ON THE UNIQUENESS AND GLOBAL DYNAMICS OF ADS SPACETIMES

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Abstract. We study global aspects of complete, non-singular asymptotically locally AdS spacetimes solving the vacuum Einstein equations whose conformal infinity is an arbitrary globally stationary spacetime. It is proved that any such solution which is asymptotically stationary to the past and future is itself globally stationary.

This gives certain rigidity or uniqueness results for exact AdS and related spacetimes.

1. Introduction

Consider geodesically complete, asymptotically simple solutions of the vacuum Einstein equations with negative cosmological constant $\Lambda < 0$ in $(n+1)$ dimensions. Up to rescaling, these are given by complete, (non-singular), metrics g, defined on manifolds of the form $M^{n+1} = \mathbb{R} \times \Sigma$, and satisfying the Einstein equations

$$Ric_g = -ng. \tag{1.1}$$

The metric g has a conformal completion, at least C^2, with conformal boundary $(\mathcal{I}, [\gamma])$, where γ is a complete Lorentz metric on $\mathcal{I}$. Topologically, conformal infinity $\mathcal{I}$ is of the form $\mathbb{R} \times \partial\Sigma$.

The canonical example is the (exact) anti-de Sitter spacetime g_{AdS}, which may be represented globally in static form as

$$g_{AdS} = -\cosh^2 r\, dt^2 + dr^2 + \sinh^2 r\, g_{S^{n-1}(1)}, \tag{1.2}$$

where $g_{S^{n-1}(1)}$ is the round metric of radius 1 on the sphere S^{n-1}. Here $M = \mathbb{R} \times \mathbb{R}^n$, with conformal infinity $\mathcal{I} = \mathbb{R} \times S^{n-1}$, with boundary metric $\gamma_0 = -dt^2 + g_{S^{n-1}(1)}$ the Einstein static cylinder. Asymptotically simple spacetimes approximate the metric g_{AdS} locally on approach to $\mathcal{I}$, and so are often also called asymptotically locally AdS spacetimes.

It is generally believed that anti-de Sitter spacetime should have an infinite dimensional space of dynamical perturbations, i.e. time-dependent, complete vacuum solutions (1.1), which have the same conformal infinity $(\mathcal{I}, \gamma_0)$ as exact AdS and which are globally close to g_{AdS}. In other words, g_{AdS} is dynamically stable. This is certainly the case at the linearized level; one may globally solve the linearized Einstein equations at g_{AdS}, with zero boundary data on $\mathcal{I}$ and arbitrary smooth Cauchy data on Σ. These linearized solutions, (or normalizable modes), remain uniformly bounded in time, cf. [1] for a detailed treatment.

Such dynamical or global stability results are well-known in the context of spacetimes with zero cosmological constant, $\Lambda = 0$. Thus, the celebrated work of Christodoulou-Klainerman [2] shows that there exist global non-singular perturbations of Minkowski spacetime, (in 3+1 dimensions), which tend to the flat Minkowski spacetime as $t \to \pm\infty$; see also [3] and the more recent work [4], as well as [5], which gives the existence of non-singular asymptotically simple global perturbations. There are also analogues of such stability results in the context of cosmological spacetimes, (in the expanding direction), where Σ is compact without boundary, cf. [6], [7].

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Similarly, in the context where $\Lambda > 0$, Friedrich [8] has proved that exact de Sitter spacetime is globally stable in a natural sense in $3 + 1$ dimensions; the same result holds in fact at least in all even dimensions, cf. [9].

In this paper, we discuss some aspects of the global dynamics of complete, asymptotically simple solutions of the Einstein equations when $\Lambda < 0$. To describe the main results, let γ be a fixed but arbitrary globally stationary metric on $\mathcal{I} \simeq \mathbb{R} \times \partial\Sigma$, with $\partial\Sigma$ compact. Let $\mathcal{E} = \mathcal{E}(\gamma)$ be the space of all geodesically complete, asymptotically simple solutions of the Einstein equations (1.1) with conformal infinity $(\mathcal{I}, \gamma)$; (this definition will be made more precise in §2). Choose a fixed diffeomorphism $M \simeq \mathbb{R} \times \Sigma$, (i.e. spacetime decomposition), and let $\mathcal{C}$ denote the corresponding space of solutions of the constraint equations on (Σ, g, K) given by solutions in $\mathcal{E}$. In particular, a metric $g \in E$ gives rise to a curve (g_t, K_t), $t \in (-\infty, \infty)$ in $\mathcal{C}$.

Theorem 1.1. *Suppose $g \in \mathcal{E}$ is weakly asymptotically stationary in the sense that there exist times $t_i \to \infty$ and $t_i \to -\infty$ such that (g_{t_i}, K_{t_i}) converges to a globally stationary solution of the vacuum constraint equations, as $i \to \infty$. Suppose further that, modulo infinitesimal diffeomorphisms, solutions of the linearized Einstein equations at g satisfy the unique continuation property at $\mathcal{I}$.*

Then (M^{n+1}, g) is globally stationary.

As explained in detail in §2, the unique continuation property at $\mathcal{I}$ means that solutions of the linearized Einstein equations are uniquely determined, up to infinitesimal diffeomorphisms, by their Cauchy data, (*not* just the boundary data), at $\mathcal{I}$. Equivalently, they are uniquely determined, up to infinitesimal diffeomorphism, by their formal series expansion at $\mathcal{I}$. This is automatically the case for solutions of the linearized equations which are analytic, (in the polyhomogeneous sense), at $\mathcal{I}$. The unique continuation property was proved to hold in general for Riemannian-Einstein metrics in [10], and we conjecture that it also always holds for Lorentzian-Einstein metrics. There certainly appears to be no physical reasons to doubt the validity of this property.

The time evolution of metrics $g \in E$ gives rise to a flow, (i.e. dynamical system), on the constraint space $\mathcal{C}$. Stationary solutions in $\mathcal{E}$ then correspond to fixed points of this flow, (for a suitable choice of spacetime foliation of M). Roughly speaking, Theorem 1.1 then states that there are no orbits of the flow on $\mathcal{C}$ which are weakly asymptotic to a fixed point at both ends, except fixed orbits.

It is possible that $(\mathcal{I}, \gamma)$ has more than one timelike Killing field, (modulo constants). This will occur when there are non-trivial spatial Killing fields on $(\mathcal{I}, \gamma)$. Theorem 1.1 applies to each Killing field and leads easily to the following corollary.

Corollary 1.2. *Under the assumptions in Theorem 1.1, suppose $(\mathcal{I}, \gamma)$ is the Einstein static cylinder and that (M^{n+1}, g) is weakly asymptotic to the exact AdS spacetime to the infinite past and infinite future. Then (M^{n+1}, g) is globally isometric to the exact AdS spacetime.*

Similar rigidity or uniqueness results hold for other spacetimes which, for example, have sufficient symmetry at conformal infinity $(\mathcal{I}, \gamma)$. Thus, the Horowitz-Myers AdS soliton [11] is unique in the sense above, cf. §3. The same applies to the AdS soliton metric of Copsey-Horowitz [12]. Previously, the uniqueness of the Horowitz-Myers soliton metric was only known within the much more restrictive class of static solutions of the Einstein equations, cf. [13], [14].

A more general result on the extension of isometries from the boundary to the bulk is proved in [10] for Riemannian (or Euclidean) Einstein metrics (1.1), by completely different methods.

The proofs of the results above are conceptually very simple. They follow from basic conservation properties of the holographic stress-energy tensor and holographic mass arising in the AdS/CFT correspondence, [15], [16], [17], [18], [19], together with a basic identity discussed in §3, (cf. (3.1)). For simplicity, we have restricted the analysis to vacuum solutions of the Einstein equations (1.1) with negative cosmological constant. Of course the results above also hold in the presence of matter terms which arise from a vacuum solution in higher dimensions via Kaluza-Klein reduction. However, in general the presence of matter terms changes the conservation properties of the stress-energy tensor [17]. In any case, we hope to discuss the situation with matter terms elsewhere.

As stated, these results do not hold for spacetimes containing black holes, or for any spacetimes containing singularities which propagate to $\mathcal{I}$. As a specific example, it is well-known that the AdS-Kerr spacetime has conformal infinity $\mathcal{I}$ given by a finite-time region $I \times S^{n-1}$ of the Einstein cylinder $\mathbb{R} \times S^{n-1}$. Not all symmetries of S^{n-1} extend to symmetries of the AdS-Kerr solution, so there are timelike Killing fields on $\mathcal{I}$ which do not extend to Killing fields on the Kerr-AdS solution. More generally, smooth global initial data on a space-like slice Σ may evolve to the future (or past) and eventually form a black hole, for example the AdS Kerr metric. Such solutions will be asymptotically stationary to the future (or past) near $\mathcal{I}$ within their domain of outer communication, but are not themselves globally stationary; see also Remark 3.1.

