} - \{3\}$, they do not lead to elliptic boundary conditions; this is probably because both systems (2.6)-(2.7) and (2.13)-(2.14) are identities in harmonic coordinates. In any case, we reexpress the divergence terms in (2.13) and (2.14) as follows. Regarding (2.13), a standard formula for the change of Ricci curvature under conformal changes gives, at ∂M,

$$Ric_{0i} = -\frac{2}{3}\partial_i H, \tag{2.15}$$

cf. again [2, (1.18)] for instance. This equation uses the fact that g is conformally Einstein; note this is 2^{nd} order in $g_{\alpha\beta}$, in contrast to (2.8), (cf. also Remark 2.4). Substituting (2.15) in (2.12) and using (2.13) then gives (2.10). For (2.14), the ambient Ricci curvature term Ric_{ij} in (2.12), $(\alpha = i > 0)$, is in fact intrinsic to the boundary, modulo lower order terms and the s term, by (2.8). Since the derivatives are also being taken tangentially, the term $e_j(Ric(e_j, \partial_i))$ is intrinsic to the boundary metric γ, modulo lower order terms and the s term. At leading order, it does not depend on $g_{0\alpha}$ and thus it may be absorbed into the Q_i term. Taking the s term in (2.8) into account, this gives (2.11).

The boundary conditions $\{1\} - \{4\}$ are the conditions that will be used below. Note that only the conditions $\{3\} - \{4\}$ use the fact that g is conformally Einstein. Given this groundwork, we are now in position to prove the following result.

Proposition 2.1. *Let $\widetilde{g}$ be a conformal compactification of an AH Einstein metric g, defined on a thickening M of $N = \partial M$, with scalar curvature $s = s_{\widetilde{g}}$ given. In boundary harmonic coordinates, the Bach equation (2.1), with the boundary conditions $\{1\}$–$\{4\}$, forms a non-linear elliptic boundary value problem, with real-analytic coefficients.*

Proof: It is clear that the operator (2.1) and boundary operators $\{1\} - \{4\}$ are real-analytic in the metric g and its derivatives. Thus, one needs to check that the conditions of Agmon-Douglis-Nirenberg [1] or Morrey [23, §6] are satisfied; we will verify the conditions of Morrey. First, ellipticity of the boundary value problem depends only on that of its linearization at any solution $\widetilde{g}$. Thus, in the work above, replace $\widetilde{g}$ by $\widetilde{g} + \lambda\widetilde{h}$ and take the derivative with respect to λ to obtain a linear system in $\widetilde{h}$; as above, henceforth we drop the tilde from the notation. We will also assume that the coordinate system is small, so that $g_{\alpha\beta}$ is close to $\delta_{\alpha\beta}$; in particular $g^{\alpha\beta} \sim g_{\alpha\beta}$.

The interior system is then essentially the same as before:

$$\Delta\Delta h + F_3(g, h) = 0, \tag{2.16}$$

where the Laplacian is with respect to (M, g) and F is of order 3 in h.

In the notation of Morrey [23,§6.1], the interior system has the form

$$L_{jk}u^k = 0 \ \text{ in } \ M.$$

Here $\{u^k\} = \{h_{\alpha\beta}\}$ so that $j, k \in \{1, ..., N\}$, with $N = \#(\alpha\beta) = 10$. The leading order term L'_{jk} of L_{jk} is given by $L'_{jk} = (\Delta\Delta)\delta_{jk}$, the biLaplacian acting diagonally.

The order of each L_{jk} is 4, and we set $t_k = 4$, for all k, $s_j = 0$, for all j. This leading order symbol of L_{jk} has $2m$ roots, each $+i$ or $-i$ (at a cotangent vector of the form $\xi + n$, where ξ is tangent to ∂M, $|\xi| = 1$, and n is the unit conormal. Hence, the system (2.16) is properly elliptic, (i.e. satisfies the root condition).

The boundary operator has the general form

$$B_{rk}u^k = f_r \ \text{ on } \ \partial M, \tag{2.17}$$

where $r \in \{1, ..., m\}$, with $m = 2N$. Thus, one has 2 boundary operators for each $h_{\alpha\beta}$. The operator B_{rk} is considered as a $2N \times N$ matrix, with each $N \times N$ block consisting of 6 horizontal rows (ij) ordered lexicographically and 4 mixed rows, ordered $(00, 01, 02, 03)$.

The order of $B_{rk} = t_k - h_r = 4 - h_r$. Thus, for $u^k = \{h_{ij}\}$, i.e. the 6 tangential components of h, one has $h_r = 4$ for $1 \leq r \leq 6$, (corresponding to the Dirichlet data $\{1\}$) and $h_r = 2$ for $11 \leq r \leq 16$, (corresponding to the Dirichlet data $\{3\}$). Similarly, for the mixed terms $h_{0\alpha}$, one has $h_r = 3$ for

$7 \leq r \leq 10$ and $h_r = 1$ for $17 \leq r \leq 20$, corresponding to the Neumann data $\{2\}$ and $\{4\}$. Thus, $h_0 = 0$ in the notation of [23,§6.1].

Since $g_{\alpha\beta} \sim \delta_{\alpha\beta}$, the positive roots $z_s^+(x,\xi)$, $s = 1, ...m = 20$, of the symbol $L(x, \xi + zn)$ are all close to $z = i|\xi|$; when $g_{\alpha\beta} = \delta_{\alpha\beta}$, the positive roots are exactly $z = i|\xi|$. Hence

$$L_0^+(x,\xi,z) \equiv \prod_1^{20}(z - z_s^+(x,\xi)) \sim (z - i|\xi|)^{20}.$$

Let L^{jk} be the matrix adjoint of L_{jk}, (the matrix of minors of L). Then $L^{jk}(x, \xi + zn) = (|\xi|^2 + |z|^2)^{18}\delta_{jk}$ + lower order terms. One then forms

$$Q_{rk}(x,\xi,z) = \sum_{j=1}^{N} B'_{rj}(x,\xi+zn)L^{jk}(x,\xi+zn) \sim B'_{rk}(x,\xi+zn)(|\xi|^2+|z|^2)^{18}, \tag{2.18}$$

where B'_{rk} is the leading order symbol of B_{rk}.

Here Q_{rk} is viewed as a polynomial in z, for any fixed x, ξ with $x \in \partial M$ and ξ a cotangent vector to ∂M; n is the unit conormal. Then the complementary condition is that the rows of Q_{rk} are linearly independent mod L_0^+, i.e.

$$\sum_{r=1}^{m} c_r Q_{rk}(x,\xi,z) = 0 \ \ mod \ \ L_0^+ \Rightarrow \{c_r\} = 0.$$

By (2.18), this is essentially equivalent to

$$\sum_{j=1}^{m} c_r B'_{rk}(x,\xi+zn) \equiv (z - i|\xi|)^2 \Rightarrow \{c_r\} = 0, \tag{2.19}$$

where the congruence is modulo polynomials in z. More precisely, this is the condition one obtains when the lower order terms in L^{jk} are ignored. Including the lower order terms leads to the addition of polynomials of higher degree on the right in (2.19), and it will be obvious from the computations below that one may safely ignore such terms.

The $2N \times N$ matrix B'_{rk} is the leading order symbol for the linearization of the boundary problems $\{1\}$-$\{4\}$ at g, with variable or unknown h. Consider this matrix M as a pair of $N \times N$ matrices, an upper block M_1 consisting of the boundary operators $\{1\}$ and $\{2\}$, and a lower block M_2, consisting of the boundary operators $\{3\}$ and $\{4\}$. The leading order symbol of the linearization of B is obtained by replacing g by h in the highest derivatives of g that appear in $\{1\}$-$\{4\}$, and ignoring all lower order terms. Further, since the ellipticity condition is open and we are working locally, one may assume that $g_{\alpha\beta} = \delta_{\alpha\beta}$. A simple inspection of the form of $\{1\}$-$\{4\}$ then leads to the following description of M.

The matrix M_1 consists of I_6, the 6×6 identity matrix, with 0 elsewhere in the first 6 rows, corresponding to the boundary operator $\{1\}$. For the next 4 rows, corresponding to the boundary operator $\{2\}$, the 4×4 block corresponding to the (0α) terms (the lower right block) has the form

$$\begin{pmatrix} z & 2\xi_1 & 2\xi_2 & 2\xi_3 \\ \frac{1}{2}\xi_1 & z & 0 & 0 \\ \frac{1}{2}\xi_2 & 0 & z & 0 \\ \frac{1}{2}\xi_3 & 0 & 0 & z \end{pmatrix} \tag{2.20}$$

To see this, the diagonal z terms come from the operator N in (2.6)-(2.7). Next, one has $H = g^{ij}A_{ij} = A_{ii} = -\partial_i h_{0i} + \frac{1}{2}\partial_0 h_{ii}$. Via (2.6), the first term here gives rise to the first row in (2.20); the term $\frac{1}{2}\partial_0 h_{ii}$, giving rise to $-zh_{ii}$ goes into the (ii) columns of the (00) row, (in the lower left block), and may ignored. The first (00) column in (2.20) comes from the first term on the right in (2.7); note that the term Hg_{0i} linearizes to 0.

The matrix M_2 has a similar description. The first 6 rows of M_2 consist of $(|z|^2 + |\xi|^2)I_6$ and 0 elsewhere, corresponding to the boundary operator $\{3\}$. The last 4×4 block (on the lower right) of the boundary operator $\{4\}$ gives a matrix of the form

$$\begin{pmatrix} z(z^2+|\xi|^2) & -\frac{1}{3}|\xi|^2\xi_1 & -\frac{1}{3}|\xi|^2\xi_2 & -\frac{1}{3}|\xi|^2\xi_3 \\ 0 & z(z^2+|\xi|^2) & 0 & 0 \\ 0 & 0 & z(z^2+|\xi|^2) & 0 \\ 0 & 0 & 0 & z(z^2+|\xi|^2) \end{pmatrix} \tag{2.21}$$

There are again terms of the form $|\xi|^2 z$ in the tangential (ii) columns of the (00) row, (in the lower left block) but again these can be ignored.

Using these forms of the matrices, together with the fact that the terms ignored above are at most first order in z, it is easy to see that there are no non-trivial solutions of (2.19). Namely, the polynomials in z on the left side of (2.19) are all of order at most 3, with no z^2 terms. Such polynomials cannot have $i|\xi|$ as a double root. This shows that all the hypotheses of [23, §6.1,6.3] are satisfied, which proves the result.

■

Having verified that the Bach equation with the boundary conditions $\{1\}$-$\{4\}$ forms an elliptic boundary value problem in local harmonic coordinates, one then has the following:

Corollary 2.2. *Let g be an AH Einstein metric on a 4-manifold M, which admits an $L^{2,p}$ conformal compactification $\widetilde{g}$ in local boundary harmonic coordinates, for some $p > 4$, with boundary metric γ.*

Let $k \geq 1$ and $q \geq 2$. If $\gamma \in L^{k+2,q}(\partial M)$ and the scalar curvature $\widetilde{s} \in L^{k,q}(M)$, with $\widetilde{s}|_{\partial M} \in L^{k,q}(\partial M)$, then the metric $\widetilde{g} \in L^{k+2,q}(M)$.

Similarly, for $m \geq 0$ and $\alpha \in (0,1)$, if $\gamma \in C^{m+2,\alpha}(\partial M)$ and $\widetilde{s} \in C^{m,\alpha}(M)$, then $\widetilde{g} \in C^{m+2,\alpha}(M)$. If γ and $\widetilde{s}$ are real-analytic, then so is $\widetilde{g}$.

Proof: The regularity hypotheses and conclusions are understood to be with respect to local boundary harmonic coordinates.

This result follows from Proposition 2.1 and the regularity theory for elliptic systems, cf. [23, §6]. Suppose first that the compactification $\widetilde{g}$ is $L^{4,p}$ or $C^{4,\alpha}$, so that $\widetilde{g}$ is a classical solution of the Bach equation (2.1) and satisfies the boundary conditions $\{1\}$-$\{4\}$, with the given control on $\widetilde{s}$. Then the result follows from boundary regularity for such elliptic systems, see [23, Thm. 6.3.7]. Here, the coefficients of the interior operator L and boundary operator B are frozen to obtain a linear elliptic system, and the usual boostrap argument is used to obtain regularity. In the notation of [23], one sets $h_0 = 0$, $h = 1$, and proceeds iteratively. The real-analytic case follows from [23, Thm. 6.7.6′].

In the case $\widetilde{g} \in L^{2,p}$, the metric $\widetilde{g}$ satisfies the Bach equation and boundary conditions weakly. Using the fact that these equations are of divergence-form at leading order, (since they come from a natural variational problem), one then applies [23, Thm. 6.4.8] to prove that $\widetilde{g}$ has higher regularity, according to the regularity of $\widetilde{s}$ and the boundary data. This process is then iterated until $\widetilde{g}$ is a classical solution, as above.

