EVERY CIRCLE HOMEOMORPHISM IS THE COMPOSITION OF TWO CONFORMAL WELDINGS

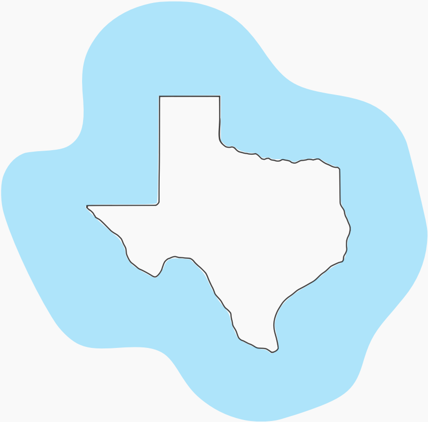
Alex Rodriguez

- Advisor: Chris Bishop -

Dallas - February 19, 2025

Stony Brook University

CONFORMAL WELDING HOMEOMORPHISMS



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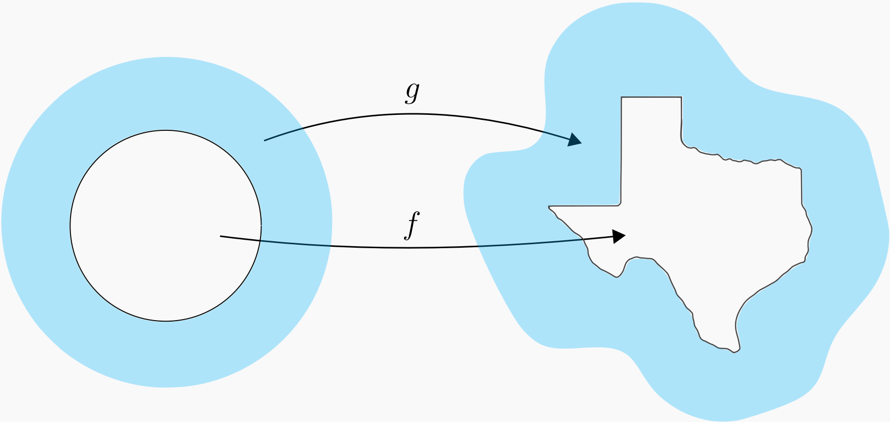

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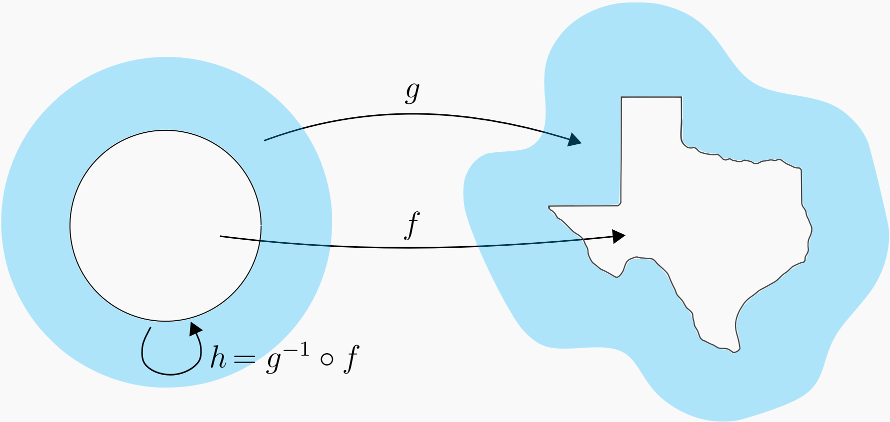

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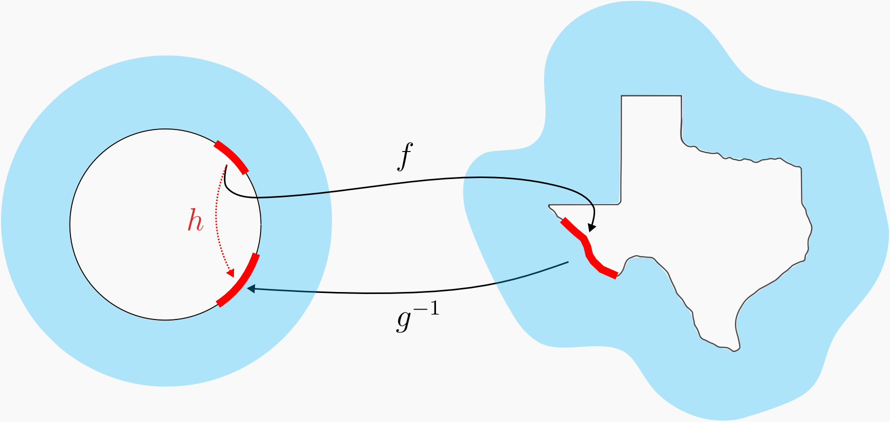