The contents of the paper are briefly as follows. In §2, we present some basic background material and elementary results needed to establish the main results. These results are then proved in §3. In §4, we conclude with further discussion and interpretation of the results.

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2. Background Material.

Let Σ be a compact n-manifold with boundary $\partial\Sigma$ and $M = \mathbb{R} \times \Sigma$, where $\mathbb{R}$ represents the time direction. We consider geodesically complete solutions (M, g) of the vacuum Einstein equations (1.1) which are asymptotically simple. Asymptotic simplicity is equivalent to the existence of a (reasonably smooth) conformal completion. Thus, let $\widetilde{\rho}$ be a defining function for the boundary $\partial M = \mathcal{I} = \mathbb{R} \times \partial\Sigma$, i.e. $\widetilde{\rho}$ is a coordinate function for ∂M in M, vanishing on ∂M. One then requires that the conformally equivalent (unphysical) metric

$$\widetilde{g} = \widetilde{\rho}^2 g \tag{2.1}$$

extends at least C^2 to $\mathcal{I}$; (a stronger smoothness condition will be required below). It is easy to see that this implies that the sectional curvatures K of g satisfy $|K+1| = O(\widetilde{\rho}^2)$, so that the metric g locally approaches the AdS metric to order $O(\widetilde{\rho}^2)$ near $\mathcal{I}$; for this reason, asymptotically simple spacetimes are also often called asymptotically locally AdS. The space $\mathcal{I}$ is called conformal infinity.

Let

$$\gamma = \widetilde{g}|_{\mathcal{I}} \tag{2.2}$$

be the metric induced on $\mathcal{I}$. It is well-known that causality arguments imply that γ is a Lorentz metric on $\mathcal{I}$. Different choices of defining function $\widetilde{\rho}$ lead to conformally equivalent metrics γ on $\mathcal{I}$, so that only the conformal structure $[\gamma]$ on $\mathcal{I}$ is uniquely determined by (M, g).

If (M, g) is asymptotically simple, then it is standard, (and easily seen), that each choice of boundary metric $\gamma \in [\gamma]$ determines a unique geodesic defining function ρ, for which the integral curves of $\bar{\nabla}\rho$ in the compactified metric

$$\bar{g} = \rho^2 g \tag{2.3}$$

are (spacelike) geodesics orthogonal to $\mathcal{I}$. In the following, we work only with such geodesic compactifications. (The integral curves of $\nabla \log\rho$ are also geodesics with respect to the metric g).

The metric g splits in the ρ-direction, so that

$$g = \rho^{-2}(d\rho^2 + g_\rho), \tag{2.4}$$

where g_ρ is a curve of Lorentz metrics on the level sets $S(\rho)$ of ρ. As $\rho \to 0$, $g_\rho \to \gamma \equiv g_{(0)}$. The Fefferman-Graham expansion [20] is the expansion of the curve g_ρ in a Taylor-type series in ρ. The exact form of the expansion depends on whether n is even or odd. If n is odd, then one has

$$g_\rho \sim g_{(0)} + \rho^2 g_{(2)} + \cdots + \rho^{n-1} g_{(n-1)} + \rho^n g_{(n)} + \cdots \tag{2.5}$$

while if n is even,

$$g_\rho \sim g_{(0)} + \rho^2 g_{(2)} + \cdots + \rho^{n-1} g_{(n-1)} + \rho^n \log\rho\,\mathcal{H} + \rho^n g_{(n)} + \cdots \tag{2.6}$$

Below order n, the expansions are even in powers of ρ, and all coefficients $g_{(2k)}$, $2k < n$, as well as the coefficient $\mathcal{H}$ in (2.6) are determined by the boundary metric $\gamma = g_{(0)}$ and its derivatives up to order $2k$, (respectively n); thus they do not depend on the particular bulk Einstein metric g. The series (2.5) is a formal power series, in powers of ρ, while the series (2.6) is a series in powers of ρ and $\log\,\rho$, (i.e. a polyhomogeneous series).

The coefficient $g_{(n)}$ is formally undetermined; thus it is not determined by γ, and depends on the bulk metric g. All the remaining (higher order) coefficients in the series (2.5)-(2.6) are completely determined by the data

$$(g_{(0)}, g_{(n)}), \tag{2.7}$$

so that these terms determine the formal expansion of the metric $\bar{g}$, and hence g, near $\mathcal{I}$. Note that the terms in the expansion (2.5)-(2.6) depend on the choice of boundary metric $\gamma \in [\gamma]$. A conformal change of γ will cause a change in the geodesic defining function ρ, and hence a change in the coefficients. Transformation formulae for these coefficients are given in [17], [21] for instance.

The boundary metric $g_{(0)}$ is (formally) free, in the sense that the Einstein equations in a neighborhood of $\mathcal{I}$ impose no conditions or constraints on $g_{(0)}$. Similarly, the transverse-traceless part of $g_{(n)}$ is formally free; however, the Einstein equations do impose constraints on the divergence and trace of $g_{(n)}$. Thus

$$\delta g_{(n)} = r_{(n)}, \quad \text{and} \quad tr g_{(n)} = s_{(n)}, \tag{2.8}$$

where the divergence δ and trace are taken with respect to $g_{(0)}$. The terms $r_{(n)}$ and $s_{(n)}$ may be explicitly computed from the boundary metric $g_{(0)}$ and its derivatives up to order n, cf. [17]. When n is odd, one has $r_{(n)} = s_{(n)} = 0$. We will call the equations (2.8) the constraint equations on $\mathcal{I}$. They arise from the Gauss and Gauss-Codazzi equations on $\mathcal{I}$ in the geodesic gauge (2.4).

Throughout the following, we assume that the metric g is asymptotically simple in the sense that the expansion (2.5) or (2.6) exists to order n, so that

$$g_\rho = g_{(0)} + \rho^2 g_{(2)} + \cdots + \rho^{n-1} g_{(n-1)} + \rho^n g_{(n)} + o(\rho^n), \tag{2.9}$$

or,

$$g_\rho = g_{(0)} + \rho^2 g_{(2)} + \cdots + \rho^{n-1} g_{(n-1)} + \rho^n \log\rho\,\mathcal{H} + \rho^n g_{(n)} + o(\rho^n), \tag{2.10}$$

where $o(\rho^n)/\rho^n \to 0$ as $\rho \to 0$.

We point out that if the free data $(g_{(0)}, g_{(n)})$ are real-analytic and satisfy the constraints (2.8), then a result of Kichenassamy [22] shows that the (formal) series expansion (2.5) or (2.6) converges to an actual solution g of the Einstein equations near $\mathcal{I}$. Of course, since the series is uniquely determined by the data (2.7), this solution is unique. (It is mathematically an open question whether there could exist other solutions with the same data (2.7) which don't have convergent expansions; this seems very unlikely however).

Throughout most of the paper, we only consider boundary metrics $\gamma = g_{(0)}$ which are globally stationary on $\mathcal{I}$. Thus, there is a complete, timelike Killing field Z, generating a free isometric $\mathbb{R}$-action on $(\mathcal{I}, \gamma)$, so that

$$\mathcal{L}_Z \gamma = 0. \tag{2.11}$$

Let $\pi : \mathcal{I} \to S$ be the projection onto the orbit space of the $\mathbb{R}$-action. The $\mathbb{R}$-bundle π is trivial, $M \simeq \mathbb{R} \times S$, and with respect to a fixed trivialization determined by a global time function t on $\mathcal{I}$, the metric γ may be written globally in the form

$$\gamma = -N^2(dt + \theta)^2 + \pi^* g_S, \tag{2.12}$$

where θ is a connection 1-form on the bundle π, $N^2 = -\gamma(Z, Z) > 0$ and g_S in the Riemannian metric induced on the orbit space S by γ.

We require that γ is $C^{n+1,\alpha}$ smooth, but otherwise impose no other conditions on γ; it need not satisfy any equations or have any other symmetries. If γ does have other symmetries, i.e. the metric g_S also admits Killing fields, the timelike Killing field Z will not be unique. The results to follow hold for any fixed choice of Z.