To verify this in more detail, in the Bach equation (2.1), the assumptions $s \in L^{2,q}$ and $g \in L^{2,p}$ imply that the right side of (2.1) is in $L^{\hat{q}}$, where $\hat{q} = q/2 > 1$. Hence,

$$D^*DRic = f \in L^{\hat{q}}.$$

(In Morrey's notation, $\hat{q}$ equals q and f is a term f_j^0 in [23, (6.4.1)]). Next, the leading term of D^*DRic is the biLaplacian $\Delta\Delta$. In local harmonic coordinates, the expression $(\Delta\Delta u)dV$ schematically has the form $\partial(g\partial(g\partial^2 u)) = \partial^2(g^2\partial^2 u) - \partial(g\partial g\partial^2 u)$; here g or g^2 denotes some algebraic expression in the metric g. The interior operator $L(u)$ is now chosen to be the first term, (the leading order term of the biLaplacian),

$$L(u) = \partial^2(g^2\partial^2 u),$$

while the second term $\partial(g\partial g\partial^2 u)$ is treated as a term f_j^1 in [23, (6.4.1)]. Note that $g\partial g\partial^2 u \in L^{\hat{q}}$ again, when one sets $u = g$. The remaining lower order terms in D^*DRic are then treated in exactly the same way; it should be noted here that all 3$^{\text{rd}}$ order terms in the metric in D^*DRic are total derivatives, i.e. of divergence form, as above.

In Morrey's notation, one now chooses, $h = h_0 = -2$, and $h' = -1$ and for the interior operator sets, $m_j = 2$ and $s_j = 0$ for all j, while $t_k = 4$ for all k. The h-μ conditions of [23, Def. 6.4.1], or more precisely the h'-μ conditions with $h' = -1$, require only $g \in C^{1,\mu}$ which is satisfied by hypothesis. The hypotheses [23, (6.4.2)-(6.4.3)] are also satisfied.

Essentially the same manipulations are performed on the boundary system. Consider for instance the (most complicated) 3$^{\text{rd}}$ order boundary operator $\{4\}$. One commutes $q^{0\beta}N$ with the Laplacian to obtain schematically an operator of the form $\partial(g\partial(g\partial u)) = \partial^2(g^2\partial u) - \partial(g\partial g\partial u)$. The first, leading order, term $g^2\partial u$ forms one of the boundary operators $B_{rk\gamma}$ in [23,(6.4.15)] with $|\gamma| = 2$. The h'-μ conditions on this boundary operator again require only $g \in C^{1,\mu}$, which is satisfied. The second term $g\partial g\partial u$ forms one of the $g_{r\gamma}$ terms in [23,(6.4.15)] with $|\gamma| = 1$. Setting $u = g$, this term is in $L^{1,\hat{q}}$, for some $\hat{q} > 0$, as required by [23, Thm. 6.4.8]. For this part of $B_{rk\gamma}$, one has $h_r = 1$ and $p_r = 2$, so that $h_r + p_r \geq 3$, as required by [23, (6.4.17)].

Carrying out the same procedure as needed for the remaining boundary operators gives a system of boundary operators with $p_r = 0$ for the operators $\{0\}$ and $\{1\}$, while $p_r = 1, 2$ for the operators $\{3\}$ and $\{4\}$ respectively. The terms h_r are already defined as following (2.17), and one thus has $h_r + p_r \geq 3$ for all boundary operators, as required by [23, (6.4.17)]. As above, it is easily seen that the boundary coefficients satisfy the h'-μ conditions. This shows that the hypotheses of [23, Thm. 6.4.8] are satisfied, and one concludes that $g \in L^{3,p}$, (assuming corresponding regularity in $\widetilde{s}$ and the boundary data). Given this regularity boost, one then iterates this process as needed to obtain $g \in L^{4,p}$ or $g \in C^{4,\alpha}$.

■

Corollary 2.2 leads to the following boundary regularity result.

Theorem 2.3. *Let g be an AH Einstein metric on a 4-manifold M, which admits an $L^{2,p}$ conformal compactification $\widetilde{g} = \rho^2 g$, $p > 4$, with respect to a given background C^∞ atlas $\{y^\mu\}$ for M near ∂M, where $\rho = \rho(y^\mu)$ is an $L^{3,p}(y^\mu)$ defining function.*

If, for a given $m \geq 2$ and $\alpha \in (0,1)$, or $m = \infty$, the boundary metric $\gamma = \widetilde{g}|_{\partial M}$ is in $C^{m,\alpha}(y^\mu)$, then g admits a $C^{m,\alpha}$ conformal compactification $\hat{g} = \hat{\rho}^2 g$, with respect to a $C^{m+1,\alpha}$ atlas $\{x^\mu\}$ consisting of local boundary $\hat{g}$-harmonic coordinates, with the same boundary metric. Further, $\hat{\rho} = \hat{\rho}(x^\mu) \in C^{m+1,\alpha}(x^\mu)$. If $\gamma \in C^\omega(y^\mu)$, then $\hat{g} \in C^\omega(x^\mu)$.

Moreover, the x-coordinates are at least $L^{3,p}(y)$ functions of the y-coordinates.

Proof: Let $\hat{g}$ be a constant scalar curvature metric conformal to $\widetilde{g}$ on M with $\hat{g}|_{\partial M} = \gamma$. Thus, for $\hat{g} = u^2\widetilde{g}$, the function $u > 0$ is a solution of the Dirichlet problem for the Yamabe equation

$$u^3\mu = -6\widetilde{\Delta}u + \widetilde{s}u \tag{2.22}$$

on M, with $u = 1$ on ∂M and $\hat{s} = \mu = const.$ It is simplest to choose $\mu = -1$. Standard methods in elliptic PDE then give an $L^{2,p}(y)$ solution to this Dirichlet problem, cf. [21]. Thus, the metric $\hat{g}$ is $L^{2,p}(y)$ conformally compact, with constant scalar curvature. Let $\{x\} = \{x^\mu\}$ be a system of local boundary $\hat{g}$-harmonic coordinates near ∂M. Then $\hat{g} \in L^{2,p}(x)$, (since harmonic coordinates have optimal regularity), and

$$x \in L^{3,p}(y). \tag{2.23}$$

Moreover, when restricted to ∂M, $x \in C^{m+1,\alpha}(y)$. It follows from Corollary 2.2 that $\hat{g}$ then has the same regularity as the boundary metric γ in the x-coordinates, i.e. $\hat{g} \in C^{m,\alpha}(x)$.

To prove that $\hat{\rho} \in C^{m+1,\alpha}(x)$, standard formulas relating the Ricci curvature of $\hat{g}$ with that of g, cf. [2, (1.4-(1.5)] or [7, Ch.1J], give

$$\hat{z} \equiv \hat{R}ic - \frac{\hat{s}}{4}\hat{g} = -2\hat{\rho}^{-1}(\hat{D}^2\hat{\rho} - \frac{\hat{\Delta}\hat{\rho}}{4}\hat{g}),$$

so that $\hat{D}^2\hat{\rho} - \frac{\hat{\Delta}\hat{\rho}}{4}\hat{g} = -\frac{1}{2}\hat{\rho}\hat{z}$. Now apply the divergence operator $\hat{\delta}$ to both sides of this equation. On the one hand, a simple computation gives, (dropping the hats from the notation), $\delta D^2 f = -d\Delta f - Ric(\nabla f)$ and $\delta(fg) = -df$. On the other hand, for metrics of constant scalar curvature, $\delta(fz) = f\delta z - z(\nabla f) = -z(\nabla f)$, where the last equation follows from the contracted Bianchi identity. These calculations then give $\frac{3}{4}d\hat{\Delta}\hat{\rho} = -\frac{\hat{s}}{4}d\hat{\rho} - \frac{3}{2}\hat{z}(d\hat{\rho})$, or equivalently

$$\hat{\Delta}d\hat{\rho} = -\frac{\hat{s}}{3}d\hat{\rho} - 2\hat{z}(d\hat{\rho}). \tag{2.24}$$

Since $\hat{s}$ is constant and d commutes with $\hat{\Delta}$, it follows that $\hat{z}(d\hat{\rho})$ is exact, so that

$$-2\hat{z}(d\hat{\rho}) = d\phi, \tag{2.25}$$

for some function ϕ. Thus (2.24) is equivalent to

$$\hat{\Delta}\hat{\rho} + \frac{\hat{s}}{3}\hat{\rho} = \phi, \tag{2.26}$$

(where an undetermined constant has been absorbed into ϕ). This is an elliptic equation for $\hat{\rho}$, with $\hat{\rho} = 0$ on ∂M, and so one may use elliptic boundary regularity results to determine the smoothness of $\hat{\rho}$. To do this, recall that $\hat{g} \in C^{m,\alpha}(x)$ and $\hat{z} \in C^{m-2,\alpha}(x)$. Suppose first that

$$\hat{\rho} \in C^{k,\alpha}(x), \tag{2.27}$$

for some k, $1 \leq k \leq m$. Then $d\hat{\rho} \in C^{k-1,\alpha}(x)$ and since $\hat{z} \in C^{m-2,\alpha}(x)$, it follows from (2.25) that $d\phi \in C^{\ell,\alpha}$, where $\ell = \min(m-2, k-1)$. Hence, $\phi \in C^{\ell+1,\alpha}(x)$. In the x-coordinates, the Laplacian $\hat{\Delta}$ has the form $\hat{\Delta} = \hat{g}^{\mu\nu}\partial_{x_\mu}\partial_{x_\nu}$, and the Schauder elliptic boundary estimates, (cf. [14] for instance), for the equation (2.26) then give

$$\hat{\rho} \in C^{\ell+3,\alpha}(x), \tag{2.28}$$

provided $\ell + 1 \leq m$. This gives an increase in the regularity of $\hat{\rho}$ by 2 derivatives over (2.27), and hence by induction it follows that

$$\hat{\rho} \in C^{m+1,\alpha}(x) \tag{2.29}$$

provided, (for instance), $\hat{\rho} \in C^{1,\alpha}(x)$.

To prove this last statement, note that $\hat{\rho} = u\rho$, (since $g = \hat{\rho}^{-2}\hat{g} = \hat{\rho}^{-2}u^2\widetilde{g} = \hat{\rho}^{-2}u^2\rho^2 g$). One has $\rho \in L^{3,p}(y)$ by assumption and $u \in L^{2,p}(y)$, so that $\hat{\rho} \in L^{2,p}(y)$. Then (2.23) gives $\hat{\rho} \in L^{2,p}(x) \subset C^{1,\alpha'}(x)$, for some $\alpha' > 0$. (The fact that α' may be less than α is of no consequence). ■

One expects that the regularity conclusions in Theorem 2.3 are optimal. Namely, it seems unlikely that the regularity of $\widetilde{g}$ itself can be improved without further hypotheses, for example on the scalar curvature $\widetilde{s}$ or on the conformal factor u relating $\hat{\rho}$ with ρ.

Remark 2.4. **(i).** Proposition 2.1, Corollary 2.2 and Theorem 2.3 have all been phrased globally. However, the proofs of these results are completely local, and so local versions of these results hold equally well.

(ii). We point out here that the proof of [2, Thm.2.4] contains a small gap. Namely, [2, Lemma 1.3] does not hold for the mixed components Ric_{0i} of the Ricci curvature, when $A \neq 0$. The mixed components Ric_{0i} are not determined by $\widetilde{s}$ and the boundary metric, modulo lower order terms as in (2.8) but instead are given by (2.15), which is 2^{nd} order in the ambient metric. Since [2, Thm.2.4] uses the Yamabe gauge for which $A \neq 0$, one does not directly obtain a regularity estimate for Ric_{0i} in this gauge. My thanks to Robin Graham and Dylan Helliwell for pointing out this gap.

The proof of Theorem 2.3 above fixes this gap, via the boundary condition $\{4\}$ above. Alternately, it is straightforward to verify that one can also prove [2, Thm.2.4] by the same methods used there by adding the boundary conditions $\{4\}$. Very briefly, in place of the single Neumann-type boundary condition $\{2\}$ used in [2, Thm.2.4], one uses the pair of Neumann-type boundary conditions $\{2\}$ and $\{4\}$, to obtain regularity in the normal and mixed directions. The proof of regularity in the tangential directions remains the same.

(iii). A version of Theorem 2.3 has been proved in all even dimensions recently by Dylan Helliwell, [16]. The proof uses the ideas of the proof above in dimension 4, together with the Fefferman-Graham ambient obstruction tensor in higher dimensions, in place of the Bach tensor.

From certain perspectives, the best compactifications are *geodesic* compactifications, defined by the property that

$$\bar{g} = t^2 g, \tag{2.30}$$

where $t(x) = dist_{\bar{g}}(x, \partial M)$. The integral curves of $\bar{\nabla} t$ are then geodesics, orthogonal to ∂M and so the Gauss Lemma gives the splitting

$$\bar{g} = dt^2 + g_t, \tag{2.31}$$

near ∂M, where g_t may be identified as a curve of metrics on ∂M with $g_0 = \gamma$. Similarly, the metric g splits as $g = d\log t^2 + t^{-2}g_t$, so that $r = -\log t$ is a geodesic parameter on (M, g). It is well-known that C^2 conformally compact Einstein metrics admit a geodesic compactification, cf. [12] or [15]. Theorem 2.3 gives the following result on the smoothness of the geodesic compactification.

Corollary 2.5. *If g is an $L^{2,p}$ conformally compact Einstein metric on a 4-manifold M, with $C^{m,\alpha}$ boundary metric γ, then the geodesic compactification $\bar{g} = t^2 g$ is $C^{m-1,\alpha}$ smooth, in harmonic coordinates. The same result holds with respect to C^∞ and C^ω.*

Proof: By Theorem 2.3, there exists a $C^{m,\alpha}$ compactification $\widetilde{g} = \rho^2 g$ of g in harmonic coordinates. Writing $t = \omega\rho$, the defining equation for t, i.e. $|\bar{\nabla} t|^2_{\bar{g}} = 1$, is equivalent to

$$2(\widetilde{\nabla}\rho)\log\omega + \rho|\widetilde{\nabla}\log\omega|^2_{\widetilde{g}} = \rho^{-1}(1 - |\widetilde{\nabla}\rho|^2_{\widetilde{g}}).$$

This is a first order, non-characteristic PDE, with coefficients in $C^{m,\alpha}$ and right hand side in $C^{m-1,\alpha}$. Hence, the solution ω is in $C^{m-1,\alpha}(\bar{M})$.