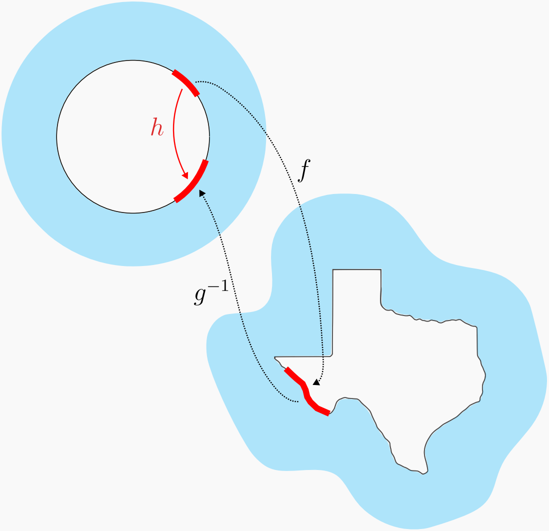

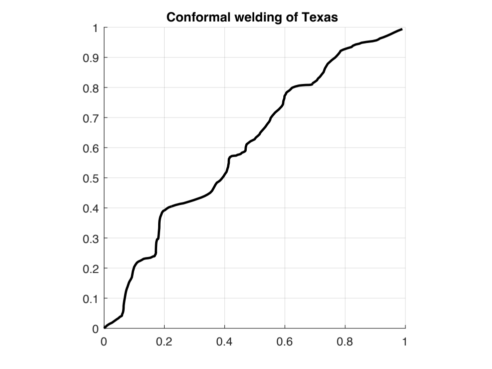

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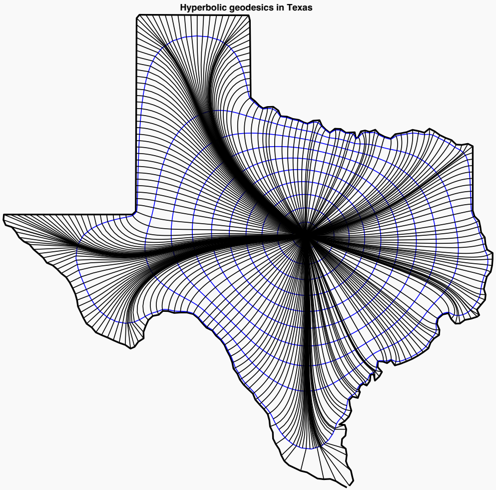
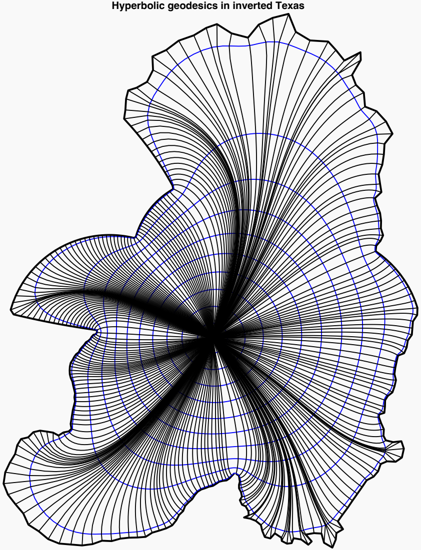
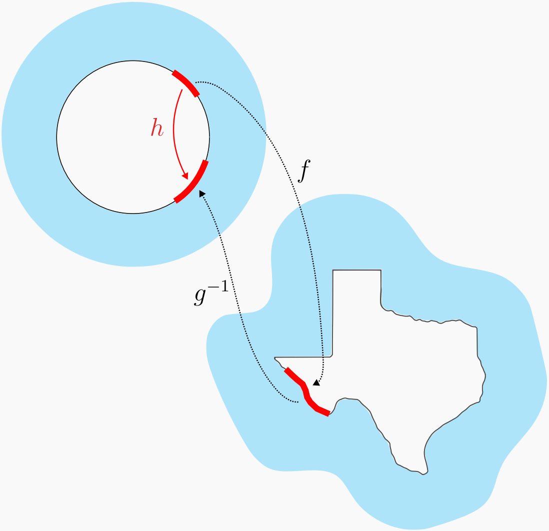