It follows that if g is an asymptotically simple solution of the Einstein equations with boundary metric γ, then the determined coefficients in (2.5)-(2.6), i.e. $g_{(k)}$ with $k \leq n-1$, and the logarithmic coefficient $\mathcal{H}$, are also invariant under the isometric action generated by Z,

$$\mathcal{L}_Z g_{(k)} = 0, \quad \mathcal{L}_Z \mathcal{H} = 0. \tag{2.13}$$

Definition 2.1. The space $\mathcal{E} = \mathcal{E}_\gamma$ is the space of all geodesically complete, asymptotically simple solutions of the vacuum Einstein equations which have the fixed stationary metric γ in (2.12) as boundary metric and for which the expansions (2.9)-(2.10) hold to order $n + 1 + \alpha$, $\alpha \in (0, 1)$. One may define a natural $\mathcal{C}^{n+1,\alpha}$ polyhomogeneous topology on $\mathcal{E}$.

Let t be a fixed smooth global time function on M, which restricts to the time function t above on $\mathcal{I}$. This gives a trivialization $M = \mathbb{R} \times \Sigma$, with fibers Σ_t given by the level sets of t. With respect to this foliation Σ_t, the metric g may be written in local coordinates (t, x^i) as

$$g = -u^2 dt^2 + g_{ij}(dx^i + \xi^i dt)(dx^j + \xi^j dt), \tag{2.14}$$

where (N, ξ) is the lapse-shift vector with respect to ∂_t. Since the boundary metric of g with respect to ρ is γ, one has $u\rho \to N$, for N as in (2.12), as $\rho \to 0$.

Let $\mathcal{C}$ be the space of solutions of the constraint equations on a given Σ, say Σ_0 as above, induced by the global metrics $g \in \mathcal{E}$. Thus, an element in $\mathcal{C}$ is a triple (Σ, g_0, K), where g_0 is a complete, conformally compact metric on Σ, and K is a symmetric bilinear form, (the 2nd fundamental form), with $K = g_0 + O(\rho^2)$. The data (g_0, K) satisfy the vacuum constraint equations

$$R - |K|^2 + \kappa^2 = n(n-1), \tag{2.15}$$

$$\delta K + d\kappa = 0,$$

where $\kappa = trK$ is the mean curvature.

Note that $\mathcal{C}$ does not consist of all solutions of the constraint equations (2.15); many solutions of (2.15) will only give rise to local-in-time solutions of the vacuum equations (1.1). The space $\mathcal{C}$ may be given a $\mathcal{C}^{n+1,\alpha} \times \mathcal{C}^{n,\alpha}$ polyhomogeneous topology induced from the topology of $\mathcal{E}$.

Given the spacelike foliation Σ_t above, let g_t be the metric induced by $g \in \mathcal{E}$ on Σ_t. Choosing the zero-shift gauge gives a diffeomorphism $\phi_t : \Sigma = \Sigma_0 \to \Sigma_t$, and we will let g_t also denote the induced metric $\phi_t^*(g_t)$ on Σ. Thus, the Einstein flow from Σ to any Σ_t defines a flow, denoted

$$g_0 \to g_t, \; K_0 \to K_t \tag{2.16}$$

on the constraint space $\mathcal{C}$.

Next we discuss briefly the linearized Einstein equations. The linearization of the vacuum equations (1.1) at $g \in \mathcal{E}$ is given by

$$D^*Dh - 2Rh + \delta^*\delta\bar{h} = 0. \tag{2.17}$$

Here h is a symmetric bilinear form, $\bar{h} = h - \frac{1}{2}trh\, g$, $D^*D = trD^2$ is the wave operator and R is the curvature tensor acting on symmetric bilinear forms; all metric quantities in (2.17) are with respect to the background metric g. Altering h by infinitesimal diffeomorphisms $h \to h + \delta^*X$, it is well-known that one can solve (2.17) in the transverse gauge $\delta\bar{h} = 0$, by solving the coupled system

$$D^*D(h + \delta^*X) - 2R(h + \delta^*X) = 0, \tag{2.18}$$

$$\delta(\bar{h} + \bar{\delta}^*X) = 0, \tag{2.19}$$

in the variables (h, X), cf. [23].

Consider the initial boundary value problem for (2.18):

$$D^*Dh - 2Rh = 0,\ h|_{\Sigma} = h_0,\ \nabla_{\partial_t}h = h_1,\ \text{and}\ h|_{\mathcal{I}} = h_{(0)}, \tag{2.20}$$

where the boundary data $h_{(0)} \in C^\infty(\mathcal{I})$ and the initial data h_0, h_1 are C^∞ polyhomogeneous on $\Sigma = \Sigma_0$ up to the boundary. We assume h_0, h_1 and $h_{(0)}$ match in a smooth polyhomogeneous sense at the corner $\partial\Sigma$. In this generality, it is not known if there is a global C^∞ polyhomogeneous solution h of (2.20) defined on all of M.[1] However, it suffices for our purposes, (namely regarding the unique continuation property), that one has a smooth solution in the interior of the domain of dependence $D(\Sigma) \subset M$, which is $C^{n+1,\alpha}$ polyhomogeneous smooth at the boundary $\partial\Sigma \subset \mathcal{I}$. The existence of such solutions follows from standard energy estimates, cf. [24] for instance. Such energy estimates are carried out in the Sobolev spaces H^s; note that Sobolev embedding gives $C^{n+1,\alpha} \subset H^s$, for $s > \frac{3}{2}n + 2$. In fact, it suffices for our purposes to know that there exist solutions to the linearization of the constraint equations (2.15) which are $C^{n+1,\alpha}$ polyhomogeneous at $\partial\Sigma$; this follows from the work of [25].

A standard identity (Weitzenbock formula) gives $\delta\bar{\delta}^*X = \frac{1}{2}D^*DX - RicX$, so that (2.19) is equivalent to the vector system

$$\tfrac{1}{2}D^*DX - RicX = -\delta\bar{h}, \tag{2.21}$$

which is of the same form as (2.20). Thus, as above, one may solve the initial boundary value problem for (2.21) within $D(\Sigma)$.

The symmetric bilinear forms h satisfying

$$D^*Dh - 2Rh = 0, \tag{2.22}$$

$$\delta\bar{h} = 0,$$

which satisfy $h_{(0)} = 0$ on $\mathcal{I}$ may be viewed as defining the formal tangent space $T_g\mathcal{E}$ to $\mathcal{E}$, (modulo diffeomorphisms). However, this is formal; no claim is made that $\mathcal{E}$ is a manifold, corresponding to the linearization stability of the Einstein equations within the space $\mathcal{E}$. Moreover, solutions of (2.22) with

$$h_{(0)} \neq 0 \text{ on } \mathcal{I},$$

are certainly not in the formal tangent space $T\mathcal{E}$, since these deformations don't preserve the boundary metric.

We now define precisely the unique continuation property (iii) of Theorem 1.1. This is the statement that, for $g \in \mathcal{E}$, any solution h of the linearized Einstein equations (2.22) which vanishes

[1] The logarithmic terms in the expansion of h at $\mathcal{I}$ may propagate into the bulk of M and could, apriori, lead to singularities of the solution in the bulk. I am grateful to P. Chruściel for discussions on this point.

to infinite order at $\mathcal{I}$ is necessarily zero. For solutions h of (2.22) which are C^∞ polyhomogeneous up to $\mathcal{I}$, this is clearly equivalent to the statement that if h has zero Cauchy data on $\mathcal{I}$, i.e. if

$$h_{(0)} = 0 \text{ and } h_{(n)} = 0, \tag{2.23}$$

then

$$h \equiv 0. \tag{2.24}$$

This follows from the properties of expansions the (2.5)-(2.6); the conditions (2.23) imply that the formal series solution of (2.22) vanishes. Note that it suffices to have the unique continuation property at a cut $\partial\Sigma$ of $\mathcal{I}$, i.e. within the domain of dependence $D(\Sigma)$. For if h vanishes to 1st order on Σ and $h_{(0)} = 0$, then $h \equiv 0$, by uniqueness of solutions to the initial boundary value problem (2.22).

Next we return to the expansions (2.5)-(2.6). The undetermined term $g_{(n)}$ is closely related to the stress-energy tensor on $\mathcal{I}$, and is an important feature of the AdS/CFT correspondence. Thus, as shown in [16], [18], there is a symmetric bilinear form $r_{(n)}$, determined by the boundary metric γ and its derivatives up to order n, such that the form

$$\tau_{(n)} = g_{(n)} - r_{(n)}, \tag{2.25}$$

is divergence-free with determined trace, i.e.

$$\delta_\gamma \tau_{(n)} = 0, \quad tr_\gamma \tau_{(n)} = a_{(n)}. \tag{2.26}$$

The term $\tau_{(n)}$ is obtained by a covariant (intrinsic) renormalization of the Brown-York quasi-local stress-energy tensor. Via the AdS/CFT correspondence, $\tau_{(n)}$ corresponds to the expectation value of the stress-energy tensor of the QFT dual to (M, g). The term $a_{(n)}$ is proportional to the conformal anomaly [26], and is determined by the boundary metric γ. When n is odd, $r_{(n)} = 0$, so $\tau_{(n)} = g_{(n)}$ and $a_{(n)} = 0$.