∎

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1.

We first set up the local Cauchy problem for the Bach equation (2.1). As local coordinates, choose geodesic coordinates (t, x^i) where, given a compact metric g on $\bar{M}$, $t(x) = dist_g(x, \partial M)$ and x^i are local coordinates on ∂M extended into M to be invariant under the flow of $T = \nabla t$. Thus, the metric splits in these coordinates as in (2.31). (The bar has been dropped from the notation). In particular, $g_{0i} = 0$ and $g_{00} = 1$ in these coordinates. Note however that the Bach equation (2.1) is *not* an elliptic system in these coordinates.

Since (2.1) is a 4^{th} order equation, Cauchy data consist of prescribing g, or equivalently g_t in (2.31), and its first three Lie derivatives with respect to T at $t = 0$. This data may be freely chosen at ∂M, but we choose data agreeing with that of the Fefferman-Graham expansion (1.4) of a conformally compact Einstein metric. Thus, set

$$g_{(0)} = g_0 = \gamma, \quad g_{(1)} = \mathcal{L}_T g|_{t=0} = 0, \tag{2.32}$$

where γ is an arbitrary C^ω Riemannian metric on ∂M. For a conformally compact Einstein metric, the term $g_{(2)}$ is intrinsically determined by γ, (via the Einstein equations (1.2)), as

$$(\mathcal{L}_T g)^2|_{t=0} = g_{(2)} = -\tfrac{1}{2}(Ric_\gamma - \frac{s_\gamma}{4})\gamma. \tag{2.33}$$

Finally let

$$g_{(3)} = (\mathcal{L}_T g)^3|_{t=0} = \sigma \tag{2.34}$$

be an arbitrary transverse-traceless C^ω symmetric bilinear form on $(\partial M, \gamma)$, cf. again the discussion following (1.4). This set of Cauchy data is clearly non-characteristic on ∂M. We recall that all higher order terms in the expansion (1.4) are determined by $g_{(0)}$ and $g_{(3)}$. In fact, if one defines g^k by $g^k = t^{-2}\bar{g}^k$ and

$$\bar{g}^k \equiv dt^2 + g_{(0)} + tg_{(1)} + t^2 g_{(2)} + t^3 g_{(3)} + \cdots + t^k g_{(k)},$$

so that $\bar{g}^k$ is a truncation of the Taylor series of $\bar{g}$, then the coefficients $g_{(j)}$ are uniquely determined by the property that

$$||Ric_{g^k} + 3g^k||_{\bar{g}} = O(t^{k-2}). \tag{2.35}$$

With the exception of $g_{(0)}$ and $g_{(3)}$, one finds that $g_{(j)}$ depends on the lower order terms $g_{(l)}$, $l < j$, and their x-derivatives up to second order, cf. [12].

Now the system (2.1) has real-analytic coefficients, and the Cauchy data above are real-analytic. Of course the boundary $\{t = 0\}$ at ∂M is real-analytic in the given coordinates (t, x^i). Hence, the Cauchy-Kovalewsky theorem, cf. [17], implies there is a unique C^ω metric g, given in the form (2.31) and defined on a thickening $M = [0, \varepsilon) \times \partial M$ of ∂M, which satisfies the Bach equation (2.1), and satisfies the prescribed Cauchy data (2.32)-(2.34).

Since the curve of metrics g_t on ∂M as in (2.31) is real-analytic in t, it is given by its Taylor expansion at $t = 0$. Now recall that conformally Einstein metrics are Bach-flat, and so are solutions of the equations (2.1). Via the Bach equations, the higher order coefficients $g_{(n)}$, $n \geq 4$, in the Taylor expansion of the solution g are determined inductively by the lower order terms $g_{(j)}$, $0 \leq j \leq 3$ and their x-derivatives. Since, by construction in (2.32)-(2.34), these lower order terms are determined by the Einstein equations, it follows immediately by uniqueness of analytic solutions that the higher order terms $g_{(n)}$ are also determined by the Einstein equations. Hence the Taylor series of g_t is the same as the Fefferman-Graham series (1.4). Equivalently, via (2.35), one sees that the compactified metric $\bar{g}$ is conformally Einstein, to infinite order at ∂M. Analyticity then implies that $\bar{g}$ is exactly conformally Einstein, and moreover that $g = t^{-2}\bar{g}$ is an AH Einstein metric defined near ∂M.

If g' is any other AH Einstein metric with $L^{2,p}$ conformal compactification, and with given boundary data (γ, σ), then by Corollary 2.5, the geodesic compactification of g' is real-analytic. Hence $g' = g$ up to isometry, so that g is unique among AH Einstein metrics (with a weak compactification). ■

As described in [4], the solution to the Einstein equations given by Theorem 1.1 can be analytically continued past $N = \partial M$ onto the "other side", to obtain a deSitter-type vacuum solution of the Einstein equations. This is a Lorentz metric $\mathfrak{g}$, satisfying the Einstein equations with positive cosmological constant, i.e.

$$Ric_{\mathfrak{g}} = 3\mathfrak{g}. \tag{2.36}$$

This Lorentz metric is C^{ω} conformally compact, and defined at least in the region $M = \partial M \times [0, \varepsilon)$, for some $\varepsilon > 0$. Hence, the solution is geodesically complete to the future of some Cauchy surface, with real-analytic $\mathcal{I}^{+}$.

Thus, the Lorentzian version of Theorem 1.1 is the following:

Theorem 2.6. *Let N be a closed 3-manifold, and let (γ, σ) be a pair consisting of a real-analytic Riemannian metric γ on N, and a real-analytic symmetric bilinear form σ on N satisfying $\delta_{\gamma}\sigma = tr_{\gamma}\sigma = 0$. Then there exists a unique vacuum solution to the Einstein equations (2.36) with cosmological constant $\Lambda = 3$, which is C^{ω} conformally compact, defined in a neighborhood of $\mathcal{I}^{+}$, and for which the geodesic compactification $\bar{\mathfrak{g}} = t^2\mathfrak{g}$ satisfies*

$$\bar{\mathfrak{g}} = (-dt^2 + \gamma - t^2 g_{(2)} - t^3\sigma + t^4 g_{(4)} + ...). \tag{2.37}$$

Proof: Given the analyticity from Theorem 1.1 and Corollary 2.5, this is proved in [4]. The Fefferman-Graham expansion (1.4) and its basic properties holds equally well for Lorentzian deSitter-type vacuum solutions of the Einstein equations, cf. [12]. The terms $g_{(j)}$ in (2.37) are the same as those given for the Riemannian AH Einstein metrics in (2.35). Note then that formally, the expansion (2.37) is obtained from the expansion (1.5) by replacing t by it, and dropping any i factors, giving a form of "Wick rotation" in this situation. This is explained in more detail in [4].

Alternately, one can prove Theorem 2.6 directly, since a Lorentzian vacuum solution (2.36) is also Bach-flat. The proof of Theorem 1.1 given above in the Riemannian AH setting then carries through in the Lorentzian deSitter-type setting in exactly the same way. ■

Remark 2.7. This result gives a simple proof of a result of H. Friedrich [13], obtained by solving the conformal Einstein equations, in the special case of analytic initial data. A third proof of this result has recently been given by A. Rendall [25], using degenerate Fuchsian systems, analogous to the original arguments of Fefferman-Graham [12].

3. Infinitesimal Einstein Deformations and Diffeomorphisms.

This section is a bridge between the previous and next sections. We begin with a brief discussion of the Fefferman-Graham expansion [12] in all dimensions and then discuss a weak nondegeneracy result from [6] which will be needed in the proof of Theorem 1.2.

Let g be a conformally compact Einstein metric on a compact $(n+1)$-manifold M with boundary ∂M which has a C^2 geodesic compactification as in (2.30). The metric $\bar{g}$ then splits in geodesic boundary coordinates, as in (2.31):

$$\bar{g} = dt^2 + g_t, \tag{3.1}$$

near ∂M. Each choice of boundary metric $g_0 = \gamma \in [\gamma]$ determines a unique geodesic defining function t. Now suppose for the moment that the boundary metric γ is C^{∞} smooth. Then by Corollary 2.5 when $n = 3$, or by [10] for general n, $\bar{g}$ is C^{∞} smooth when n is odd, and is C^{∞} polyhomogeneous when n is even. Hence, the curve g_t has a Taylor-type series in t - the Fefferman-Graham expansion [12]. The exact form of the expansion depends on whether n is odd or even. If n is odd, one has a power series expansion

$$g_t \sim g_{(0)} + t^2 g_{(2)} + \cdots + t^{n-1} g_{(n-1)} + t^n g_{(n)} + \cdots , \tag{3.2}$$

while if n is even, the series is polyhomogeneous,

$$g_t \sim g_{(0)} + t^2 g_{(2)} + \cdots + t^n g_{(n)} + t^n \log t \ \mathcal{H} + \cdots . \tag{3.3}$$

In both cases, this expansion is even in powers of t, up to t^n. The coefficients $g_{(2k)}$, $k \leq [n/2]$, as well as the coefficient $\mathcal{H}$ when n is even, are explicitly determined by the boundary metric $\gamma = g_{(0)}$

and the Einstein condition (1.2), cf. [11], [12]. For n even, the series (3.3) has terms of the form $t^{n+k}(\log t)^m$.

For any n, the divergence and trace (with respect to $g_{(0)} = \gamma$) of $g_{(n)}$ are determined by the boundary metric γ; in fact there is a symmetric bilinear form $r_{(n)}$ and scalar function $a_{(n)}$, both depending only on γ and its derivatives up to order n, such that

$$\delta_\gamma(g_{(n)} + r_{(n)}) = 0, \quad \text{and} \quad tr_\gamma(g_{(n)} + r_{(n)}) = a_{(n)}. \tag{3.4}$$

For n odd, $r_{(n)} = a_{(n)} = 0$. However, beyond the relations (3.4), the term $g_{(n)}$ is not determined by $g_{(0)}$; it depends on the "global" structure of the metric g. The higher order coefficients $g_{(k)}$ of t^k and coefficients $h_{(km)}$ of $t^{n+k}(\log t)^m$, are then determined by $g_{(0)}$ and $g_{(n)}$ via the Einstein equations. The equations (3.4) are constraint equations, and arise from the Gauss-Codazzi and Gauss and Riccati equations on the level sets $S(t) = \{x : t(x) = t\}$ in the limit $t \to 0$.

Now suppose k is an infinitesimal Einstein deformation of (M, g), so that k satisfies

$$L_E(k) \equiv 2\frac{d}{ds}(Ric_{g+sk} + n(g + sk)) = D^*Dk - 2R(k) - 2\delta^*\beta(k) = 0, \tag{3.5}$$

where β is the Bianchi operator $\beta(k) = \delta k + \frac{1}{2}dtrk$. Suppose for the moment that k is C^∞ polyhomogeneous smooth up to ∂M and preserves the geodesic boundary coordinates near ∂M, so that $k_{0\alpha} = 0$. If

$$k = O(t), \tag{3.6}$$

on approach to ∂M, then the discussion above on the Fefferman-Graham expansion implies the stronger decay

$$k = O(t^n). \tag{3.7}$$

If moreover one assumes the stronger condition that

$$k = o(t^n), \tag{3.8}$$

then the induced variation of the terms $g_{(0)}$ and $g_{(n)}$ in (3.2)-(3.3) vanishes and, again in view of the discussion on the expansions above, one has

$$k = o(t^\nu), \tag{3.9}$$

for all $\nu < \infty$. In this situation, one would expect that $k \equiv 0$ near ∂M. More generally, if k is as above but is not necessarily in geodesic gauge, then near ∂M, k should be a "pure gauge" deformation, i.e. $k = \delta^*Z$, for some vector field Z on M with $X = 0$ on ∂M. These expectations do in fact hold, and are proved in [6]; this corresponds to a unique continuation property at infinity for AH Einstein metrics and their linearizations.

For the work to follow in §4, we need to discuss this in somewhat more detail. Thus, suppose (M, g) is a $C^{2,\alpha}$ conformally compact Einstein metric. In view of (3.5), the simplest gauge choice for infinitesimal Einstein deformations κ of g is the Bianchi gauge

$$\beta(\kappa) = 0. \tag{3.10}$$

In this case, κ satisfies the elliptic equation

$$L(\kappa) = D^*D\kappa - 2R(\kappa) = 0. \tag{3.11}$$

A well-known result of Biquard [8] also gives a converse to this statement. Namely, if κ is a solution of (3.11) satisfying (3.6), then (3.10) holds. In fact, one then has

$$\delta\kappa = 0 \quad \text{and} \quad tr\kappa = 0. \tag{3.12}$$

To prove (3.12), the trace of (3.11) gives the equation

$$-\Delta tr\kappa + 2ntr\kappa = 0.$$

Since $|tr\kappa| \to 0$ at infinity, it follows immediately from the maximum principle that $tr\kappa = 0$. Combining this with (3.10) shows that $\delta\kappa = 0$ also.

