

On estimating the quality of synaptic transmission in an Oja learning network

February 25, 2006

There are a few plausible ways in which we could think of mathematizing our learning process at synaptic level, so that we illustrate its dynamics / plasticity. This section will elaborate two such possible synaptic models, using basic knowledge of the mechanism of synaptic activation.

Model 1: Recall that one of our main goals is to estimate the quality factor Q of a learning network. To do this, we fix our attention on the postsynaptic neuron and its dendritic arbor. We are less interested in the actual geometry and ramification of its dendrites, so for simplification we will think of it as a unique dendrite with length L . Along this dendrite, there are a considerable number of dendritic spines capable of developing the chemical material necessary for creating a synaptic site with any presynaptic axon, if situated in enough proximity. These sites, which we will call from now *silent sites* (or *potential synapses*) are not active, but could develop into an active synapse almost instantaneously when the request to do so exists. The request to activate such a site acts like a switch that turns the synapse from the silent state (*off*) to the active state (*on*).

Say the output cell receives the signal to strengthen the connection with a prescribed presynaptic axon at a target site A . The activation signal will trigger (by back-propagation) the release of a messenger (in this case Ca^{2+}) which in ideal conditions should appear along the dendrite only at the target site A . Instead, we expect that the messenger is not locally contained and will spill away from A , down the dendritic spine. A small portion \sqrt{a} of it ($a \sim 10^{-1}$) will reach the dendritic “cable” and propagate along it in both directions, such that its concentration at a distance x from the target spine follows a function $C(x) = e^{-x/\lambda}$ (so that the quantity of messenger diffused within the specified dendritic segment is expected to decrease towards zero with the distance to the target). This messenger could reach along the dendrite to some arbitrary silent synaptic site B and some proportion of it (again \sqrt{a}) will travel up the spine, arriving at the *silent* postsynaptic density. There, in order to activate the synaptic mechanism and switch the

site from inactive (state *off*) to active (state *on*), it would have to bind to a sufficient number h of molecules of **enzyme** (typically $h = 4$).

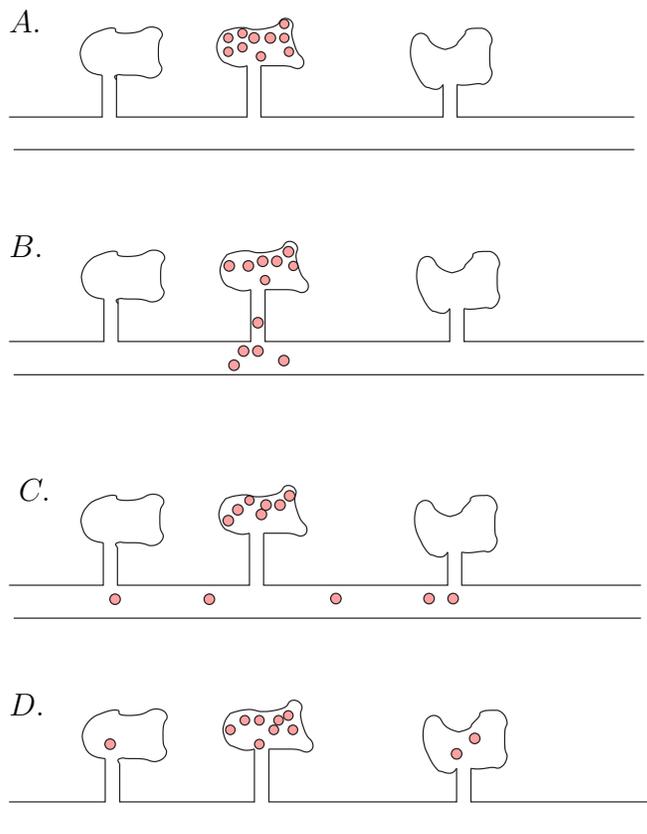


Figure 1: *The stunted tent map f^a , with critical point $\frac{1}{2}$ and critical value plateau at height a .*

Mathematically, we express the probability for a Ca^{2+} ion to get from the target site A to an arbitrary site B situated at distance x from A along the dendrite as:

$$\sqrt{a} e^{-x/\lambda} \sqrt{a} dx = a e^{-x/\lambda} dx$$

The probability for the site B to be turned from *off* to *on* also depends on the number h of molecules of enzyme required to bind the Calcium ions, so that:

$$p_x dx = a^h e^{-hx/\lambda} dx$$

We can subsequently calculate the probability for a silent synapse to be turned on anywhere along the dendritic arbor (considered to be linear of

length L), assuming that the distribution of the silent synapses along the dendrite is homogeneous, with density function $\rho = \frac{1}{L}$:

$$P = \int_0^L p_x \rho dx = \frac{1}{L} \int_0^L a^h e^{-\frac{hx}{\lambda}} dx = \frac{a^h \lambda}{hL} (1 - e^{-\frac{hL}{\lambda}})$$