It is important to note that the construction of $\tau_{(n)}$ is background-independent; there is no normalization with respect to a background "standard" solution. Indeed, there are no such standard background solutions to which (M, g) can be compared in the generality of the current discussion. For instance, apriori, there may not be any stationary metric on M with conformal infinity $(\mathcal{I}, \gamma)$. A more recent and efficient construction of the stress-energy tensor $\tau_{(n)}$ is given in [18], [19], cf. also [27], [28].

Given $g \in \mathcal{E}$, the (holographic) mass, cf. [16], [17], of the cut $\partial\Sigma_t \subset \mathcal{I}$ is defined by

$$m_{\partial\Sigma_t} = \int_{\partial\Sigma_t} \tau_{(n)}(Z, \nu) dV_\gamma, \tag{2.27}$$

where ν is the future unit normal and Z is the (future-oriented) Killing field on $\mathcal{I}$. By its definition, this mass is independent of spacelike hypersurfaces $\Sigma_t \subset (M, g)$ giving the same cut $\partial\Sigma_t$ at $\mathcal{I}$. Both the holographic mass m and the stress-energy tensor $\tau_{(n)}$ depend on the choice of boundary metric, (or equivalently on the choice of defining function $\widetilde{\rho}$). However, as noted above, the boundary metric γ is chosen to be the fixed stationary metric (2.12).

Since γ is stationary, a standard application of (2.26) and the divergence theorem implies that

$$m = m_{\partial\Sigma_t}, \tag{2.28}$$

is independent of the cut $\partial\Sigma_t$, (and hence t), so that the mass depends only on the solution $(M, g) \in \mathcal{E}$, (given the fixed choice of boundary metric), and the choice of timelike Killing field Z on $(\mathcal{I}, \gamma)$. In other words, the mass is conserved. If γ has a larger space of timelike Killing fields, one may consider the mass (2.27) defined with respect to each choice, or take the "canonical" Killing field with zero angular velocity, cf. [19]. For the rest of the paper, Z denotes any fixed timelike Killing field.

There have been numerous definitions of mass and other conserved quantities for asymptotically AdS and asymptotically locally AdS spacetimes, cf. [29], [30], [31], for example; cf. [32] for an overview. In the generality considered here, the holographic mass in the only suitable definition, again since it is background-independent and covariant or intrinsic to the metric (M, g), given a fixed choice of boundary metric. Many of the various definitions of mass have recently been shown to be equivalent to the holographic mass, cf. [33] and in particular [19] for a very clear analysis.

Next, consider variations of the mass with respect to variations of the metric $g \in \mathcal{E}$. Suppose then that h is an infinitesimal Einstein deformation, with induced boundary variation $h_{(0)}$. It is convenient to put the variation h in the geodesic gauge for a fixed geodesic defining function ρ, so that h respects the splitting (2.4), i.e. $h_{0\alpha} = 0$. This can always be accomplished by a suitable gauge transformation, (i.e. infinitesimal diffeomorphism). Since the curve $g_u = g + uh$ is Einstein to 1st order in u, the constraint equations (2.26) give

$$\delta'(\tau_{(n)}) + \delta\sigma_{(n)} = 0, \tag{2.29}$$

where δ' is the variation of the divergence δ in the direction of the variation of the boundary metric γ and $\sigma_{(n)} = \tau'_{(n)} = h_{(n)} - r'_{(n)}$. Suppose the boundary metric γ is kept fixed, i.e. $h_{(0)} = 0$, so that, formally, $h \in T_g\mathcal{E}$. Then

$$\delta\sigma_{(n)} = 0. \tag{2.30}$$

By definition, one has

$$dm_{\partial\Sigma_t}(h) = \frac{dm_{u,t}}{du}|_{u=0} = \int_{\partial\Sigma_t} \frac{d}{du}(\tau_{(n)})_{g+uh}(Z, \nu)dV = \int_{\partial\Sigma_t} \sigma_{(n)}(Z, \nu)dV, \tag{2.31}$$

again since the boundary metric is fixed. Thus, via (2.30) and the divergence theorem, $dm_{\partial\Sigma}$ is independent of the cut $\partial\Sigma$, i.e. is conserved, for any fixed h vanishing on $\mathcal{I}$. So for such variations, both the mass as well as its variation, are conserved.

The following discussion will be important in the proof of Theorem 1.1 in §3. For an arbitrary complete Lorenztian boundary metric γ on $\mathcal{I} = \mathbb{R} \times \partial\Sigma$, let $\mathcal{T}$ be the space of (smooth) symmetric bilinear forms on $\mathcal{I}$ which satisfy the constraint equations (2.26) with respect to γ; when n is odd, these are the transverse-traceless (TT) forms. Thus $\mathcal{T}$ is naturally an affine bundle

$$\pi : \mathcal{T} \to Met(\mathcal{I})$$

over the space of Lorentz metrics on $\mathcal{I}$. An element (γ, τ), $\tau \in \mathcal{T}_\gamma$, in $\mathcal{T}$ then defines a unique formal series solution of the Einstein equations (1.1), defined near $\mathcal{I}$. This follows immediately from the discussion following (2.5)-(2.6). If the pair (γ, τ) are real-analytic, the formal series converges to an actual solution g of the Einstein equations (1.1), again defined in a neighborhood of $\mathcal{I}$.

Proposition 2.2. *At any $(\gamma, \tau) \in \mathcal{T}$, the map π is a submersion, i.e. its derivative is surjective, and so π is locally surjective.*

Proof: Given $(\gamma, \tau) \in \mathcal{T}$, one needs to show that for any variation $\gamma' = h_{(0)}$ of γ, there exist solutions τ' of the linearized constraint equations (2.26), i.e.

$$\delta'\tau + \delta\tau' = 0, \tag{2.32}$$

$$tr'\tau + tr\,\tau' = a'_{(n)}.$$

Let $\sigma = \tau'$, so that it suffices to solve

$$\delta\sigma = \phi_1, \quad tr\sigma = \phi_2, \tag{2.33}$$

for arbitary ϕ_1, ϕ_2. Consider for instance σ of the form $\sigma = \bar{\delta}^* V + f\gamma$, where V is a vector field. Then (2.33) becomes the system $\delta\bar{\delta}^* V - df = \phi_1$, $tr(\bar{\delta}^* V) + nf = \phi_2$, so that

$$\delta\bar{\delta}^* V + \tfrac{1}{n} dtr\bar{\delta}^* V = \phi_1 - \tfrac{1}{n} d\phi_2.$$

One has $\delta\bar{\delta}^* V = \frac{1}{2} D^* DV - Ric V$, while tr $\bar{\delta}^* V = (\frac{n}{2} - 1)\delta V$. Hence, the equation above is equivalent to

$$\tfrac{1}{2} D^* DV + (\tfrac{1}{2} - \tfrac{1}{n}) d\delta V - Ric V = \phi_1 - \tfrac{1}{n} d\phi_2. \tag{2.34}$$

This is a linear hyperbolic system for V on $\mathcal{I}$ with $\partial\Sigma$ a compact Cauchy surface. It is standard that the Cauchy problem for (2.34) has global solutions V on $(\mathcal{I}, \gamma)$ with arbitrary initial data. Given V, one may then solve the trace equation above to obtain f and the resulting pair σ satisfying (2.32).

∎

Just as before with pairs (γ, τ) satisfying the constraint equations (2.26), the space of pairs $(h_{(0)}, \sigma)$, for σ satisfying (2.32), corresponds exactly to the space of formal series solutions of the linearized Einstein equations defined near $\mathcal{I}$, (in the geodesic gauge). Again if $(h_{(0)}, \sigma)$ are real-analytic, the formal series converges to an actual solution of the linearized Einstein equations defined in a neighborhood of $\mathcal{I}$.