We also note the well-known fact, proved via elliptic regularity in weighted Hölder spaces associated to the equation (3.11), that for (M, g) as above, if κ satisfies (3.11) and either $\kappa \in L^2(M, g)$ or (3.6) holds for κ, then (3.7) holds, i.e.

$$\kappa = O(t^n). \tag{3.13}$$

In addition, an analysis of the behavior of the indicial roots of (3.11) shows that one also has

$$\kappa(N, \cdot) = O(t^{n+1}), \tag{3.14}$$

where $N = \rho\partial_\rho$, with ρ the given defining function for ∂M in M; the decay estimates (3.13)-(3.14) are proved in [8], [20] or [22].

Given this background, the following result is proved in [6, Cor. 4.4], and will be used in the proof of Theorem 4.1.

Proposition 3.1. *Let g be a $C^{3,\alpha}$ conformally compact Einstein metric on M, and suppose*

$$\pi_1(M, \partial M) = 0. \tag{3.15}$$

If κ is a symmetric bilinear form in $L^2(M, g)$ satisfying (3.11) *and* (3.8) *holds in a neighborhood of ∂M, then*

$$\kappa \equiv 0, \tag{3.16}$$

on (M, g).

Remark 3.2. Proposition 3.1 proves a weak nondegeneracy property conjectured in [22]. As noted above, it corresponds to a unique continuation property at infinity for solutions of the linearized AH Einstein equations. A local version of this result also holds, (where (3.8) holds only on approach to a portion of the boundary); this will not be used here however. The topological condition (3.15) is needed to ensure that the (iterative) use of the local unique continuation property for solutions of (3.11) extends consistently to give a global uniqueness on the full manifold M.

We expect that Proposition 3.1 is false in general if the assumption $\pi_1(M, \partial M) = 0$ is dropped, for example if ∂M is not connected. However, this is not known and it would be of interest to find some concrete counterexamples.

On the other hand, if ∂M is connected and (M, g) has no local Killing fields, (i.e. there are no Killing fields on the universal cover $\widetilde{M}$ of M), then the proof Proposition 3.1 in [6] holds without the assumption (3.15). In particular, it follows that Proposition 3.1 holds for generic AH Einstein metrics on M provided ∂M is connected.

Finally for the application in §4, we note that the pointwise assumption (3.8) in Proposition 3.1 may be weakened to the analogous assumption on the L^2 norm of κ over the spheres $S(t)$ as $t \to 0$. This follows again from elliptic regularity associated with the equation (3.11), cf. [6, Lemma 4.2] for further details.

4. The Banach Manifold $\mathcal{E}_{AH}$.

In this section we prove that the moduli space of AH Einstein metrics on a given $(n+1)$-manifold is naturally an infinite dimensional Banach manifold, assuming it is non-empty. This is essentially the content of Theorem 1.2, but the full version is proved in §5. The work in this section uses the methods developed by Graham-Lee [15] and Biquard [10], as well as the work of White [26], [27].

We begin by describing the function spaces to be used. First, let ρ_0 be a fixed C^ω defining function for ∂M in M. Throughout §4, the defining function ρ_0 will be kept fixed and only compactifications with respect to ρ_0, will be considered, i.e.

$$\widetilde{g} = \rho_0^2 \cdot g. \tag{4.1}$$

The situation where ρ_0 varies over the family of smooth defining functions is discussed in §5. Given ρ_0, define the function $r = r(\rho_0)$ on M by

$$r = -\log(\tfrac{\rho_0}{2}). \tag{4.2}$$

Let $Met^{m,\alpha}(\partial M)$ be the space of $C^{m,\alpha}$ Riemannian metrics on ∂M, so that $Met^{m,\alpha}$ is an open cone in the Banach space $\mathbb{S}^{m,\alpha}(\partial M)$ of symmetric bilinear forms on ∂M. The space $Met^{m,\alpha}(\partial M)$ is given the $C^{m,\alpha'}$ topology, for a fixed $\alpha' < \alpha$, so that bounded sequences in the $C^{m,\alpha}$ norm have convergent subsequences. In this topology, $Met^{m,\alpha}$ is separable, cf. [26]. Next let $\mathbb{S}^{k,\beta}(M)$ be the Banach space of $C^{k,\beta}$ symmetric bilinear forms on M, and let $\mathbb{S}^{k,\beta}(\bar{M})$ be the corresponding space of forms on the closure $\bar{M}$, again with the $C^{k,\beta'}$ topology, $\beta' < \beta$.

Forms in $\mathbb{S}^{k,\beta}(M)$ have no control or restriction on their behavior on approach to ∂M, while those in $\mathbb{S}^{k,\beta}(\bar{M})$ of course by definition extend $C^{k,\beta}$ up to ∂M. Thus, (m, α) determines the regularity of the boundary data, while (k, β) determines the regularity in the interior M. These are not necessarily related, unless one has boundary regularity results, i.e. regularity of the data up to and including the boundary. We will always assume that $m + \alpha \geq k + \beta$, and $k \geq 2$, $\alpha, \beta \in (0, 1)$.

Let g be a complete Riemannian metric of bounded geometry on M, i.e. g has bounded sectional curvature and injectivity radius bounded below on M. Following [15] and [8], define the weighted Hölder spaces $\mathbb{S}^{k,\beta}_{\delta}(M) = \mathbb{S}^{k,\beta}_{\delta}(M, g)$ to be the Banach space of symmetric bilinear forms h on M such that

$$h = e^{-\delta r} h_0, \tag{4.3}$$

where $h_0 \in \mathbb{S}^{k,\beta}(M)$ satisfies $||h_0||_{C^{k,\beta}(M,g)} \leq C$, for some constant $C < \infty$. Here the norm is the usual $C^{k,\beta}$ norm with respect to the metric g, given by

$$||h_0||_{C^{k,\beta}(M,g)} = \sum_{j \leq k} ||\nabla^j h_0||_{C^0(M,g)} + ||\nabla^k h_0||_{C^{\beta}(M,g)}. \tag{4.4}$$

Thus $h \in \mathbb{S}^{k,\beta}_{\delta}(M)$ implies that h and its derivatives up to order k with respect to g decay as $e^{-\delta r}$ as $r \to \infty$. The weighted norm of h is then defined as

$$||h||_{C^{k,\beta}_{\delta}(M)} = ||h_0||_{C^{k,\beta}(M)}. \tag{4.5}$$

The norms in (4.4) and (4.5) depend only on $C^{k,\beta}$ quasi-isometry class of g; two metrics g and g' are $C^{k,\beta}$ quasi-isometric if, in a fixed local coordinate system, the linear map $g^{-1}g'$ is bounded away from 0 and ∞ in $C^{k,\beta}(M)$. Hence the spaces $\mathbb{S}^{k,\beta}_{\delta}(M)$ depend only on the $C^{k,\beta}$ quasi-isometry class of g.

Now suppose the metric g is conformally compact, with compactification $\widetilde{g}$ as in (4.1). One may then define the $C^{k,\beta}$ norm of h above also with respect to $\widetilde{g}$. Using standard formulas for conformal changes of metric gives, for any j, $\beta \geq 0$,

$$||\widetilde{\nabla}^j h||_{C^{\beta}(\widetilde{g})} = ||\rho_0^{-2-j-\beta} \nabla^j h||_{C^{\beta}(g)} + \quad \text{lower order terms.} \tag{4.6}$$

Given these preliminaries, one can construct a natural or "standard" AH metric associated to any boundary metric $\gamma \in Met^{m,\alpha}(\partial M)$. This is first done in a collar neighborhood U of ∂M on which $d\rho_0$ is non-zero, and then later extended to a metric on all of M. Choose a fixed identification of U with $I \times \partial M$ so that ρ_0 corresponds to the variable on I. Recalling that ρ_0 is fixed, define the $C^{m,\alpha}$ hyperbolic cone metric $g_U = g_U(\gamma, \rho_0)$ in U by

$$g_U = dr^2 + \sinh^2 r \cdot \gamma, \tag{4.7}$$

for r as in (4.2). Observe that the dependence of g_U is C^{ω} in γ, (and also in ρ_0). Also if γ_1 and γ_2 are $C^{m,\alpha}$ quasi isometric boundary metrics, then $g_U(\gamma_1)$ and $g_U(\gamma_2)$ are $C^{m,\alpha}$ quasi isometric.

If $\{e_i\}$ is a local orthonormal frame for g_U, with $e_1 = \partial_r$, then one easily verifies that the sectional curvatures K_{ij} of g_U in the direction (e_i, e_j) are given by

$$K_{1i} = -1, K_{jk} = \frac{1}{\sinh^2 r}(K_\gamma)_{jk} - \coth^2 r,$$

where i, j, k run from 2 to $n+1$ and K_γ is the sectional curvature of γ. This implies that the curvature of g_U decays to that of the hyperbolic space $H^{n+1}(-1)$ at a rate of $O(e^{-2r}) = O(\rho_0^2)$. The same decay holds for the covariant derivatives of the curvature, up to order $m-2+\alpha$. In particular by (4.3)-(4.5)

$$Ric_{g_U} + n \cdot g_U \in \mathbb{S}_2^{m-2,\alpha}(M). \tag{4.8}$$

The metric g_U is $C^{m,\alpha}$ conformally compact. In fact if $\widetilde{g}_U$ is the compactification (4.1) of g_U, then a simple computation gives

$$\widetilde{g}_U = d\rho_0^2 + (1 - \tfrac{1}{4}\rho_0^2)^2 \cdot \gamma.$$

The metric g_U will be viewed as a background metric with which to compare other conformally compact metrics with the same boundary metric. Thus suppose g' is any conformally compact metric on M, with compactification $\widetilde{g}'$ as in (4.1). Then one may write

$$g'|_U = g_U + h, \tag{4.9}$$

and we will assume that $h \in \mathbb{S}^{k,\beta}(M)$. This implies that

$$\widetilde{g}'|_U = d\rho_0^2 + (1 - \tfrac{1}{4}\rho_0^2)^2 \cdot \gamma + \rho_0^2 h, \tag{4.10}$$

so that if $\rho_0^2|h| \to 0$ on approach to ∂M, then $\widetilde{g}'$ is a C^0 compactification of $g'|_U$ with boundary metric γ; here $|h|$ is the pointwise norm of h with respect to any smooth metric on $\bar{M}$. The compactification $\widetilde{g}'$ is $C^{k,\beta}$ when $\rho_0^2 h \in \mathbb{S}^{k,\beta}(\bar{M})$. Using the relations (4.3)-(4.6), observe that

$$h \in \mathbb{S}_\delta^{k,\beta}(M) \ \text{ with } \ \delta \geq k + \beta \Rightarrow \widetilde{h} = \rho_0^2 h \in \mathbb{S}^{k,\beta}(\bar{M}). \tag{4.11}$$

However, if $\delta < k + \beta$ and $h \in \mathbb{S}_\delta^{k,\beta}(M)$, then in general, i.e. without further restrictions, $\rho_0^2 h$ will not be in $\mathbb{S}^{k,\beta}(\bar{M})$; this is essentially the issue of boundary regularity, and will be discussed at the end of §4 and in §5.

The standard metrics g_U may be naturally extended to all of M as follows. Let $\eta = \eta(r)$ be a fixed cutoff function on M, with $\eta \equiv 1$ on U, $\eta \equiv 0$ on $M \setminus U'$, where U' is a thickening of U on which $d\eta$ is also non-vanishing. If g_C is any smooth Riemannian metric on the compact manifold $M \setminus U$, (so g_C is incomplete), then define

$$g_\gamma = \eta g_U + (1 - \eta) g_C. \tag{4.12}$$

Thus for any $\gamma \in Met^{m,\alpha}(\partial M)$, g_γ in (4.12) gives a standard AH metric on M, with boundary metric γ. The metric g_γ on M again depends smoothly on γ and the choices of the compact metric g_C and cutoff η. As with ρ_0, we fix the metric g_C and cutoff η once for all. With this understood, one thus has a C^ω smooth (addition) map

$$A : Met^{m,\alpha}(\partial M) \times \mathcal{U}_\delta^{k,\beta} \to Met_\delta^{k,\beta}(M), A(\gamma, h) = g \equiv g_\gamma + h, \tag{4.13}$$

where $\mathcal{U}_\delta^{k,\beta}$ is the open subset of $\mathbb{S}_\delta^{k,\beta}(M)$, consisting of those h such that $g_\gamma + h$ is a well-defined metric on M.

In view of the decay rate (4.8), the most natural choice of δ is

$$\delta = 2, \tag{4.14}$$

and we fix this choice for the remainder of this section. The map A is clearly injective and the asymptotically hyperbolic (AH) metrics (of weight $\delta = 2$) are defined to be the image of A;

$$Met_{AH}^{k,\beta} = Im A. \tag{4.15}$$

The inverse map to A, $S : Met_{AH}^{k,\beta} \to Met^{m,\alpha}(\partial M) \times \mathcal{U}_2^{k,\beta}$ gives the splitting of the AH metric g into its components g_γ and h. Let

$$E_{AH}^{k,\beta} \subset Met_{AH}^{k,\beta} \tag{4.16}$$

be the subset of AH Einstein metrics, with topology induced as a subset of the product topology. Note that, as discussed in (4.11), metrics in $E_{AH}^{k,\beta}$ are C^2 conformally compact, but not necessarily $C^{k,\beta}$ conformally compact with respect to ρ_0, for $k+\beta > 2$. Of course Einstein metrics are C^ω in local harmonic coordinates, and so (k,β) only serves to denote the ambient space $Met_{AH}^{k,\beta}$ in which E_{AH} is embedded.