Images by Chris Bishop, using Toby Driscoll's code

MAIN RESULT



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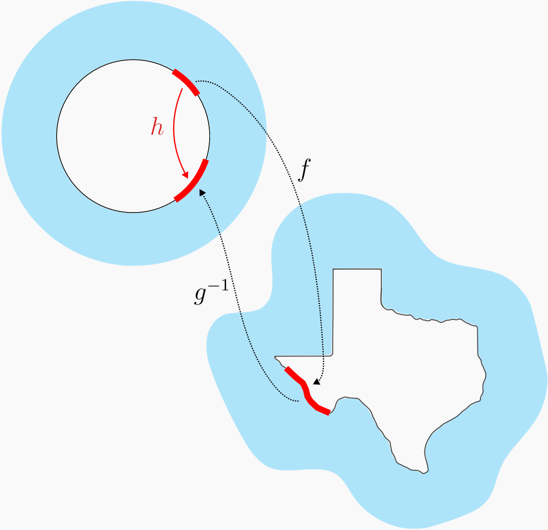

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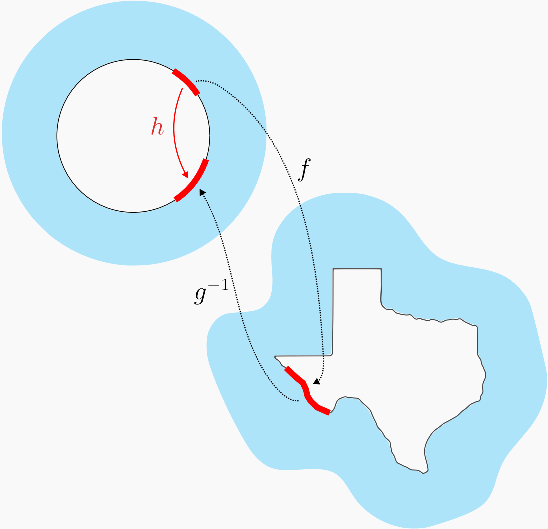

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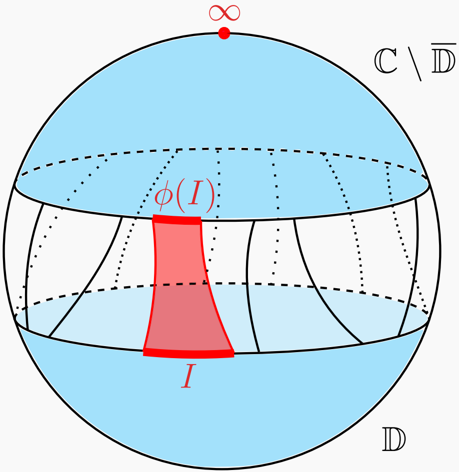
NOT every $\phi\colon \mathbb{S}^1 \to \mathbb{S}^1$ is a welding.

Theorem (R. 2025)

Every circle homeomorphism is the composition of two conformal welding homeomorphisms.

WHEN WILL $\phi\colon \mathbb{S}^1 \to \mathbb{S}^1$ BE A WELDING?



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$$M^{-1} \leq |\phi(I)|/|\phi(J)| \leq M,$$

where $I, J \subset \mathbb{S}^1$ are two adjacent arcs of equal length.

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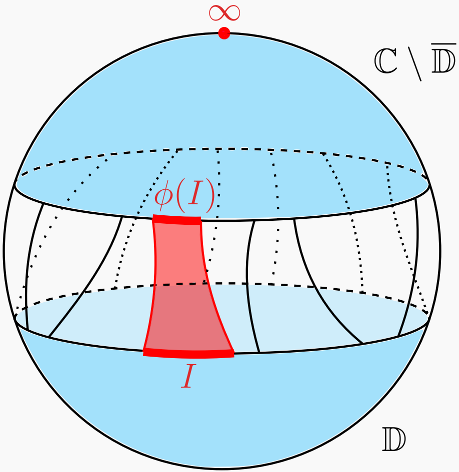

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Non-example:

$$\phi(x) = \begin{cases} x & \text{for } x \leq 0 \\ x^3 & \text{for } x \geq 0. \end{cases}$$