For a given $h_{(0)}$, the space of solutions σ of (2.32) is an affine space $\mathcal{F}_{h_{(0)}}$. Observe that the space of global solutions, (or solutions defined in $D(\Sigma)$), of the Einstein equations linearized at g with induced boundary variation $h_{(0)}$ is also an affine space $\mathcal{G}_{h_{(0)}}$, parameterized by the Cauchy data h_0, h_1 on a spacelike hypersurface $\Sigma \subset (M, g)$. Clearly,

$$\mathcal{G}_{h_{(0)}} \subset \mathcal{F}_{h_{(0)}}.$$

We do not know if the two spaces $\mathcal{G}_{h_{(0)}}$, $\mathcal{F}_{h_{(0)}}$ actually coincide, although there seems to be no compelling reason for this to be the case.

Lemma 2.3. *Suppose γ is globally stationary. Then for any smooth variation $h_{(0)}$ of compact support on $\mathcal{I}$, there exist solutions σ to (2.32) on $\mathcal{I}$ which are uniformly bounded, in that*

$$|\sigma|_{C^1} \leq K, \tag{2.35}$$

where K depends only on $(\gamma, h_{(0)})$ and is independent of t.

Proof: We will constuct a specific bounded solution, although it is clear that there will be many other possibilities. Let e_α, $0 \leq \alpha \leq n$, be a local orthonormal framing of $(\mathcal{I}, \gamma)$, with $e_0 = T = Z/|Z|$, so that $\{e_i\}$ are tangent to the orbit space S of $\mathcal{I}$. At a given point $p \in S$, assume that $\nabla_{e_i} e_j(p) = 0$, where here, and only here, the covariant derivative is on (S, γ_S). Then, at p, $-\delta\sigma = (\nabla_T \sigma)(T) + (\nabla_{e_i}\sigma)(e_i) = \nabla_T \sigma(T) - \sigma(\nabla \log N) + \nabla_{e_i}\sigma(e_i)$, since $\langle \nabla_{e_i} e_i, T\rangle = 0$. (The last equation follows since T is colinear to a Killing field orthogonal to e_i). Let σ^H be the horizontal part of σ, $\sigma^H = \sigma - \sigma(T) \cdot T$.

For simplicity, suppose $\sigma(T, e_i) = 0$, for all i. Then a simple calculation shows that the first equation in (2.33) becomes

$$-(\nabla_T \sigma(T))(T) = \phi(T), \tag{2.36}$$

$$\delta^H (N\sigma^H) = N\phi^H, \tag{2.37}$$

where $\phi^H = \phi - \phi(T)T$ is the horizontal projection of ϕ and δ^H is the divergence operator on the orbit space (S, g_S). Since $h_{(0)}$ has compact support, so does ϕ.

The equations (2.36)-(2.37) are uncoupled. The first equation may be solved directly by integration along the integral curves of T. Since $\phi(T)$ has compact support, $\sigma(T, T)$ remains uniformly bounded in the sense of (2.35).

For the second equation, let $\mathbb{S}^2(S)$ be the space of symmetric bilinear forms on S. Then $\mathbb{S}^2(S) = Im(\delta^H) \oplus Ker((\delta^H)^*)$, and the second summand corresponds to the space of Killing fields on (S, γ_S). Suppose first the orbit space (S, g_S) has no Killing fields, so that the operator $\delta^T : \mathbb{S}^2(S) \to \Omega^1(S)$ is surjective. Then (2.37) admits (many) solutions, with σ^H satisfying (2.35).

If (S, γ_S) admits Killing fields, then by linearity it suffices to solve

$$\delta\sigma = X_t, \tag{2.38}$$

where, for each fixed t, X_t is a Killing field on S and X_t has compact support on $\mathcal{I}$. Consider first the equation

$$\delta\sigma = f(t)X,$$

where X is a fixed Killing field on S. Setting $\sigma = v(t)T \cdot X$, a simple computation gives

$$-\delta(v(t)T \cdot X) = v'(t)X + \nabla_X T + \nabla_T X.$$

Straightforward computation using the Killing properties shows that $\nabla_X T + \nabla_T X$ is orthogonal to T and the space $\chi(S)$ of Killing fields on S, and hence $\nabla_X T + \nabla_T X \in Im(\delta^H)$. Setting $v' = -f$ and using linearity then shows that the first equation in (2.33) is solvable with bounded σ. A general X_t as in (2.38) has the form $X_t = \sum f_i(t)X_i$, where X_i is a basis of $\chi(S)$ and so the result for the first equation in (2.33) again follows by linearity.

To solve the second equation in (2.33), by linearity it suffices to solve $\delta\sigma = 0$ and $tr\sigma = \phi_2$. It is clear that this system has many C^1 bounded solutions.

∎

It is not clear in this generality that there exists a solution h of the linearized Einstein equations, with $h|_{\mathcal{I}} = h_{(0)}$ of compact support for which $\sigma_{(n)}$ satisfies (2.35).

3. Proofs of the Results.

The main tool used in the proof of Theorems 1.1 and 1.2 is the following identity, proved in [10]; for completeness, the proof is also given in the Appendix. Let X be any Killing field on $(\mathcal{I}, \gamma)$ and let τ be any smooth symmetric bilinear form on $\mathcal{I}$, satisfying the constraint equations (2.26). (More precisely, only the divergence-free condition in (2.26) is needed). If $h_{(0)}$ is any variation of γ of compact support on $\mathcal{I}$, then

$$\int_{\mathcal{I}} \langle \mathcal{L}_X \tau, h_{(0)} \rangle dV = -2\int_{\mathcal{I}} \langle \delta'\tau, X \rangle dV, \tag{3.1}$$

where as in (2.29), δ' is the variation of the divergence $\delta = \delta_\gamma$ in the direction $h_{(0)}$. (A simple modification of (3.1) also holds for conformal Killing fields, cf. (A.6)).

The relation (3.1) holds in particular for $\tau = \tau_{(n)}$, where $\tau_{(n)}$ is the stress-energy tensor associated to a solution $g \in \mathcal{E}$. Further, by Proposition 2.2 or Lemma 2.3, the equation

$$\delta'\tau_{(n)} = -\delta\sigma, \tag{3.2}$$

is always solvable, for some symmetric bilinear form σ on $\mathcal{I}$. Clearly σ is determined only up to a divergence-free symmetric bilinear form.

Choose $X = Z$ a timelike Killing field on $(\mathcal{I}, \gamma)$ and suppose the variation $h_{(0)}$ of the boundary metric γ has compact support; let $\partial\Sigma^\pm$ be any two cuts of $\mathcal{I}$ which enclose supp $h_{(0)}$. Suppose also that $\sigma = \sigma_{(n)}$ arises from a (global) solution h to the linearized Einstein equations, i.e. $\sigma_{(n)} = \tau'_{(n)} = h_{(n)} - r'_{(n)}$, (cf. (2.29)). Then (3.1) and the divergence theorem applied to (3.2) give

$$\int_{\mathcal{I}} \langle \mathcal{L}_Z \tau_{(n)}, h_{(0)} \rangle dV = -2\int_{\partial\Sigma^\pm} \sigma_{(n)}(Z, \nu) dV = -2[dm^+(h) - dm^-(h)], \tag{3.3}$$

where $dm^{\pm}$ is the variation of the holographic mass (2.27) at $\partial\Sigma^{\pm}$, cf. also (2.31). It is clear that this formula holds for any formal solution $h_f = (h_{(0)}, \sigma)$ of the linearized constraint equations (2.32) or (3.2):

$$\int_{\mathcal{I}} \langle \mathcal{L}_Z \tau_{(n)}, h_{(0)} \rangle dV = -2 \int_{\partial\Sigma^{\pm}} \sigma(Z, \nu) dV = -2[dm^{+}(h_f) - dm^{-}(h_f)]. \tag{3.4}$$

Proof of Theorem 1.1.

Related to the various definitions of conserved quantities for asymptotically AdS and asymptotically locally AdS spacetimes, there has been much recent discussion in the literature concerning the validity of the first law of black hole mechanics for such spacetimes, see in particular [34] and references therein and thereto. The ambiguities regarding the first law for the holographic mass (2.27) obtained by holographic or covariant renormalization have recently been resolved in a very clear analysis by Papadimitriou-Skenderis [19]. In particular, in the context of the present work, it is proved in [19] that if $g_S \in \mathcal{E}$ is globally stationary, then g is a critical point for the holographic mass (2.27), under infinitesimal Einstein variations h with fixed boundary metric, i.e.

$$\delta m = 0, \tag{3.5}$$

among infinitesimal Einstein variations h such that

$$h_{(0)} = 0. \tag{3.6}$$

This is a special case of the first law of black hole mechanics in the AdS setting, namely in the case where there is no black hole, in the sense that the bifurcate Killing horizon is empty.