Now let g_0 be a fixed (but arbitrary) background metric in $Met_{AH}^{k,\beta}$ with boundary metric γ_0. For $\gamma \in Met^{m,\alpha}(\partial M)$ close to γ_0, let

$$g(\gamma) = g_0 + \eta(g_\gamma - g_{\gamma_0}). \tag{4.17}$$

Any metric $g \in Met_{AH}^{k,\beta}$ with boundary metric γ thus has the form $g = g(\gamma) + h$, for $h \in \mathbb{S}_2^{k,\beta}$. Essentially as in [8], for any $k \geq 2$, define

$$\Phi = \Phi^{g_0} : Met_{AH}^{k,\beta} \to \mathbb{S}_2^{k-2,\beta}(M), \tag{4.18}$$

$$\Phi(g) = \Phi(g(\gamma)+h) = Ric_g + ng + (\delta_g)^*\beta_{g(\gamma)}(g),$$

where $\beta_{g(\gamma)}$ is the Bianchi operator with respect to $g(\gamma)$, (cf. (3.5)),

$$\beta_{g(\gamma)}(g) = \delta_{g(\gamma)}g + \tfrac{1}{2}d(tr_{g(\gamma)}g). \tag{4.19}$$

Observe that Φ is well-defined, by (4.8), (4.14) and the fact that $h = g - g(\gamma) \in \mathbb{S}_2^{k,\beta}$. Clearly Φ is C^ω in g.

There are several natural reasons for considering the operator Φ. First, it is proved in [8, Lemma I.1.4] that

$$Z_{AH}^{k,\beta} \equiv \Phi^{-1}(0) \cap \{Ric < 0\} \subset E_{AH}^{k,\beta}, \tag{4.20}$$

where $\{Ric < 0\}$ is the open set of metrics with negative Ricci curvature. (Here one uses the fact that $\beta_{g(\gamma)}(g) \in \mathbb{S}_2^{k-1,\beta}$). Further, if g is an AH Einstein metric, i.e. $Ric_g = -ng$, with boundary metric γ, which is close to g_0 and which satisfies $\Phi(g) = 0$, then

$$\beta_{g(\gamma)}(g) = 0. \tag{4.21}$$

As discussed later, the condition (4.21) defines the tangent space of a slice to the action of the diffeomorphism group on Met_{AH} and E_{AH}. Thus, for any $g \in E_{AH}^{k,\beta}$ near g_0, there exists a diffeomorphism ϕ such that $\phi^*g \in Z_{AH}^{k,\beta}$, cf. (4.38) below. Hence, $E_{AH}^{k,\beta}$ differs from $Z_{AH}^{k,\beta}$ just by the action of diffeomorphisms.

Second, as discussed in §3, the linearization of the Einstein operator $Ric_g + ng$ at an Einstein metric g is given by

$$\tfrac{1}{2}D^*D - R - \delta^*\beta, \tag{4.22}$$

acting on the space of symmetric 2-tensors $\mathbb{S}(M)$ on M, cf. [7]. The kernel of the elliptic self-adjoint linear operator

$$L = \tfrac{1}{2}D^*D - R \tag{4.23}$$

corresponds to the space of non-trivial infinitesimal Einstein deformations in Bianchi-free gauge, analogous to the Jacobi fields for geodesics. An AH Einstein metric g on M is called *non-degenerate* if

$$K = L^2 - KerL = 0, \tag{4.24}$$

i.e. if there are no non-trivial infinitesimal Einstein deformations of g in $L^2(M,g)$. Einstein metrics are critical points of the Einstein-Hilbert functional or action, and this corresponds formally to the condition that the critical point be non-degenerate, in the sense of Morse theory. Recall from (3.12) that elements in K are transverse-traceless.

Now the linearization of Φ at $g_0 \in Met^{k,\beta}_{AH}$ with respect to the 2^{nd} variable h has the simple form

$$(D_2\Phi)_{g_0}(\dot{h}) = \tfrac{1}{2}D^*D\dot{h} + \tfrac{1}{2}(Ric_{g_0} \circ \dot{h} + \dot{h} \circ Ric_{g_0} + 2n\dot{h}) - R_{g_0}(\dot{h}); \tag{4.25}$$

this is due to cancellation of the variation of the term $(\delta_g)^*\beta_{g(\gamma)}(g)$ with the variation of the Ricci curvature, cf. [8, (1.9)]. Hence, if g_0 is Einstein, then

$$(D_2\Phi)_{g_0} = L = \tfrac{1}{2}D^*D - R. \tag{4.26}$$

The variation of Φ at g_0 with respect to the 1^{st} variable $g(\gamma)$ has the form

$$(D_1\Phi)_{g_0}(\dot{g}(\gamma)) = (D_2\Phi)_{g_0}(\dot{g}(\gamma)) - \delta^*_{g_0}\beta_{g_0}(\dot{g}(\gamma)),$$

where $(D_2\Phi)_{g_0}(\dot{g}(\gamma))$ is given by (4.25) with $\dot{g}(\gamma)$ in place of $\dot{h}$.

The main result of this section, which leads to a version of Theorem 1.2 is the following:

Theorem 4.1. *Suppose $\pi_1(M,\partial M) = 0$. At any metric $g \in E^{k,\beta}_{AH}$ which is $C^{3,\alpha}$ conformally compact, the map Φ is a submersion, i.e. the derivative*

$$(D\Phi)_g : T_g Met^{k,\beta}_{AH} \to T_{\Phi(g)}\mathbb{S}^{k-2,\beta}_2(M) \tag{4.27}$$

is surjective and its kernel splits in $T_g Met^{k,\beta}_{AH}$.

Proof: By (4.13) and (4.15), $T_g Met_{AH} = T_\gamma Met^{m,\alpha}(\partial M) \oplus T_h\mathbb{S}^{k-2,\beta}_2(M)$. With respect to this splitting, (4.26) shows that the derivative of Φ with respect to the second (i.e. h) factor is given by

$$(D_2\Phi)_g = \tfrac{1}{2}D^*D - R : \mathbb{S}^{k,\beta}_2(M) \to \mathbb{S}^{k-2,\beta}_2(M).$$

Now by [8, Prop. I.3.5], $(D_2\Phi)_g$ is a Fredholm operator whose kernel on $\mathbb{S}^{k,\beta}_2(M)$ equals the L^2 kernel K in (4.24). Since $(D_2\Phi)_g$ is self-adjoint on L^2, it has Fredholm index 0, and the cokernel of $(D_2\Phi)_g$ is naturally identified with K in $\mathbb{S}^{k-2,\beta}_2(M)$. Thus to prove $D\Phi_g$ is surjective, it suffices to show that for any non-zero L^2 infinitesimal Einstein deformation $\kappa \in K$, there is a tangent vector $X \in T_g Met_{AH}$ such that

$$\int_M \langle (D\Phi)_g(X), \kappa \rangle dV_g \neq 0. \tag{4.28}$$

To do this, let $X = \dot{g}(\gamma)$, so that X corresponds to a variation of the boundary metric γ of g. Then $D\Phi_{g_0}(X)$ has the form

$$(D\Phi_g)(X) = \tfrac{1}{2}D^*D\dot{g}(\gamma) - R(\dot{g}(\gamma)) - \delta^*\beta(\dot{g}(\gamma)). \tag{4.29}$$

Let γ be the boundary metric induced by $\widetilde{g}$ in (4.1) on ∂M. For the following computation, it is convenient to work with the geodesic defining function t determined by γ. Set $r = -\log\frac{t}{2}$, as in (4.2) and let $B(r)$ be the r-sublevel set of the function r with $S(r) = \partial B(r)$ the r-level set. We apply the divergence theorem to the integral (4.28) over $B(r)$; twice for the Laplacian term in (4.29) and once for the δ^* term. Since $\kappa \in KerL$ and $\delta\kappa = 0$ by (3.12), it follows that the integral (4.28) reduces to an integral over the boundary, and gives

$$\int_{B(r)} \langle (D\Phi)_g(X), \kappa \rangle dV_g = \int_{S(r)} (\langle \dot{g}(\gamma), \nabla_N\kappa \rangle - \langle \nabla_N\dot{g}(\gamma), \kappa \rangle - \langle \beta(\dot{g}(\gamma)), \kappa(N) \rangle) dV_{S(r)}, \tag{4.30}$$

where $N = \nabla r = -t\partial_t$ is the unit outward normal.

To estimate the boundary integrals, the volume form of $S(r)$ satisfies

$$dV_{S(r)} = t^{-n}dV_\gamma + O(t^2),$$

where dV_γ is the volume form of the boundary metric. Let $\widetilde{\kappa} = t^{-n}\kappa$. By (3.13), $|\widetilde{\kappa}|_g|_{S(r)}$ is uniformly bounded. Setting $\hat{\kappa} = t^2\widetilde{\kappa}$, one has $|\hat{\kappa}|_{\bar{g}} = |\widetilde{\kappa}|_g$, and so the same is true for $|\hat{\kappa}|_{\bar{g}}$. A simple calculation from (4.17) gives

$$\dot{g}(\gamma) = \eta(\sinh^2 r)\dot{\gamma}, \ \ \nabla_N\dot{g}(\gamma) = O(t),$$

so that $|\dot{g}(\gamma)|_g \sim 1$ and $|\nabla_N\dot{g}(\gamma)|_g \sim O(t)$ as $t \to 0$. Hence,

$$(4.31)\qquad \begin{aligned}(\langle\dot{g}(\gamma), \nabla_N\kappa\rangle_g - \langle\nabla_N\dot{g}(\gamma), \kappa\rangle)_g dV_{S(r)} &= t^2\langle\nabla_N\kappa, \dot{\gamma}\rangle_\gamma dV_{S(r)} + O(t)\\ &= \langle\nabla_N\hat{\kappa} - (n-2)\hat{\kappa}, \dot{\gamma}\rangle_\gamma dV_\gamma + O(t).\end{aligned}$$

By (3.14), $\kappa(N) = O(t^{n+1})$, and hence the last term in (4.30) vanishes in the limit $r \to \infty$.

It follows that if (4.28) vanishes in the limit $r \to \infty$, for *all* variations $\dot{\gamma}$, then one must have

$$(4.32)\qquad \nabla_N\hat{\kappa} - (n-2)\hat{\kappa} = o(1),$$

weakly, as forms on $T(S(r))$ with respect to $\bar{g}$. Here ∇ is the covariant derivative with respect to g, not $\bar{g}$. Pairing this with the bounded form $\hat{\kappa}$, and using (3.14) again, one easily sees that (4.32) implies that $\frac{1}{2}N|\hat{\kappa}|^2 - n|\hat{\kappa}|^2 = o(1)$, where the norms are with respect to $\bar{g}$. Integrating this with respect to the volume form on $(S(t), \bar{g})$ and using the fact that $\frac{d}{dt}dV_{S(t)} = O(t)$, it follows that

$$(4.33)\qquad \tfrac{1}{2}N\int_{S(t)}|\hat{\kappa}|^2 dV_\gamma - n\int_{S(t)}|\hat{\kappa}|^2 dV_\gamma = o(1),$$

as $t \to 0$. Using the fact that $N = -t\partial t$, an elementary integration then implies that $\int_{S(t)}|\hat{\kappa}|^2 dV_\gamma \to 0$, and hence $\widetilde{\kappa} = o(t^n)$ weakly. However, under the assumption $\pi_1(M, \partial M) = 0$, this contradicts Proposition 3.1, (cf. also Remark 3.2), which thus proves that (4.28) holds.

To prove that the kernel of $D\Phi_g$ splits, i.e. it admits a closed complement in T_gMet_{AH}, it suffices to exhibit a bounded linear projection P mapping T_gMet_{AH} onto $Ker(D\Phi_g)$. We do this following [27]. Thus, one has

$$(4.34)\qquad KerD\Phi_g = \{(\dot{\gamma}, \dot{h}) : D_1\Phi(\dot{\gamma}) + D_2\Phi(\dot{h}) = 0\}.$$

From (4.26), $D_2\Phi = L$ and Im $L = K^\perp$, for K as in (4.24). Hence $D_1\Phi(\dot{\gamma}) \in K^\perp$, so that $\pi_K D_1\Phi(\dot{\gamma}) = 0$, i.e. $\dot{\gamma} \in Ker(\pi_K D_1\Phi)$, where π_K is orthogonal projection onto K. By (4.28) or more precisely its proof, $D_1\Phi$ maps onto K and hence $Im\pi_K D_1 = K$. Since the finite dimensional space K splits, we have $TMet_{AH} = K \oplus K^\perp = Im(\pi_K D_1\Phi) \oplus Ker(\pi_K D_1\Phi)$, so that $Ker(\pi_K D_1\Phi)$ splits. Hence, there is a bounded linear projection P_1 onto $Ker(\pi_K D_1\Phi)$. The operator $L + \pi_K$ is invertible and one may now define

$$P(\dot{\gamma}, \dot{h}) = (P_1\dot{\gamma}, (L + \pi_K)^{-1}(-(D_1\Phi)P_1(\dot{\gamma}) + \pi_K\dot{h})).$$

Then P is the required bounded linear projection. ■

As in Remark 3.2, it is doubtful if Theorem 4.1 remains valid in general without the assumption $\pi_1(M, \partial M) = 0$. As noted there, in the generic situation where $g \in E^{k,\beta}_{AH}$ has no local Killing fields, Theorem 4.1 does hold at g, at least when ∂M is connected. For simplicity, for the rest of this section and throughout §5, we assume $\pi_1(M, \partial M) = 0$.