Suppose now that we know the positions x_1, \dots, x_N of the N silent synapses along the postsynaptic dendrite. Notice that we can break N into simpler terms as $N = n\alpha$, where n is the number of inputs and α is the average number of silent sites that could develop into synapses with a given input neuron. The probability that exactly positions x_{j_1}, \dots, x_{j_k} are turned on with the activation of the target site A is:

$$\sum_{1 \leq j_1 < \dots < j_k \leq N} p(x_{j_1}) \dots p(x_{j_k}) [1 - p(x_{j_{k+1}})] \dots [1 - p(x_{j_N})] dx_{j_1} dx_{j_N}$$

Hence the probability (over all possible distributions of silent synapses along the dendrite) to turn on exactly k silent sites is:

$$\begin{aligned} & \frac{1}{L^N} \int_{[0,L]^N} \sum_{1 \leq j_1 < \dots < j_k \leq N} p(x_{j_1}) \dots p(x_{j_k}) [1 - p(x_{j_{k+1}})] \dots [1 - p(x_{j_N})] dx_1 \dots dx_N = \\ &= \frac{1}{L^N} \sum_{1 \leq j_1 < \dots < j_k \leq N} \int_{[0,L]^N} p(x_{j_1}) \dots p(x_{j_k}) [1 - p(x_{j_{k+1}})] \dots [1 - p(x_{j_N})] dx_1 \dots dx_N = \\ &= \frac{1}{L^N} \sum_{1 \leq j_1 < \dots < j_k \leq N} \int_{[0,L]} p(x_{j_1}) dx_{j_1} \dots \int_{[0,L]} p(x_{j_k}) dx_{j_k} \\ & \quad \int_{[0,L]} [1 - p(x_{j_{k+1}})] dx_{j_{k+1}} \dots \int_{[0,L]} [1 - p(x_{j_N})] dx_{j_N} = \\ &= C_N^k \left(\int_0^L p(x) dx \right)^k \left(1 - \int_0^L p(x) dx \right)^{N-k} = C_N^k P^k (1 - P)^{N-k} \end{aligned}$$

where $C_N^k = \frac{N!}{k!(N-k)!}$ is the number of all possible combinations to choose k elements out of a set of N .

If k sites are switched on, it means that $k + 1$ synapses have been made active (including the target site at A). In the context of obtaining the desired strengthening of the connection between the output cell with the specified presynaptic axon, a normalization process is necessary at synaptic level. This corresponds to the Euclidean normalization encompassed in Oja's dynamics of the learning process. It is necessary for the output neuron to decide which one of these $k + 1$ synapses should be kept functional and turn the other ones back off; otherwise it would obtain an additional change in synaptic strength, not acquainted for by the learning rule. This process will

be considered totally random: each of the $k + 1$ synapses that have been turned on have equal probability $\frac{1}{k+1}$ to be kept on.

With this normalization, the probability of the synapse left on to be the target synapse is equal to:

$$\frac{1}{k+1} \frac{N!}{k!(N-k)!} P^k (1-P)^{N-k}$$

We calculate the probability that the system turns and keeps on the desired target synapse, no matter how many other wrong synapses have been flickered on and off in the normalization process. This will be the sum after all values of k (from 0 to N) of the respective probabilities for A to stay on in each case. It is quite natural to call this probability the *quality Q of the information transfer*:

$$\begin{aligned} Q &= \sum_{k=0}^N \frac{1}{k+1} \frac{N!}{k!(N-k)!} P^k (1-P)^{N-k} = \\ &= \frac{1}{P(N+1)} \sum_{k=0}^N \frac{(N+1)!}{(k+1)!(N-k)!} P^{k+1} (1-P)^{N-k} = \\ &= \frac{1}{P(N+1)} \sum_{j=1}^{N+1} \frac{(N+1)!}{j!(N+1-j)!} P^j (1-P)^{N+1-j} = \\ &= \frac{1}{P(N+1)} [1 - (1-P)^{N+1}] \\ &\gtrsim (1-P)^N \end{aligned}$$

Q represents the probability that the synapse at A is turned on and remains on at the request of strengthening the connection at A . The probability for a synapse with an axon of any other “wrong” input neuron will be $\frac{1-Q}{n}$.

Model 2:

References

- [C] D. Chklovskii, *Synaptic connectivity and neuronal morphology: two sides of the same coin*, Neuron 43, pp 609-617 (2004)
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