We first observe that (3.5) also holds for all formal infinitesimal Einstein deformations h_f, i.e. formal series solutions of the linearized Einstein equations determined by $(h_{(0)}, \sigma)$, for σ satisfying (2.32) on (M, g_S). To see this, from (3.3) one has:

$$\int_{\mathcal{I}} \langle \mathcal{L}_Z \tau_{(n)}, h_{(0)} \rangle dV = -2 \int_{\mathcal{I}} \langle \delta\sigma_{(n)}, X \rangle dV = -2[\delta m_{\partial\Sigma^{+}}(h) - \delta m_{\partial\Sigma^{-}}(h)]. \tag{3.7}$$

Here h is any global linearized Einstein deformation with supp $h_{(0)}$ contained in the region of $\mathcal{I}$ between $\partial\Sigma^{-}$ and $\partial\Sigma^{+}$ and $\sigma_{(n)} = h_{(n)} - r'_{(n)}$. Since $h_{(0)} = 0$ on $\partial\Sigma^{-}$, by (3.5) one has

$$\delta m_{\partial\Sigma^{-}}(h) = 0.$$

Also, since (M, g_S) is stationary, the left side of (3.7) vanishes. On the other hand, (3.7) holds with $\sigma_{(n)}$ replaced by any σ satisfying (2.32), so that

$$\delta m_{\partial\Sigma^{+}}(h_f) = 0, \tag{3.8}$$

for any formal variation h_f determined by $(h_{(0)}, \sigma)$.

Now suppose the solution $g \in \mathcal{E}$ is asymptotically stationary, i.e. as $t_i \to \infty$, (or $t_i \to -\infty$), g_{t_i} converges to a stationary solution g_{∞}. By the discussion above, on the limit g_S, one has

$$\delta m_S(h_f) = \int_{\partial\Sigma} \sigma(Z, \nu) = 0,$$

for any formal solution $h_f = (h_{(0)}, \sigma) \in \mathcal{F}$ to the linearized Einstein equations at $\mathcal{I}$ with $h_{(0)} = 0$ on $\partial\Sigma$, (or of compact support). It follows that on the original spacetime (M, g), for $|T|$ sufficiently large, one has

$$|\delta m_T(h)| = |\int_{\partial\Sigma_T} \sigma(Z, \nu)| \leq \varepsilon, \tag{3.9}$$

for all variations $(h_{(0)}, \sigma)$ which are uniformly bounded by a fixed constant, independent of T and for which supp $h_{(0)}$ is a fixed compact set in $\mathcal{I}$. By Lemma 2.3, there exist such uniformly bounded σ, for all such variations $h_{(0)}$. Applying (3.4) once more to (M, g) with the data $h_f = (h_{(0)}, \sigma)$ gives

$$\int_{\mathcal{I}} \langle \mathcal{L}_Z \tau_{(n)}, h_{(0)} \rangle dV = -2 \int_{\mathcal{I}} \langle \delta\sigma, Z \rangle dV = -2[\delta m_{\partial\Sigma_{T^+}}(h_f) - \delta m_{\partial\Sigma_{T^-}}(h_f)]. \tag{3.10}$$

Letting $T^+ \to +\infty$ and $T^- \to -\infty$, using the boundedness condition, it follows from (3.9)-(3.10) that

$$\int_{\mathcal{I}} \langle \mathcal{L}_Z \tau_{(n)}, h_{(0)} \rangle dV = 0, \tag{3.11}$$

for all $h_{(0)}$ of compact support. Since $h_{(0)}$ is an arbitrary variation of the boundary metric γ in $\mathcal{I}_{[0,T]}$, it follows that

$$\mathcal{L}_Z g_{(0)} = 0 \text{ and } \mathcal{L}_Z \tau_{(n)} = 0 \text{ on } \mathcal{I}. \tag{3.12}$$

Thus, the Cauchy data $(g_{(0)}, \tau_{(n)})$ or $(g_{(0)}, g_{(n)})$ for g on $\mathcal{I}$ are invariant under the flow of the Killing field Z on $\mathcal{I}$.

We claim that (3.12) together with the unique continuation property at $\mathcal{I}$ implies that (M, g) is stationary. Intuitively, this is quite clear, but the proof requires some details.

To begin, extend Z to a vector field in the bulk by the following two-step process. First, extend Z to a neighhorhood W of $\mathcal{I}$ in M by requiring

$$[Z_1, \bar{\nabla}\rho] = 0,$$

with $Z_1|_{\mathcal{I}} = Z$. Thus Z is extended by the flow of $\bar{\nabla}\rho$ to a vector field Z_1, defined in the region where $\bar{\nabla}\rho$ is smooth. The corresponding form $\delta^* Z_1$ is then an infinitesimal Einstein deformation which preserves the defining function ρ. If ϕ_s denotes the flow of Z_1 and $g_s = \phi_s^* g$ denotes the corresponding curve of Einstein metrics, then the geodesic defining function for g_s with fixed boundary metric γ is the fixed function ρ. Each metric g_s thus has the expansion (2.9)-(2.10) with fixed ρ. Since γ is fixed, apriori only the $g_{(n),s}$ terms can vary, but (3.12) shows the variation of $g_{(n),s}$ vanishes to 1st order in s at $s = 0$. Hence, since the formal series (2.9)-(2.10) are determined by the $(g_{(0)}, g_{(n)})$ terms, it follows that

$$\delta^* Z_1 = O(\rho^p), \text{ for all } p < \infty, \tag{3.13}$$

formally near $\mathcal{I}$. More precisely, (3.13) holds to the extent that the metric $\bar{g}$ in (2.3) is smooth, (in a polyhomogeneous sense), up to $\mathcal{I}$. In any case, one has

$$\delta^* Z_1 = o(\rho^n). \tag{3.14}$$

Extend Z_1 outside W arbitrarily but smoothly to all of M.

Next we need to bring $\delta^* Z_1$ into the transverse-gauge (2.22). To do this, (3.14) implies that

$$\omega \equiv \delta\bar{\delta}^* Z_1 = o(\rho^n).$$

Set $Z = Z_1 - Y$, where Y is chosen to be the unique solution of the initial boundary value problem

$$\delta\bar{\delta}^* Y = \omega, \tag{3.15}$$

with zero Cauchy and boundary data; $Y|_\Sigma = \nabla_{\partial_t} Y|_\Sigma = 0$ and $Y|_{\mathcal{I}} = 0$. As discussed in §2, this equation has a unique smooth solution at least within the domain of dependence $D(\Sigma)$. The smallest indicial root of the operator $\delta\bar{\delta}^* = \frac{1}{2}D^*D - Ric$ is at least $(n+1)$, i.e. the formal expansion of a solution of (3.15) has determined coefficients up to order $(n+1)$, cf. [35]. Since $\omega = o(\rho^n)$, it follows that

$$\delta^* Y = o(\rho^n).$$

Hence the vector field Z on $D(\Sigma)$ satisfies

$$\delta\bar{\delta}^* Z = 0, \tag{3.16}$$

with $Z|_{\partial\Sigma}$ the given Killing field Z on $\mathcal{I}$ and

$$\delta^* Z = o(\rho^n). \tag{3.17}$$

Since $\delta^* Z$ is an infinitesimal Einstein deformation, it follows from (2.17) that $\delta^* Z$ satisfies the equations (2.22), i.e.

$$D^* D(\delta^* Z) - 2R(\delta^* Z) = 0. \tag{3.18}$$

Since (3.17) holds, the unique continuation property at $\mathcal{I}$ for (3.18) implies that

$$\delta^* Z = 0, \tag{3.19}$$

in a neighborhood U of $\partial\Sigma$ in $D(\Sigma)$.

To show that Z extends to a Killing field on all of $D(\Sigma)$, let $U_s = \{x \in D(\Sigma) : \rho(x) \geq s > 0\}$ and let $C_s = \partial U_s$, so that C_s is a timelike cylinder. For s sufficiently small, one has $\delta^* Z = 0$ to infinite order on C_s and $\delta^* Z$ satisfies (3.18) throughout $D(\Sigma)$. The equations (3.18) are a hyperbolic system of PDEs, and at leading order are a diagonal system of scalar wave equations for which the boundary C_s is strictly pseudoconvex. A unique continuation result of Tataru [36], then implies that $\delta^* Z = 0$ in a neighborhood of C_s within U_s. One may then iterate this process a finite number of times to cover a neighborhood of the initial surface Σ. Of course one must change the distance function ρ near regions where ρ becomes singular and use instead smooth distance functions, but the arguments are otherwise the same.