Corollary 4.2. *For any $C^{3,\alpha}$ conformally compact metric $g \in E^{k,\beta}_{AH}$, the local space $Z^{k,\beta}_{AH}$ is an infinite dimensional C^∞ separable Banach manifold. In fact, via the splitting* (4.13)*, $Z^{k,\beta}_{AH}$ is a C^∞ Banach submanifold of $Met^{m,\alpha}(\partial M) \times \mathbb{S}^{k,\beta}_2(M)$ and as such*

$$(4.35)\qquad T_gZ^{k,\beta}_{AH} = Ker(D\Phi)_g.$$

Proof: This is an immediate consequence of the definition (4.20), Theorem 4.1 and the implicit function theorem in Banach spaces, cf. [18]. $Z^{k,\beta}_{AH}$ is separable since it is a submanifold of $Met^{m,\alpha}(\partial M) \times \mathbb{S}^{k,\beta}_2(M)$, each of which are separable Banach spaces in the topologies defined at the beginning of §4. ∎

Locally, near any given $g_0 \in E^{k,\beta}_{AH}$, the boundary map taking an AH Einstein metric g to its boundary metric γ with respect to the compactification (4.1) is given simply by projection on the first factor:

$$\Pi : E^{k,\beta}_{AH} \to Met^{m,\alpha}(\partial M); \quad \Pi(g) = \Pi(g_\gamma + h) = \gamma. \tag{4.36}$$

Clearly, this map is C^∞ smooth.

The spaces $Met^{k,\beta}_{AH}$ and $E^{k,\beta}_{AH}$ are invariant under the action of suitable diffeomorphisms. In §5, we will consider larger diffeomorphism groups, but for now we restrict to the group $\mathcal{D}_2 \equiv \mathrm{Diff}^{m+1,\alpha}(\bar{M})$ of $C^{m+1,\alpha}$ diffeomorphisms ϕ of $\bar{M}$ such that

$$\phi|_{\partial M} = id_{\partial M}, \quad \text{and} \quad \lim_{\rho_o \to 0}\Big(\frac{\phi^*\rho_0}{\rho_0}\Big) = 1, \tag{4.37}$$

where ρ_0 is the fixed defining function. If $g \in E^{k,\beta}_{AH}$ and $\widetilde{g} = \rho_0^2 g$ is the compactification as in (4.1), then for $\phi \in \mathcal{D}_2$, the compactification of $\phi^* g$ is given by

$$\widetilde{\phi^* g} = \rho_0^2 \phi^* g = (\phi^* \widetilde{g})\Big(\frac{\rho_0}{\phi^*\rho_0}\Big)^2.$$

Hence (4.37) implies that g and $\phi^* g$ have the same boundary metric with respect to ρ_0. However, the normal vectors of the compactified metrics $\widetilde{g}$ and $\phi^*\widetilde{g}$ are different in general.

The action of $\mathcal{D}_2$ preserves the spaces $Met^{k,\beta}_{AH}$ and $E^{k,\beta}_{AH}$. This is because $|D\phi - id|_{\widetilde{g}}$ extends $C^{m,\alpha}$ smoothly up to ∂M, and hence $|D\phi - id|_g = O(e^{-r})$, so that $|\phi^* g - g|_g = O(e^{-2r})$. Note also that since $m \geq k$, for g a $C^{k,\beta}$ metric (in a smooth atlas for M), and $\phi \in \mathcal{D}_2$, $\phi^* g \in C^{k,\beta}$.

Observe that the action of $\mathcal{D}_2$ on $Met^{k,\beta}_{AH}$ or $E^{k,\beta}_{AH}$ is free, since any isometry ϕ of a metric inducing the identity on ∂M must itself be the identity; this is most easily seen by working in a geodesic compactification $\bar{g}$. It is also standard that the action of $\mathcal{D}_2$ on $Met^{k,\beta}_{AH}$ and $E^{k,\beta}_{AH}$ is proper.

It is well-known however that the action of $\mathcal{D}_2$ on $Met^{k,\beta}_{AH}$ is not smooth; for a 1-parameter group of diffeomorphisms ϕ_t with $\phi_0 = id$ and infinitesimal generator X, one has $\frac{d}{dt}(\phi_t^* g)|_t = \phi_t^* \mathcal{L}_X g$. For $g \in Met^{k,\beta}_{AH}$ and $X \in T_{id}\mathcal{D}_2$, the form $\mathcal{L}_X g$ is only $C^{k-1,\beta}$ smooth and so not an element of $T_g Met^{k,\beta}_{AH}$. However, as noted following (4.16), Einstein metrics g are C^∞ smooth in a smooth atlas for M, and in such coordinates, $\mathcal{L}_X g$ is $C^{k,\beta}$, (in fact $C^{m,\alpha}$), smooth. Thus, there is no loss-of-derivatives for Einstein metrics.

Now it is proved in [8, Prop. I.4.6] that the set of metrics $g \in E^{k,\beta}_{AH}$ near a given $g_0 \in E^{k,\beta}_{AH}$ such that

$$\beta_{g_0}(g) = 0 \tag{4.38}$$

is a local slice for the action of $\mathcal{D}_2$ on $E^{k,\beta}_{AH}$. Thus, a neighborhood $\mathcal{U}$ of any given $g_0 \in E^{k,\beta}_{AH}$ is homeomorphic to a product $Z^{k,\beta}_{AH} \times \mathcal{V}$, where $\mathcal{V}$ is a neighborhood of the identity in $\mathcal{D}_2$. The homeomorphism is given by

$$\psi_0(g) = (\phi_0^* g, \phi_0),$$

where ϕ_0 is the unique element of $\mathcal{D}_2$ such that $\phi_0^* g \in Z^{k,\beta}_{AH}$. To consider the corresponding overlap maps, let g_0 and g_1 be background metrics in $E^{k,\beta}_{AH}$ which are sufficiently close, and let Z_i be the space (4.20) determined by g_i. Then $\psi_0(g) = (\phi_0^* g, \phi_0)$, $\psi_1(g) = (\phi_1^* g, \phi_1)$ with $\phi_i^* g \in Z_i$, and hence the overlap map is given by

$$\psi_{01}(g_0, \phi_0) = (g_1, \phi_1) = ((\phi_1^*(\phi_0^{-1})^*)g_0, (\phi_1 \circ \phi_0^{-1})\phi_0),$$

where $g_i \in Z_i$ and $\phi_1 = \phi_1(g_0, \phi_0)$ is defined as the unique solution of the equation $\beta_{g_1}(\phi_1^*(\phi_0^{-1})^* g_0) = 0$. By the discussion preceding (4.38), ϕ_1 is differentiable in g_0 and ϕ_0 and in fact is C^∞ smooth in these variables. It follows that the overlap maps are C^∞ and hence the global space $E_{AH}^{k,\beta}$ is a C^∞ smooth separable Banach manifold, as is the quotient

$$\mathcal{E}_{AH}^{(2)} = E_{AH}^{k,\beta}/\mathcal{D}_2^{m+1,\alpha}. \tag{4.39}$$

Two metrics g_1 and g_2 in $\mathcal{E}_{AH}^{(2)}$ are equivalent if there is a $C^{m+1,\alpha}$ diffeomorphism ϕ of weight 2, i.e. satisfying (4.37), such that $\phi^* g_1 = g_2$. In particular, g_1 and g_2 must have the same boundary metric with respect to ρ_0.

When $E_{AH}^{k,\beta}$ is viewed as subset of the product $Met^{m,\alpha}(\partial M) \times \mathbb{S}_2^{k,\beta}(M)$ via S, since $\mathcal{D}_2^{m+1,\alpha}$ acts trivially on the first factor, one has

$$\mathcal{E}_{AH}^{(2)} \subset Met^{m,\alpha}(\partial M) \times (\mathbb{S}_2^{k,\beta}(M)/\mathcal{D}_2^{m+1,\alpha}). \tag{4.40}$$

This inclusion sends $[g] = [g_\gamma + h]$ to $(\gamma, [h])$, and, given a fixed $g_0 \in E_{AH}^{k,\beta}$, a slice representative for $[h]$ is that unique $h \in [h]$ satisfying (4.38). Via (4.36), Π descends to a smooth map

$$\Pi : \mathcal{E}_{AH}^{(2)} \to Met^{m,\alpha}(\partial M), \quad \Pi([g]) = \gamma. \tag{4.41}$$

We summarize the analysis above in the following:

Proposition 4.3. *Near any $C^{3,\alpha}$ conformally compact Einstein metric g, the space $\mathcal{E}_{AH}^{(2)}$ is a smooth separable Banach manifold. The map $\Pi : \mathcal{E}_{AH}^{(2)} \to Met^{m,\alpha}(\partial M)$ is a C^∞ Fredholm map of index 0, with*

$$Ker(D\Pi)_g = K_g, \tag{4.42}$$

where as in (4.24), K_g is the space of L^2 infinitesimal Einstein deformations at g. Consequently, $Im\Pi \subset Met^{m,\alpha}(\partial M)$ is a variety of finite codimension.

Proof: One only needs to verify that Π is Fredholm, with kernel given by (4.42). By construction, one has

$$KerD\Pi = T\mathcal{E}_{AH}^{(2)} \cap Ker\Pi_1,$$

where Π_1 is the linear projection on the first factor in the splitting (4.40). Since $D\Pi_1 = Id$ on the first factor,

$$T\mathcal{E}_{AH}^{(2)} \cap Ker\Pi_1 = T\mathcal{E}_{AH}^{(2)} \cap T(\mathbb{S}_2^{k,\beta}/\mathcal{D}_2^{m+1,\alpha}).$$

This intersection just consists of the classes $[h]$ satisfying (4.38), and so by (4.35) and (4.26),

$$KerD\Pi = KerL,$$

where the kernel is taken in $\mathbb{S}_2^{k,\beta}$. But this is the same as the L^2 kernel K by [8, Prop.I.3.5].

For the cokernel, one has

$$Im(D\Pi) = \Pi(T\mathcal{E}_{AH}^{(2)}) = \Pi(KerD\Phi) = Ker(\pi_K D_1\Phi),$$

where the second equality is from (4.35) and the last equality follows from (4.34) and the discussion following it. Again, as following (4.34), $Ker\pi_K D_1\Phi = K^\perp$ is closed, and has codimension $k = dimK$. Hence Π is Fredholm of index 0. ∎

Remark 4.4. This result shows that one has the following dictotomy: either there exist no conformally compact Einstein metrics on M, or the moduli space of such metrics is at least infinite dimensional, with $Im\Pi$ a variety of finite codimension in $Met(\partial M)$.

If there exist Einstein metrics $g \in E_{AH}$ which are non-degenerate, so that $K_g = \{0\}$, then Π is a local diffeomorphism in a neighborhood of g. This is the result of Biquard [8], extending earlier work of Graham-Lee [15]. In other words, Π is an open map on the open submanifold E'_{AH} of non-degenerate metrics.

Note that $T\mathcal{E}^{(2)}_{AH}$ is the space of (essential) infinitesimal asymptotically hyperbolic Einstein deformations, (not necessarily preserving the boundary metric as is the case with the L^2 kernel K). The fact that $\mathcal{E}^{(2)}_{AH}$ is a smooth Banach manifold implies that any infinitesimal AH Einstein deformation may be integrated to a (local) curve of AH Einstein metrics. Apriori, it is not clear if this remains the case when the boundary metric is required to be fixed, i.e. an L^2 infinitesimal Einstein deformation in K might not integrate to a curve of AH Einstein metrics with the same boundary metric.

Observe that all the results above are valid in any dimension.

We complete this section with a discussion of the boundary regularity of metrics in E_{AH}. The Einstein metrics in $E^{k,\beta}_{AH}$ have C^2 and hence $L^{2,p}$ compactifications. Suppose dim $M = n+1 = 4$. Then Theorem 2.3 implies that any $g \in E^{k,\beta}_{AH}$ is $C^{m,\alpha}$ conformally compact, for any $m \geq 2$, see the discussion following (4.2). Thus

$$E^{k,\beta}_{AH} = E^{m,\alpha}_{AH}, \tag{4.43}$$

and $E^{m,\alpha}_{AH}$ is the space of AH Einstein metrics on M which are $C^{m,\alpha}$ conformally compact with respect to the defining function ρ_0 as in (4.1). The space $E^{m,\alpha}_{AH}$ is a smooth separable Banach manifold, and boundary regularity implies that the topology on $E^{m,\alpha}_{AH}$ defined by (4.16) is equivalent to the $\mathbb{S}^{m,\mu}_2(\bar{M})$ topology on the compact manifold $\bar{M}$, for a fixed $\mu < \alpha$. This corresponds to the definition in the Introduction. With this understood, one has the following version of Theorem 1.2:

Proposition 4.5. *If dim $M = 4$ and $m \geq 3$, then $E^{m,\alpha}_{AH}$ is the space of $C^{m,\alpha}$ conformally compact Einstein metrics on M. If $\pi_1(M, \partial M) = 0$, then $E^{m,\alpha}_{AH}$ is a smooth separable Banach manifold and the map Π is a C^{∞} map*

$$\Pi : E^{m,\alpha}_{AH} \to Met^{m,\alpha}(\partial M). \tag{4.44}$$

■

Of course, Proposition 4.5 also holds on the quotient $\mathcal{E}^{(2)}_{AH} = \mathcal{E}^{(2),m,\alpha}_{AH}$. An analogous but somewhat weaker result holds in all higher dimensions $n+1 > 4$; in fact there are two versions in higher dimensions, although neither version is quite as strong as Proposition 4.5, cf. Theorems 5.5 and 5.6 for further details.