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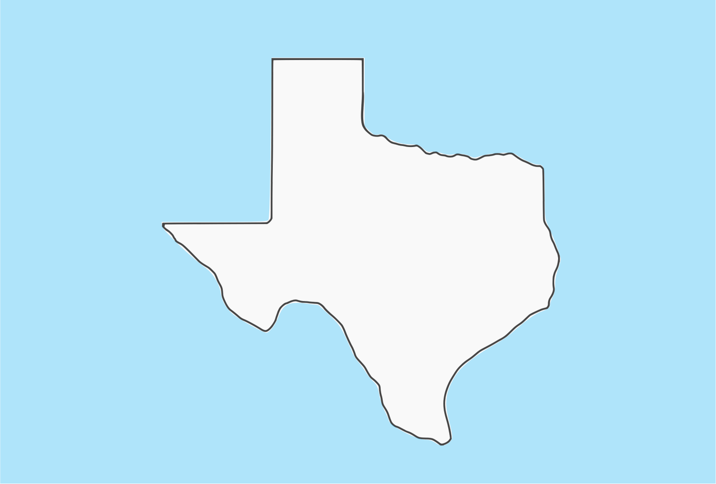
Theorem (Pfluger, 1960):
Every quasisymmetric $\phi\colon \mathbb{S}^1 \circlearrowleft$ is a welding. The Jordan curve is a quasicircle.

Conformal welding is important in:

- Teichmüller theory.
- Kleinian groups.
- Complex dynamics.
- Random geometry (Gluing of Liouville Quantum Gravity disks and SLE).
- Computer vision (work of Mumford).

REGULARITY OF THE WELDING



Harmonic measure: hitting distribution of Brownian motion.

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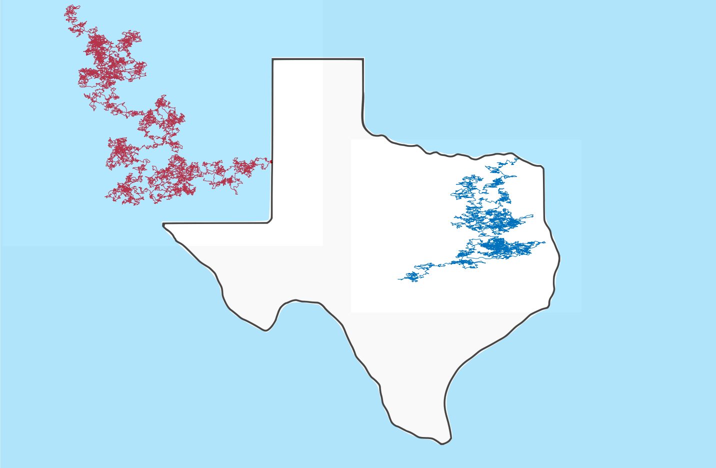

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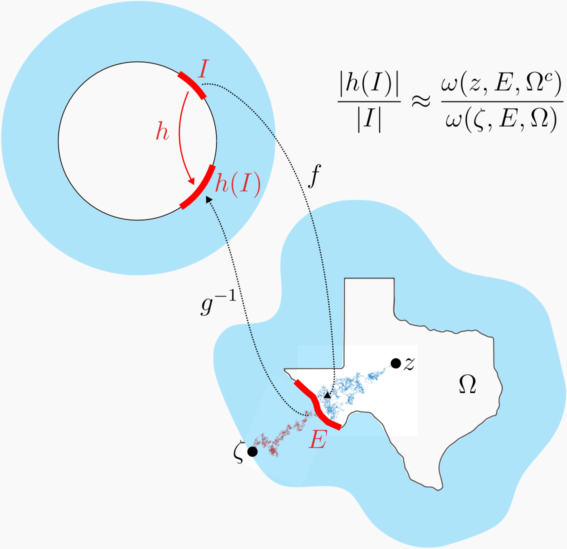

