1. Some answers to problems from §3.1

3) The equilibrium position of a string is clearly that when the string is at rest and its vertical displacement is trivial, that is $u(x,t)=0$. This is just common sense. That is reflected in the answer to this problem.

For if $\nu(x)$ is a solution of the wave equation with homogeneous boundary conditions, then we must have

$$\begin{array}{rcl} \ddot{\nu}(x) & = & 0\,, \quad 0<x<a\,, \\ \nu(0) & = & 0\,, \\ \nu(a) & = & =0\,. \end{array}$$

The general solution to the ode is given by $\nu(x)=c_1x+c_2$, for arbitrary constants c_1, c_2. If we search for the solution that vanishes at 0 and a, we must have $0=c_1\cdot 0+c_2$ and $0=c_1a+c_2$. Thus, $c_1=c_2=0$, and $\nu(x)\equiv 0$ is the equilibrium solution (as common sense indicates).

2. Some answers to problems from §3.2

2) The general solution with arbitrary initial conditions f and g is given in the textbook as

$$u(x,t)=\sum_{n=1}^{\infty}\sin(\lambda_n x)(a_n\cos(\lambda_n ct)+b_n\sin(\lambda_n ct))$$

where

$$a_n=\frac{2}{a}\int_0^a f(x)\sin(\lambda_n x)\,dx\,,$$

and

$$b_n\lambda_n=\frac{2}{a}\int_0^a g(x)\sin(\lambda_n x)\,dx\,.$$

Here $\lambda_n=n\pi/a$.

In the specific case we are asked to analyze, $g(x)=0$. Therefore, the integral above for $b_n\lambda_n$ is zero for all n, and so $b_n=0$. On the other hand, $f(x)=\sin(\pi x/a)=\sin(\lambda_1 x)$ and we have

$$a_n=\frac{2}{a}\int_0^a \sin(\lambda_1 x)\sin(\lambda_n x)\,dx\,.$$

We have seen that $\sin(\lambda_1 x)$ and $\sin(\lambda_n x)$ are orthogonal to each other for $n>1$. Therefore, for all $n>1$ we have $a_n=0$, and so, only the a_1 survives in the series above. Since

$$a_1=\frac{2}{a}\int_0^a \sin(\lambda_1 x)\sin(\lambda_1 x)=\frac{2}{a}\int_0^a\frac{1-\cos(2\lambda_1 x)}{2}\,dx=1\,,$$

we obtain that

$$u(x,t)=\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi ct}{a}\right)=\frac{1}{2}\sin\left(\frac{\pi(x-ct)}{a}\right)+\frac{1}{2}\sin\left(\frac{\pi(x+ct)}{a}\right),$$

the sum of two waves, one moving to the right and the other moving to the left.

5a) We begin b writing the solution as

$$p(x,t)=p_0(x)+\bar{p}(x,t)\,,$$

where $p_0(x)$ also satisfies the wave equation and has boundary conditions p_0 at both, $x = 0$ and $x = a$. This implies that $p_0(x) = p_0$, a constant, and we have

$$p(x,t) = p_0 + \bar{p}(x,t)\,.$$

Since $p(x,t)$ is assumed to be a solution to the wave equation with boundary condition p_0 at both ends, the function $\bar{p}(x,t)$ must satisfy the equation

$$\frac{\partial^2 \bar{p}}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \bar{p}}{\partial t^2}\,,$$

and its boundary conditions are

$$\bar{p}(0,t) = 0\,, \quad \bar{p}(a,t) = 0\,.$$

We apply the method of separation of variables in order to find the *associated eigenvalue problem*, and the corresponding eigenvalues and eigenfunctions, that is what the question is asking for.

Hence, suppose $\bar{p}(x,t) = \varphi(x)T(t)$ is a solution of the problem above. Therefore,

$$\ddot{\varphi}(x)T(t) = \frac{1}{c^2}\varphi(x)\ddot{T}(t)\,,$$

and separating the variables x and t, we obtain that

$$\frac{\ddot{\varphi}(x)}{\varphi(x)} = -\lambda^2 = \frac{\ddot{T}(t)}{c^2 T(t)}\,,$$

for some constant λ. Since we want $\varphi(x)T(t)$ to vanish at $x = 0$ and $x = a$, this leads to the eigenvalue problem

$$\begin{aligned} \ddot{\varphi} + \lambda^2\varphi &= 0\,, \\ \varphi(0) &= 0\,, \\ \varphi(a) &= 0\,, \end{aligned} \tag{1}$$

while $T(t)$ must be a solution of the equation

$$\ddot{T} + \lambda^2 c^2 T = 0\,.$$

The most general solution of the differential equation in (1) is

$$\varphi(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x)\,.$$

Since $\varphi(0) = 0$, this forces c_1 to be zero. Hence, we must only consider solutions of the form $\varphi(x) = c_2 \sin(\lambda x)$. Now, since $\varphi(a) = c_2 \sin(\lambda a)$ must be zero, we obtain that

$$\sin(\lambda a) = 0\,,$$

and therefore, λa is forced to be a non-trivial multiple of π: Thus, the eigenvalues of our problem are

$$\lambda_n = \frac{n\pi}{a}\,, \quad n = 1, 2, 3, \ldots$$

The corresponding eigenfunctions, the solutions to (1) associated with λ_n, is given by

$$\varphi(x) = \varphi_n(x) = \sin\left(\frac{n\pi x}{a}\right).$$

3. Some answers to problems from §3.3

12) We have a function $u(x,t)$ defined by

$$u(x,t) = \psi(x+ct) + \phi(x-ct)\,,$$

where $\psi(s)$ and $\phi(s)$ are two functions of one real variable, each one of which has at least two derivatives.

By the chain rule, we have that

$$\frac{\partial u}{\partial x} = \dot{\psi}(x+ct)\frac{\partial x+ct}{\partial x} + \dot{\phi}(x-ct)\frac{\partial x-ct}{\partial x} = \dot{\psi}(x+ct) + \dot{\phi}(x-ct)\,,$$

and

$$\frac{\partial u}{\partial t} = \dot{\psi}(x+ct)\frac{\partial x+ct}{\partial t} + \dot{\phi}(x-ct)\frac{\partial x-ct}{\partial t} = c\dot{\psi}(x+ct) - c\dot{\phi}(x-ct)\,.$$

We apply the chain rule once again to compute the second partial derivatives of u. For example, we have that

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}((c\dot{\psi}(x+ct) - c\dot{\phi}(x-ct)) = c(c\ddot{\psi}(x+ct) - (-c)\ddot{\phi}(x-ct)) = c^2(\ddot{\psi}(x+ct) + \ddot{\phi}(x-ct))\,.$$

Similarly,

$$\frac{\partial^2 u}{\partial x^2} = \ddot{\psi}(x+ct) + \ddot{\phi}(x-ct)\,.$$

Comparing these last two results, we conclude that

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}\,,$$

as desired.

Notice that for these computations to make sense, the functions ψ and ϕ have to be differentiable twice. That is how the hypothesis on these functions is used.