5. The Spaces $\mathcal{E}^{m,\alpha}$, Diffeomorphisms and Stability.

In §4, the defining function ρ_0 was fixed, thus giving a fixed boundary metric γ for an AH Einstein metric g on M. In this section, we consider the situation where ρ varies over all smooth defining functions, and the corresponding variation of the boundary metrics. This is closely related to the action of diffeomorphisms on $\bar{M}$. These issues are discussed in §5.1, together with the proof of Theorem 1.2 and its versions in higher dimensions. In §5.2, we prove that the spaces $\mathcal{E}^{m,\alpha}_{AH}$ are all diffeomorphic and stable in a natural sense.

§5.1. Let $\mathcal{D}_1 = \mathcal{D}^{m+1,\alpha}_1(\bar{M})$ be the group of orientation preserving $C^{m+1,\alpha}$ diffeomorphisms of $\bar{M}$ which restrict to the identity map on ∂M. Recall from (4.37) that $\mathcal{D}_2 \subset \mathcal{D}_1$ is the subgroup of diffeomorphisms ϕ satisfying $\lim_{\rho_0 \to 0}(\phi^*\rho_0/\rho_0) = 1$. It is easily seen that $\mathcal{D}_2$ is a normal subgroup of

$\mathcal{D}_1$. With respect to ρ_0, one has the splitting $TM|_{\partial M} \cong T(\partial M) \oplus \mathbb{R}$, where the $\mathbb{R}$ factor is identified with the span of $\partial/\partial\rho_0$. The groups $\mathcal{D}_1$ and $\mathcal{D}_2$ act on $T(\partial M) \oplus \mathbb{R}$ by the map $\phi \to D\phi|_{\partial M}$, and so induce subgroups of $Hom(TM|_{\partial M}, TM|_{\partial M})$. Since $\mathcal{D}_2 \subset \mathcal{D}_1$ is defined solely by a 1^{st} order condition at ∂M, the quotient group $\mathcal{D}_1/\mathcal{D}_2$ is isomorphic to the corresponding quotient group in $Hom(TM|_{\partial M}, TM|_{\partial M})$.

Lemma 5.1. *The quotient group $\mathcal{D}_1/\mathcal{D}_2$ is naturally isomorphic to the group of $C^{m,\alpha}$ positive functions λ on ∂M.*

Proof: With respect to the splitting $TM|_{\partial M} \cong T(\partial M) \oplus \mathbb{R}$, the linear map $D\phi|_{\partial M}$, for $\phi \in \mathcal{D}_1$, has the form

$$\begin{pmatrix} 1 & * \\ 0 & \lambda \end{pmatrix}$$

where $\lambda = \lim_{\rho_0 \to 0}(\phi^*\rho_0/\rho_0)$. For $\phi \in \mathcal{D}_2$, $D\phi$ is the same, except that the entry λ is 1. It follows that the quotient group is identified with the multiplicative group of functions $\lambda : \mathbb{R} \to \mathbb{R}$, acting in the $\partial/\partial\rho_0$ direction. Since $D\phi$ is non-singular, λ cannot vanish and hence $\lambda > 0$. ■

As in §4, let $\mathcal{E}_{AH}^{(2)} = E_{AH}/\mathcal{D}_2$ be the space of isometry classes of AH Einstein metrics, among diffeomorphisms in $\mathcal{D}_2$, and similarly, let $\mathcal{E}_{AH}^{(1)} = E_{AH}/\mathcal{D}_1$; here $E_{AH} = E_{AH}^{k,\beta}$, as in (4.16), (or (4.43)). There is a natural projection map $\mathcal{E}_{AH}^{(2)} \to \mathcal{E}_{AH}^{(1)}$ with fiber $\mathcal{D}_1/\mathcal{D}_2$. As in §4, $\mathcal{D}_1$ acts freely on E_{AH}, with local Bianchi slice as in (4.38) so that as following (4.38), $\mathcal{E}_{AH}^{(1)}$ is a C^∞ separable Banach manifold.

Next, let $\mathcal{C} = \mathcal{C}^{m,\alpha}$ be the space of conformal classes of $C^{m,\alpha}$ metrics on ∂M. Again, $\mathcal{C}$ has the structure of an infinite dimensional Banach manifold, with tangent spaces given by the space of trace-free symmetric bilinear forms. There is a natural projection map $Met^{m,\alpha}(\partial M) \to \mathcal{C}$, with fiber the space of $C^{m,\alpha}$ conformally equivalent metrics on ∂M.

Proposition 5.2. *The boundary map Π descends to a C^∞ boundary map on the base spaces, i.e.*

$$\Pi : \mathcal{E}_{AH}^{(1)} \to \mathcal{C}. \tag{5.1}$$

This map Π is Fredholm, of index 0, with Ker $D\Pi = K$, as in (4.42).

Proof: Let g_1 and g_2 be AH Einstein metrics on M with $\phi^*g_2 = g_1$, for $\phi \in \mathcal{D}_1$, and set $\lambda = \lim_{\rho_0 \to 0}(\phi^*\rho_0/\rho_0)$. Let $\bar{g}_i$ be the compactification of g_i, $i = 1, 2$, with respect to ρ_0, as in (4.1), and let γ_i be the induced boundary metrics. If $\hat{g}_2$ is the ρ_0-compactification of ϕ^*g_2, then one has

$$\hat{g}_2 = \rho_0^2\phi^*(\rho_0^{-2})\phi^*(\rho_0^2 g_2) = (\frac{\rho_0}{\phi^*(\rho_0)})^2\phi^*(\rho_0^2 g_2).$$

Hence, the boundary metric $\hat{\gamma}_2$ of $\hat{g}$, which must equal γ_1, is given by

$$\gamma_1 = \hat{\gamma}_2 = \lambda^{-2}\phi^*\gamma_2.$$

Since $\phi = id$ on ∂M, it follows that $\gamma_2 = \lambda^2\gamma_1$, so that the boundary metrics are conformal. It follows that the boundary map Π in (4.41) descends to the map Π in (5.1) and is smooth.

Further, observe that Lemma 5.1 shows that the converse of the proof above also holds, i.e. if γ_1 and γ_2 are conformally equivalent metrics in $Met^{m,\alpha}(\partial M)$, so that $\gamma_2 = \lambda^2\gamma_1$, then there is a diffeomorphism $\phi \in \mathcal{D}_1$ such that $\phi^*g_2 = g_1$, where g_i are any AH Einstein metrics on M with boundary metrics γ_i with respect to the ρ_0-compactification. Hence, Π maps the fibers $\mathcal{D}_1/\mathcal{D}_2$ diffeomorphically onto the fibers of $Met^{m,\alpha}(\partial M) \to \mathcal{C}$.

The proof that Π is Fredholm of index 0, with $KerD\Pi = K$, is thus exactly the same as in Proposition 4.3. ■

Remark 5.3. Recall that the map $\Pi : \mathcal{E}_{AH}^{(2)} \to Met(\partial M)$ in (4.41) depends on a choice of the defining function ρ_0 from (4.1). The reduced map Π in (5.1) is now independent of the choice of ρ_0. To see this, let ρ_1 be any other defining function, so that $\rho_1 = \lambda \cdot \rho_0$, for some function $\lambda > 0$ on M. Let

$$\bar{g} = \rho_0^2 g, \quad \text{and} \quad \widetilde{g} = \rho_1^2 g$$

be compactifications of g with respect to ρ_0 and ρ_1. The boundary metrics are related by $\widetilde{\gamma} = \lambda^2\gamma$, where $\lambda = \lim_{\rho\to 0}(\rho_1/\rho_0)$. As in the proof of Proposition 5.2. there is a diffeomorphism $\phi \in \mathcal{D}_1$ satisfying, (along integral curves of $\partial/\partial\rho_0$), $d\phi(\rho_0)/d\rho_0 = \lambda$ at ∂M. Hence

$$\widetilde{(\phi^* g)} = \rho_1^2 \phi^* g = \lambda^2 \rho_0^2 \phi^* g,$$

while

$$\phi^* \widetilde{g} = \phi^*(\rho_0^2)\phi^* g = \lambda^2 \rho_0^2 \phi^* g,$$

near ∂M. Thus, the ρ_1 compactification of $\phi^* g$ is the same as the ρ_0 compactification of g, pulled back by ϕ.

Theorem 1.2 is now essentially an immediate consequence of the work above and in §4.

Proof of Theorem 1.2.

The discussion following Lemma 5.1 shows that $\mathcal{E}_{AH}^{(1)} = \mathcal{E}_{AH}^{(1),k,\beta}$ and $\mathcal{C} = \mathcal{C}^{m,\alpha}$, are C^∞ smooth separable Banach manifolds and by Proposition 5.2, Π is a C^∞ Fredholm map of index 0. The boundary regularity result in Proposition 4.5, cf. (4.43), shows that by setting $(k, \beta) = (m, \alpha)$, $\mathcal{E}_{AH}^{(1)}$ is the space of AH Einstein metrics on M which admit a $C^{m,\alpha}$ compactification, with topology that of $\mathbb{S}_2^{m,\mu}(\bar{M})$, $\mu < \alpha$. ■

Remark 5.4. For certain purposes, it is useful to consider quotients by larger diffeomorphism groups, and we discuss this briefly here. Thus, let $\mathcal{D}_0$ be the group of ($C^{m+1,\alpha}$) diffeomorphisms of $\bar{M}$ such that the induced diffeomorphism on ∂M is isotopic to the identity. Again, the group $\mathcal{D}_1 \subset \mathcal{D}_0$ is a normal subgroup, and one may form

$$\mathcal{E}_{AH}^{(0)} = E_{AH}/\mathcal{D}_0 = \mathcal{E}_{AH}^{(1)}/(\mathcal{D}_0/\mathcal{D}_1). \tag{5.2}$$

Similarly, let $\mathcal{T}$ denote the quotient space $\mathcal{T} = \mathcal{C}/\mathcal{D}_0$. This is the space of *marked* conformal structures on ∂M, analogous to the Teichmüller space of conformal structures on surfaces. The group $\mathcal{D}_0$ however does not act freely on $\mathcal{C}$. Elements in $\mathcal{C}$ having a non-trivial isotropy group $\mathcal{D}_0[\gamma]$ are the classes $[\gamma]$ which have a non-trivial group of conformal diffeomorphisms, i.e. $\mathcal{D}_0[\gamma]$ consists of diffeomorphisms $\phi \in \mathcal{D}_0$ such that

$$\phi^*\gamma = \lambda^2 \cdot \gamma,$$

for some positive function λ on ∂M. A well-known theorem of Obata [24] implies that the isotropy group $\mathcal{D}_0[\gamma]$ of $[\gamma]$ is always compact, with the single exception of $(\partial M, [\gamma]) = (S^{n-1}, [\gamma_0])$, where γ_0 is the round metric on S^{n-1}.

Similarly, the elements g of E_{AH} which have non-trivial isotropy groups $\mathcal{D}_0(g)$ in $\mathcal{D}_0$ are AH Einstein metrics which have a non-trivial group of isometries. Such isometries $\phi \in \mathcal{D}_0$ induce a diffeomorphism ϕ of ∂M, which is a conformal isometry of the conformal infinity $[\gamma]$ of g. It follows that the boundary map Π in (5.1) descends further to a boundary map

$$\Pi : \mathcal{E}_{AH}^{(0)} \to \mathcal{T}. \tag{5.3}$$

At any class $[g]$ where $\mathcal{D}_0[g] = id$, the quotient space $\mathcal{E}_{AH}^{(0)}$ is a smooth infinite dimensional Banach manifold, and similarly for $\mathcal{T}$. At those classes $[g]$ or $[\gamma]$ where $\mathcal{D}_0[g]$ or $\mathcal{D}_0[\gamma]$ is compact, the quotients $\mathcal{E}_{AH}^{(0)}$ and $\mathcal{T}$ are smooth orbifolds, and Π is an orbifold smooth map. Only at the exceptional class (B^n, g_{-1}) of the Poincaré metric on the ball is the quotient $\mathcal{T}$ not well-behaved, and possibly non-Hausdorff.

Finally, one may carry out the same quotient construction with respect to the full group $\mathcal{D}$ $=$ Diff$(\bar{M})$ of diffeomorphisms of $\bar{M}$ mapping ∂M to itself, so that $\mathcal{E}_{AH} = E_{AH}/\mathcal{D}$, while $\mathcal{T}$ is replaced by the moduli space of conformal structures $\mathcal{M} = \mathcal{T}/\Gamma$, where $\Gamma = \mathcal{D}(\partial M)/\mathcal{D}_0(\partial M)$ is the subgroup of the mapping class group of ∂M consisting of diffeomorphisms of ∂M which extend to diffeomorphisms of M.

Next we discuss two versions of Theorem 1.2 in higher dimensions. Let M be an $(n+1)$ dimensional manifold with boundary, $n > 3$. When n is even, the Fefferman-Graham expansion (3.3) in general has log terms appearing at order n, i.e. of the form $t^n \log t$, and at higher order as well. Thus, one cannot expect a smooth boundary regularity result when dim M is odd. On the other hand, a result of Lee [20] gives boundary regularity below order n.

To describe the first version of Theorem 1.2, let E^2_{AH} be the space of AH Einstein metrics which are C^2 conformally compact, with respect to a smooth defining function ρ_0, as in (4.16). Suppose the boundary metric $\gamma \in C^{m,\alpha}$. Then Lee's result [20] states that any $g \in E^2_{AH}$ is $C^{k,\mu}$ conformally compact, where $k + \mu = \min(m + \alpha, n - 1 + \beta)$, for any $\beta \in (0, 1)$.