Since $\delta^* Z$ thus vanishes to 1st order on all of Σ and vanishes on $\mathcal{I}$, it follows from uniqueness of the initial boundary value problem for (3.18) that $\delta^* Z = 0$ on all of M.

Observe that Z cannot become null anywhere in M. For if Z is null at some point $p \in M$, then the Killing equation (3.19) implies that flow line σ of Z through p is a null line, i.e. Z remains null along σ. Since such null lines must intersect $\mathcal{I}$, this implies Z is null somewhere on $\mathcal{I}$, which is impossible. Thus Z is timelike throughout M, so that (M, g) is globally stationary.

■

Proof of Corollary 1.2.

Theorem 1.1 applies to any time-like Killing field on $(\mathcal{I}, \gamma)$. Let $\mathcal{K}$ be the cone of time-like Killing fields on $(\mathcal{I}, \gamma)$. Taking linear combinations, the cone $\mathcal{K}$ generates the full space of Killing fields on $\mathcal{I}$. Consider the space $\mathcal{S}_{\mathcal{K}}$ of solutions in $\mathcal{E}$ which are invariant under an effective $\mathcal{K}$-action, restricting to the action of $\mathcal{K}$ at conformal infinity. Then Theorem 1.1 implies that any solution $(M, g) \in \mathcal{E}$ which to the future and past is weakly asymptotic to (possibly distinct) elements in $\mathcal{S}_{\mathcal{K}}$ is necessarily a fixed solution in $\mathcal{S}_{\mathcal{K}}$.

Applying this to metrics in $\mathcal{E}$ whose conformal infinity is the Einstein static cylinder, it follows that the component of the identity of the isometry group of the Einstein static cylinder $\mathbb{R} \times S^{n-1}$ extends to a group of isometries of any $(M, g) \in \mathcal{E}$. In particular, any such (M, g) has an isometric $\mathbb{R} \times SO(n)$ action. This implies that the Einstein equations (1.1) reduce to a system of ODE's and it is standard that the only globally smooth solution of this system is the exact AdS solution.

■

Exactly the same arguments can be applied to spacetimes $(M, g) \in \mathcal{E}$ whose conformal infinity is homogeneous, i.e. $(\mathcal{I}, \gamma)$ has a transitive group of isometries. The Einstein equations for (M, g) then reduce to a system of ODE's, (in the variable ρ), and the requirement that the solutions are smooth in the interior typically gives either a unique solution, or uniqueness up to a set of parameters which determine the topology of (M, g).

To illustrate on some concrete examples, consider the AdS soliton metric of Horowitz-Myers [11]. In the toroidal compactification, $(\mathcal{I}, \gamma)$ is the flat product metric on $\mathbb{R} \times T^{n-1}$ on the $(n-1)$-torus. This is clearly homogeneous. The corresponding ODE's may be solved explicitly and have a unique smooth solution on $(M, g) \simeq \mathbb{R} \times D^2 \times T^{n-1} \in \mathcal{E}$ up to the choice of topology, (a choice of the disc D^2 bounding an $S^1 \subset T^{n-1}$). This proves uniqueness of the AdS soliton metric among all (dynamical) metrics in $\mathcal{E}$ with the given conformal infinity and topology which are asymptotic at $t = +\infty$ and $t = -\infty$ to an AdS soliton metric. (This argument can be extended without difficulty to the case where $\mathcal{I}$ is compactified to a single circle instead of the full $(n-1)$-torus). Exactly the same results hold for the recent AdS soliton metric analysed by Copsey-Horowitz [12].

Similar uniqueness results also hold with respect to perturbations of such homogeneous conformal infinities. For example, suppose (M, g) is an asymptotically simple, globally static solution of the Einstein equations (1.1) with conformal infinity $(\mathcal{I}, \gamma)$ which is non-degenerate, (e.g. (M, g) has non-positive curvature). Given a static or stationary perturbation $(\mathcal{I}, \widetilde{\gamma})$ of the boundary data $(\mathcal{I}, \gamma)$, there is a unique globally static or stationary asymptotically simple solution $(M, \widetilde{g})$ close to (M, g) with conformal infinity $(\mathcal{I}, \widetilde{\gamma})$, cf. [37], [38]. Theorem 1.1 then implies the solution $(M, \widetilde{g})$ is unique among all dynamical solutions to the Einstein equations in $\mathcal{E}$ which are asymptotic to the future and past to the given static or stationary solution.

Remark 3.1. It is worth emphasizing that the results above require the solutions (M, g) of the Einstein equations to be globally defined and non-singular. On the one hand, this is apparent from the proof. Theorem 1.1 uses the global vanishing (3.5) of the variation of mass on stationary spacetimes; per se, this is false if there are inner boundaries in addition to the boundary at conformal infinity.

In fact, Theorem 1.1 or Corollary 1.2 are false for solutions of the Einstein equations defined only in a neighborhood of conformal infinity $(\mathcal{I}, \gamma)$. For example, let $(\mathcal{I}, \gamma_0)$ be the Einstein static cylinder and let $\tau_{(n)}$ be any analytic symmetric bilinear form on $\mathcal{I}$ which is asymptotic to 0 as $t \to \pm\infty$, and which satisfies the constraint equations (2.26). The result of Kichenassamy [22] mentioned above gives the existence of a solution of the Einstein equations (1.1) defined in a neighborhood of $\mathcal{I}$ with the given $\tau_{(n)}$, i.e. $g_{(n)}$ term, on $\mathcal{I}$. This solution is asymptotic to the exact AdS solution at $t = \pm\infty$, but is not exact AdS unless $\tau_{(n)} \equiv 0$.

4. Discussion

In the context of the Euclidean (or Riemannian) version of the AdS/CFT correspondence, it is important to know to what extent the boundary data $(\partial M, \gamma)$ determine the bulk solutions (M, g) of the Einstein equations (1.1). Although it is possible in general that there are infinitely many topological types for (M, g), or that, fixing the topology, the space of solutions has infinitely many components, there is only at most a finite dimensional moduli space of solutions when one fixes the topology and component. This follows essentially from the elliptic character of the Einstein equations (1.1). Thus, the "Cauchy data" $(g_{(0)}, g_{(n)})$ uniquely determine the bulk solution (M, g) up to local isometry, (cf. [10]), and given $g_{(0)}$, although the stress-energy term $g_{(n)}$ may not quite be uniquely determined by the boundary metric $g_{(0)}$, it is determined up to a finite dimensional space of parameters, (given a choice of topology and deformation component).

In this Euclidean context, the fact that isometries of the boundary $(\partial M, \gamma)$ necessarily extend to isometries of any smooth global bulk solution (M, g) is a simple and clear illustration of (elementary or classical) aspects of the AdS/CFT correspondence.

On the other hand, in the context of the Lorentzian solutions of the Einstein equations (1.1) in $\mathcal{E}$, there is an infinite dimensional space of normalizable modes, i.e. L^2 solutions of the linearized Einstein equations. At least in some situations, for example that of exact AdS spacetime, these linearized solutions remain uniformly bounded in time, i.e. g_{AdS} is linearization stable. One expects that that g_{AdS} is in fact dynamically stable, with the behavior of the nonlinear exact solutions nearby to g_{AdS} well-modeled on the linearized behavior.

In fact, Friedrich [39] has shown that one may solve the initial boundary value problem for the Einstein equations (1.1) at least locally in time, in 3+1 dimensions. Thus, suppose the boundary data $(\mathcal{I}, \gamma)$ are globally stationary, and that Σ is a Cauchy surface with initial data (g_0, K) satisfying the constraint equations (2.15) and matching γ at the corner $\partial\Sigma \subset \mathcal{I}$, (cf. also [40]). Then there is a solution g of the Einstein equations (1.1), with conformal infinity $(\mathcal{I}, \gamma)$ defined on a thickening $I \times \Sigma$ of Σ, realizing the given Cauchy data. This gives an infinite dimensional space of local-in-time exact solutions. If the data (g_0, K) are (arbitrarily) close to the data $(g_{-1}, 0)$ of exact AdS spacetime, where g_{-1} is the hyperbolic metric on the n-ball, then the resulting solution will exist for an (arbitrarily) long time, to the future and past. One would expect, although this remains to be proved, that such solutions extend to global-in-time solutions which remain globally asymptotically simple and globally close to g_{AdS}.