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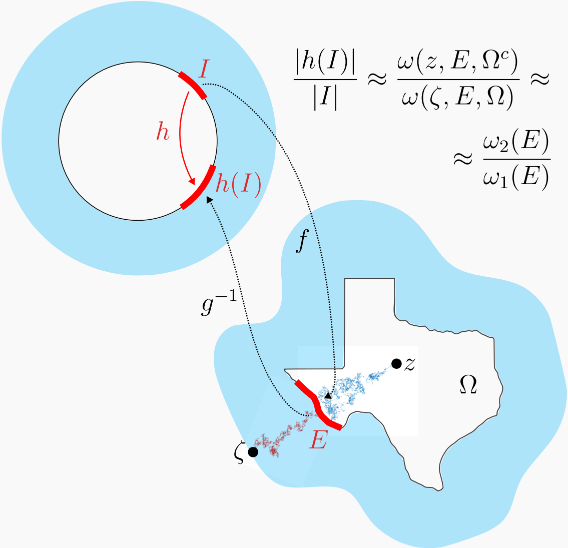

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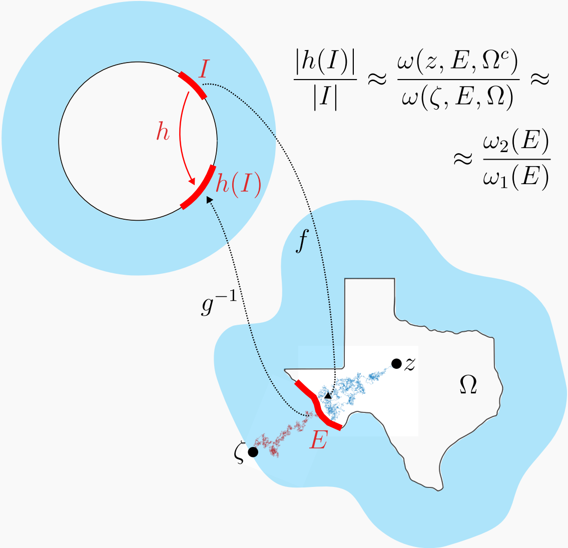

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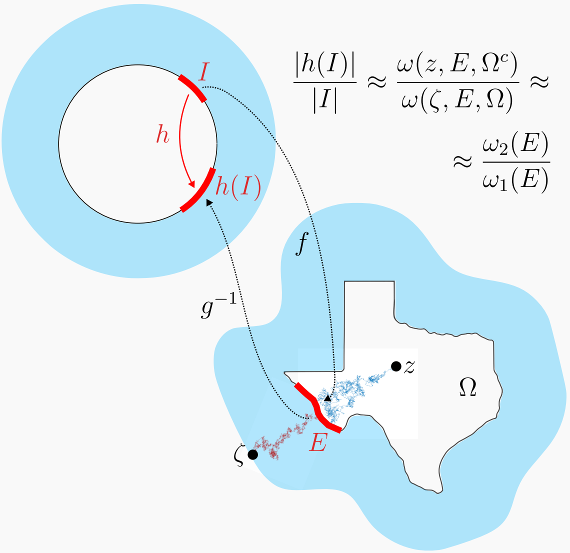

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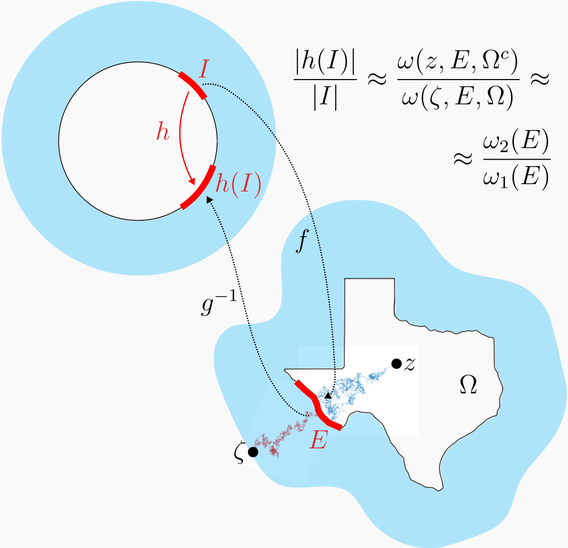

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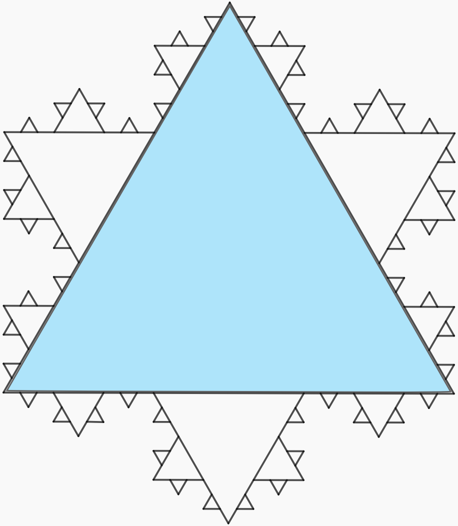