Combining this result with the results above in §5, and with Proposition 4.3 and the discussion preceding Proposition 4.5, gives the following:

Theorem 5.5. *Let M be a compact, oriented $(n+1)$-manifold with boundary ∂M, $n > 3$, with $\pi_1(M, \partial M) = 0$. If, for a given (m, α), with $3 \leq m \leq n-1$, $\hat{\mathcal{E}}_{AH} = \hat{\mathcal{E}}^{m,\alpha}_{AH}$ is non-empty, then $\hat{\mathcal{E}}_{AH}$ is a smooth infinite dimensional Banach manifold. Further, the boundary map*

$$\Pi : \hat{\mathcal{E}}_{AH} \to \mathcal{C} = \mathcal{C}^{m,\alpha} \tag{5.4}$$

is a C^∞ smooth Fredholm map of index 0.

■

Thus, the statement of Theorem 5.5 is equivalent to that of Theorem 1.2, provided $m \leq n - 1$. For the second version of Theorem 1.2, a result of Chruściel et al. [10] gives an optimal boundary regularity result for C^∞ boundary metrics γ. Thus, if g is an AH Einstein metric with a C^2 conformal compactification to a C^∞ boundary metric γ, then if n is odd, g is C^∞ conformally compact. If n is even, g is C^∞ polyhomogeneous, i.e. g has a compactification $\bar{g}$ which is a smooth function of $(t, t^n \log t, y)$, where $y \in \partial M$. In either case even/odd, let $\widetilde{\mathcal{E}}_{AH}$ be the space of such metrics, and $\mathcal{C}^\infty$ the space of C^∞ conformal classes.

The same proof as Theorem 5.5 gives:

Theorem 5.6. *Let M be a compact, oriented $(n+1)$-manifold with boundary ∂M with $\pi_1(M, \partial M) = 0$. If $\widetilde{\mathcal{E}}_{AH}$ is non-empty, then $\widetilde{\mathcal{E}}_{AH}$ is a smooth infinite dimensional Frechet manifold. Further, the boundary map*

$$\Pi : \widetilde{\mathcal{E}}_{AH} \to \mathcal{C} = \mathcal{C}^\infty \tag{5.5}$$

is a C^∞ smooth Fredholm map of index 0.

■

§5.2. In this section, we compare the structure of the spaces $\mathcal{E}^{m,\alpha}_{AH}$ over varying m, α. In dimension 4, the spaces $\mathcal{E}^{m,\alpha}_{AH}$ are defined as in (4.43), while in dimensions greater than 4, $\mathcal{E}^{m,\alpha}_{AH}$ is defined as preceding (5.4) with $3 \leq m \leq n-1$. (The spaces $\widetilde{\mathcal{E}}_{AH}$ are only defined for C^∞ boundary data).

Clearly, one has inclusions

$$\mathcal{E}^{\omega}_{AH} \subset \mathcal{E}^{\infty}_{AH} \subset \mathcal{E}^{m',\alpha'}_{AH} \subset \mathcal{E}^{m,\alpha}_{AH}, \tag{5.6}$$

for any (m', α') with $m' + \alpha' > m + \alpha$. Here we recall that the topology on $\mathcal{E}_{AH}^{m,\alpha}$ is that induced by the $C^{m,\mu}$ topology on $\mathbb{S}_2^{m,\alpha}(\bar{M})$, for a fixed $\mu < \alpha$, cf. the discussion preceding Proposition 4.4.

The inclusions (5.6) correspond formally to the much simpler inclusions of the conformal classes $\mathcal{C}^{\omega} \subset \mathcal{C}^{\infty} \subset \mathcal{C}^{m',\alpha'} \subset \mathcal{C}^{m,\alpha}$ of conformal classes of metrics on ∂M. It is essentially clear that the spaces $\mathcal{C}^{m,\alpha}$ are diffeomorphic, for all (m, α), including $m = \infty$ or $m = \omega$. Further, each $\mathcal{C}^{m',\alpha'}$ is dense in $\mathcal{C}^{m,\alpha}$.

For later purposes, it is worthwhile to verify these claims explicitly. With respect to a fixed real-analytic atlas for ∂M, metrics in $Met^{m,\alpha}(\partial M)$ are given by a collection of $C^{m,\alpha}$ functions $g_{ij} : U \to \mathbb{R}$, where U is an open set in $\mathbb{R}^n$. Hence the topology on $Met^{m,\alpha}(\partial M)$ is determined by the standard $C^{m,\alpha}$ topology on $C^{m,\alpha}(U, \mathbb{R})$. These local spaces are all diffeomorphic in a natural sense, as (m, α) vary and induce diffeomorphisms of the global spaces $Met^{m,\alpha}(\partial M)$. This argument also holds when passing to the associated spaces $\mathcal{C}^{m,\alpha}$ of conformal classes. The fact that $\mathcal{C}^{m,\alpha}$ is dense in $\mathcal{C}^{m',\alpha'}$ also follows from the fact that the local spaces $C^{m',\alpha'}(U, \mathbb{R})$ are dense in $C^{m,\alpha}(U, \mathbb{R})$.

Theorem 5.7. *For any (m, α), with $m \geq 3$ and including $m = \infty$ and $m = \omega$, the spaces $\mathcal{E}_{AH}^{m,\alpha}$ are all diffeomorphic. Further $\mathcal{E}_{AH}^{\omega}$, and hence $\mathcal{E}_{AH}^{m',\alpha'}$, is dense in $\mathcal{E}_{AH}^{m,\alpha}$ so that if $\overline{\mathcal{E}_{AH}^{\omega}}$ denotes the completion of $\mathcal{E}_{AH}^{\omega}$ in $\mathcal{E}_{AH}^{m,\alpha}$, then*

$$\overline{\mathcal{E}_{AH}^{\omega}} = \mathcal{E}_{AH}^{m,\alpha}. \tag{5.7}$$

Proof: It suffices to work with the spaces $\mathcal{E}_{AH}^{(2),m,\alpha}$ in (4.39) and $Met^{m,\alpha}(\partial M)$, using a fixed C^{ω} defining function ρ_0 as in §4. In the following, we will drop the superscript (2) from the notation. Suppose first that $g \in \mathcal{E}_{AH}^{m,\alpha}$ is a regular point of Π, so that $D\Pi_g$ is an isomorphism. The inverse function theorem implies that there are neighborhoods $\mathcal{U}$ of g in $\mathcal{E}_{AH}^{m,\alpha}$ and $\mathcal{V}$ of $\gamma = \Pi(g)$ in $Met^{m,\alpha}(\partial M)$ such that $\Pi : \mathcal{U} \to \mathcal{V}$ is a diffeomorphism. Since $\mathcal{V}$ is an open set in a Banach space, $\Pi|_{\mathcal{U}}$ is a chart for $\mathcal{E}_{AH}^{m,\alpha}$. It follows that $\mathcal{V}^{m',\alpha'} = Met^{m',\alpha'}(\partial M) \cap \mathcal{V} \subset Im\Pi$ and by boundary regularity that $\Pi^{-1}(\mathcal{V}^{m',\alpha'}) \cap \mathcal{U} = \mathcal{U}^{m',\alpha'}$ is an open set in $\mathcal{E}_{AH}^{m',\alpha'}$. Hence Π induces a chart for $\mathcal{E}_{AH}^{m',\alpha'}$ and so $\mathcal{E}_{AH}^{m',\alpha'}$ is locally diffeomorphic to $\mathcal{E}_{AH}^{m,\alpha}$.

Next suppose that g is a singular point of Π, and let $K = KerD\Pi_g \subset T_g Met_{AH}^{m,\alpha}(M)$, with $H = (CokerD\Pi_g)^{\perp} \subset T_{\Pi(g)} Met_{AH}^{m,\alpha}(\partial M)$, where the orthogonal complement is taken with respect to the L^2 inner product. By the implicit function theorem, i.e. Theorem 4.1, a neighborhood $\mathcal{U}$ of g in $\mathcal{E}_{AH}^{m,\alpha}$ may be written as a graph over a domain $\mathcal{V}$ in $K \oplus H$. This gives a local chart on $\mathcal{U}$ and for the same reasons as above, one thus obtains a local chart structure for the open set $\mathcal{U}^{m',\alpha'} = \mathcal{E}_{AH}^{m',\alpha'} \cap \mathcal{U} \subset \mathcal{E}_{AH}^{m',\alpha'}$.

These local chart structures patch together to give the spaces $\mathcal{E}_{AH}^{m,\alpha}$ the Banach manifold structure. Since the local charts for $\mathcal{E}_{AH}^{m',\alpha'}$ are just those obtained by restricting the charts of $\mathcal{E}_{AH}^{m,\alpha}$ to subdomains, it follows that the spaces $\mathcal{E}_{AH}^{m,\alpha}$ are all diffeomorphic. Similarly, (5.7) follows from the density of the corresponding local charts, i.e. the density of $C^{\omega}(U, \mathbb{R})$ in $C^{m,\alpha}(U, \mathbb{R})$.

■

References

[1] S. Agmon, A. Douglis and L. Nirenberg, Estimates near the boundary for the solutions of elliptic partial differential equations satisfying general boundary values, I, II, Comm. Pure Appl. Math. **12**, (1959), 623-727 and **17**, (1964), 35-92.

[2] M. Anderson, Boundary regularity, uniqueness and non-uniqueness for AH Einstein metrics on 4-manifolds, Adv. in Math., **179**, (2003), 205-249.

[3] M. Anderson, A. Katsuda, Y. Kurylev, M. Lassas and M. Taylor, Boundary regularity for the Ricci equation, geometric convergence, and Gel'fand's inverse boundary problem, Inventiones Math., **158**, (2004), 261-321.

[4] M. Anderson, On the structure of asymptotically de Sitter and anti-de Sitter spaces, Adv. Theor. Math. Phys., **8**, (2005), 861-894.

[5] M. Anderson, Einstein metrics with prescribed conformal infinity on 4-manifolds, Geom. & Funct. Analysis, **18**, (2008), 305-366.
[6] M. Anderson and M. Herzlich, Unique continuation results for Ricci curvature and applications, Jour. Geom. Physics, **58**, (2008), 179-207.
[7] A. Besse, Einstein Manifolds, Ergebnisse Series, 3:10, Springer Verlag, New York, (1987).
[8] O. Biquard, Metriques d'Einstein asymptotiquement symmetriques, Asterisque, **265**, (2000).
[9] R. Böhme and A.J. Tromba, The index theorem for classical minimal surfaces, Annals of Math. **113**, (1981), 447-499.
[10] P. T. Chruściel, E. Delay, J.M. Lee and D.N. Skinner, Boundary regularity of conformally compact Einstein metrics, Jour. Diff. Geom., **69**, (2005), 111-136.
[11] S. de Haro, K. Skenderis and S.N. Solodukhin, Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence, Comm. Math. Phys. **217**, (2001), 595-622.
[12] C. Fefferman and C. R. Graham, Conformal invariants, in Elie Cartan et les Mathematiques d'Aujourd'hui, Asterisque, 1985, Numero Hors Serie, Soc. Math. France, Paris, 95-116.
[13] H. Friedrich, Existence and structure of past asymptotically simple solutions of Einstein's field equations with positive cosmological constant, Jour. Geom. Physics, **3**, (1986), 101-117.
[14] D. Gilbarg and N. Trudinger, Elliptic Partial Differential Equations of Second Order, Second Edition, Gpringer Verlag, New York, (1983).
[15] C. R. Graham and J.M. Lee, Einstein metrics with prescribed conformal infinity on the ball, Adv. in Math. **87**, (1991), 186-225.
[16] D. Helliwell, Boundary regularity for conformally compact Einstein metrics in even dimensions, Comm. PDE, **33**, (2008), 842-880.
[17] F. John, Partial Differential Equations, Appl. Math. Sci. vol. 1, Springer Verlag, New York, (1975)
[18] S. Lang, Differential Manifolds, Springer Verlag, New York, (1985).
[19] C. LeBrun, $\mathcal{H}$-space with a cosmological constant, Proc. Royal Soc. London, Ser. A, **380**, (1982), 171-185.
[20] J.M. Lee, Fredholm operators and Einstein metrics on conformally compact manifolds, Memoirs Amer. Math. Soc., **183**, (2006), No. 864.
[21] M. Li, The Yamabe problem with Dirichlet data, C.R. Acad. Sci. Paris, Ser. 1, **320**, (1995), 709-712.
[22] R. Mazzeo and F. Pacard, Maskit combinations of Poincaré-Einstein metrics, Adv. in Math., **204**, (2006), 379-412.
[23] C. B. Morrey, Jr., Multiple Integrals in the Calculus of Variations, Grundlehren Series, vol. 130, Springer Verlag, Berlin, 1966.
[24] M. Obata, The conjectures on conformal transformations of Riemannian manifolds, Jour. Diff. Geom, **6**, (1971), 247-258.
[25] A. Rendall, Asymptotics of solutions of the Einstein equations with positive cosmological constant, Annales Henri Poincaré, **5**, (2004), 1041-1064.
[26] B. White, The space of m-dimensional surfaces that are stationary for a parametric elliptic functional, Ind. Univ. Math. Jour. **36**, (1987), 567-603.
[27] B. White, The space of minimal submanifolds for varying Riemannian metrics, Ind. Univ. Math. Jour. **40**, (1991), 161-200.

February 2008/October 2009

Department of Mathematics
S.U.N.Y. at Stony Brook
Stony Brook, NY 11794-3651
anderson@math.sunysb.edu