Theorem 1.1 implies that such global solutions cannot be both future and past asymptotic to the exact AdS spacetime. This is of course in strong contrast to the asymptotically flat situation $(\Lambda = 0)$, where the Christodoulou-Klainerman theorem [2] implies that small global perturbations of Minkowski spacetime disperse in time and tend to the flat solution, preserving, (at least to a certain degree), asymptotic simplicity and boundedness; in some situations asymptotic simplicity and boundedness is preserved to all orders, [5]. To a certain extent, it is the global conformal or causal structure which leads to these differences. Thus, observe that the Bondi mass on $\mathcal{I}^+$ is not preserved in time, but decreases monotonically. While the ADM mass is preserved under time evolution, it is defined with respect to the singular structure at spacelike infinity ι^0, which does not match smoothly with the geometry of null infinity $\mathcal{I}^+$.

Similarly, with regard to the global stability results of de Sitter spacetimes in [8] and [9], here it is not clear if there is even a reasonable definition of mass, since $\mathcal{I}$ is spacelike. Given a definition of mass, as for instance in [41], it is not directly related to the dynamical evolution of the spacetime; it is a fixed quantity at future or past spatial infinity.

On the other hand, Theorem 1.1 is consistent with the known oscillatory behavior of the Einstein equations linearized at exact AdS, cf. [1]. It is unknown if general asympotically locally AdS Einstein spacetimes are linearization stable, or what conditions will guarantee linearization stability.

To conclude, we discuss informally the implication of Theorem 1.1 on the global dynamical behavior of bounded solutions in $\mathcal{E}$. Thus, consider the subspace $\mathcal{E}_K$ of uniformly bounded solutions in $\mathcal{E}$, i.e. solutions which remain uniformly bounded by a fixed constant K in time, in a suitable smooth norm. Then $\mathcal{E}_K$ is weakly compact, in that in any sequence in $\mathcal{E}_K$ has a subsequence converging in a weaker topology to a limit solution again in $\mathcal{E}_K$. Standard results in dynamical systems, (cf. [42] for example), then imply that the resulting Einstein flow on the constraint space $\mathcal{C}_K$ has (many) uniformly recurrent points: thus there exist times $t_i \to \infty$, (or $t_i \to -\infty$), with $t_{i+1} - t_i \leq T$, for some fixed T, and points $(g_{t_i}, K_{t_i}) \in \mathcal{C}_K$ such that

$$dist((g_{t_{i+1}}, K_{t_{i+1}}), (g_{t_i}, K_{t_i})) \leq \varepsilon,$$

where the distance is taken in the weaker norm. Here ε may be chosen arbitrarily small, but then with T becoming possibly arbitrarily large.

Theorem 1.1 implies that no orbits of the flow on $\mathcal{C}_K$ are asymptotic to fixed points as $t \to \pm\infty$. This indicates that most all orbits in $\mathcal{C}_K$ should be almost periodic.

It is an open question whether there exist periodic (non-stationary) solutions in $\mathcal{E}$. The proof of Theorem 1.1 can be easily adapted to rule out periodic solutions provided, for any periodic boundary variation $h_{(0)}$, there exists a periodic solution σ of the linearized constraints as following (3.7). However, it is not clear that such periodic σ always exist; I am very grateful to Bob Wald for discussions on this point.

Appendix

Let X be a Killing field with respect to a Lorentz metric $(\mathcal{I}, \gamma)$, and let τ be a divergence-free symmetric bilinear form on $\mathcal{I}$, $\delta\tau = 0$. We give here a proof of the identity (3.1) from [10], i.e.

$$\int_{\mathcal{I}} \langle \mathcal{L}_X \tau, \kappa \rangle dV = -2 \int_{\mathcal{I}} \langle \delta'(\tau), X \rangle dV, \tag{A.1}$$

where $\kappa = \frac{d}{ds}(\gamma + u\kappa)|_{s=0}$ is a variation of γ of compact support and $\delta' = \frac{d}{ds}\delta_{\gamma+s\kappa}|_{s=0}$ is the variation of the divergence. The proof of (A.1) below holds for metrics of any signature.

To prove (A.1), we use the following standard formulas, cf. [43] for example:

$$\mathcal{L}_V \phi = \nabla_V \phi + 2\nabla V \circ \phi, \tag{A.2}$$

$$(\delta^*)'V = \tfrac{1}{2}\nabla_V \kappa + \delta^* V \circ \kappa, \tag{A.3}$$

for any vector field V. Here $\phi \circ \psi$ is the symmetrized product; $(\phi \circ \psi)_{ij} = \frac{1}{2}(\langle \phi_i, \psi_j \rangle + \langle \phi_j, \psi_i \rangle)$ and $(\delta^* V)_{ij} = \frac{1}{2}(V_{i,j} + V_{j,i})$.

To begin, by (A.2), $\int_{\mathcal{I}} \langle \mathcal{L}_X \tau, \kappa \rangle = \int_{\mathcal{I}} \langle \nabla_X \tau, \kappa \rangle + 2\langle \nabla X \circ \tau, \kappa \rangle$. Since κ is symmetric, $\langle \nabla X \circ \tau, \kappa \rangle = \langle \delta^* X \circ \tau, \kappa \rangle$. For the first term, write $\langle \nabla_X \tau, \kappa \rangle = X\langle \tau, \kappa \rangle - \langle \tau, \nabla_X \kappa \rangle$. The first term here integrates to $\delta X \langle \tau, \kappa \rangle$, (here we use the fact that κ is of compact support), while by (A.3), the second term is $-\langle \tau, \nabla_X \kappa \rangle = -2\langle \tau, (\delta^*)'X \rangle + 2\langle \tau, \delta^* X \circ \kappa \rangle$. Hence

$$\int_{\mathcal{I}} \langle \mathcal{L}_X \tau, \kappa \rangle dV = -2 \int_{\mathcal{I}} \langle \tau, (\delta^*)'X \rangle dV + 4 \int_{\mathcal{I}} \langle \tau, \delta^* X \circ \kappa \rangle + \int_{\mathcal{I}} \delta X \langle \tau, \kappa \rangle dV.$$

Next, a straightforward computation using the fact that $\delta\tau = 0$ gives

$$\int_{\mathcal{I}} \langle \tau, (\delta^*)'X \rangle dV = \int_{\mathcal{I}} \langle (\delta')(\tau), X \rangle dV + 2 \int_{\mathcal{I}} [\langle \tau \circ \delta^* X, \kappa \rangle - \tfrac{1}{2}\langle \tau, \delta^* X \rangle tr\kappa] dV :$$

the last two terms come from variation of the metric and volume form. Combining these computations gives

$$\int_{\mathcal{I}} \langle \mathcal{L}_X \tau, \kappa \rangle dV = -2 \int_{\mathcal{I}} \langle \delta'(\tau), X \rangle dV + \int_{\mathcal{I}} [\delta X \langle \tau, \kappa \rangle + \langle \tau, \delta^* X \rangle tr\kappa] dV. \tag{A.4}$$

This gives (A.1) when X is Killing, i.e. $\delta^* X = 0$.

Let $\hat{\mathcal{L}}_X \gamma$ denote the conformal Killing operator: $\hat{\mathcal{L}}_X \gamma = \mathcal{L}_X \gamma - \frac{2divX}{n}\gamma$, (where $divX = -\delta X$). Then the calculations above give, (again for divergence-free τ),

$$\int_{\mathcal{I}} \langle \mathcal{L}_X \tau + divX\tau, \kappa \rangle dV = -2 \int_{\mathcal{I}} \langle \delta'(\tau), X \rangle dV + \tfrac{1}{2} \int_{\mathcal{I}} \langle \tau, \hat{\mathcal{L}}_X \gamma \rangle dV + \tfrac{1}{n} \int_{\mathcal{I}} divX\, a \cdot tr\, \kappa dV, \tag{A.5}$$

where $a = tr\tau$. If σ satisfies the linearized constraint equations (2.32) then a simple calculation from (A.5), gives

$$\int_{\mathcal{I}} \langle \mathcal{L}_X \tau + [1 - \tfrac{2}{n}] divX\, \tau, \kappa \rangle dV = \tag{A.6}$$

$$\int_{\mathcal{I}} \langle \sigma + \tfrac{1}{2} tr\kappa\, \tau, \hat{\mathcal{L}}_X \gamma \rangle dV - 2 \int_{\mathcal{I}} div(\sigma(X)) dV + \tfrac{1}{n} \int_{\mathcal{I}} divX(a \cdot tr\, \kappa + 2a') dV.$$

When X is a conformal Killing field on $(\mathcal{I}, \gamma)$, the first term on the right vanishes. When $\tau = \tau_{(n)}$, the second term gives the difference of the variation of the mass when X is timelike by the divergence theorem while the integrand on the left is related to the transformation properties of the stress-energy tensor $\tau_{(n)}$ under conformal changes, cf. [17].

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