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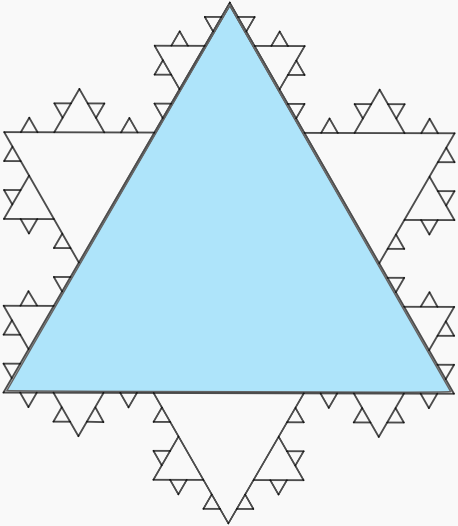

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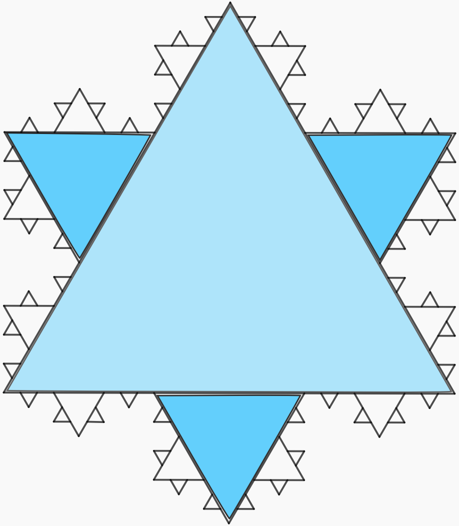

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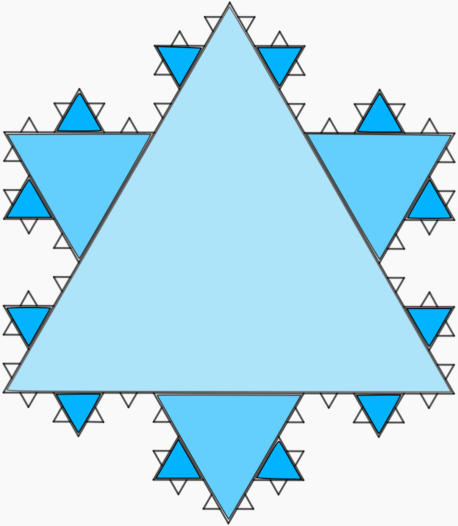

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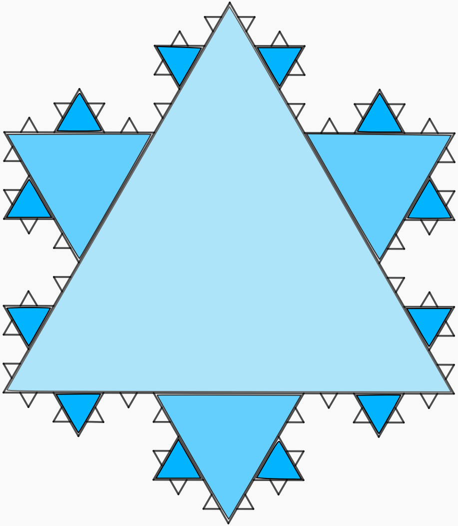

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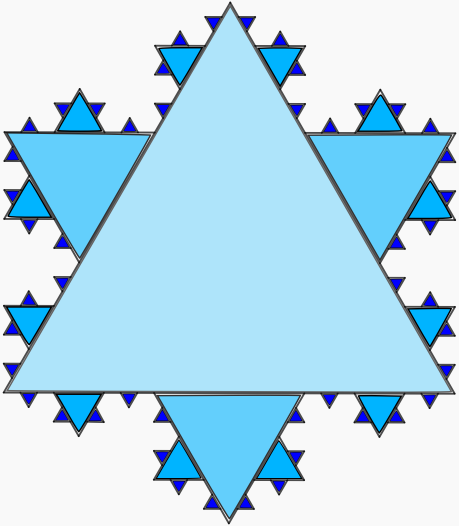

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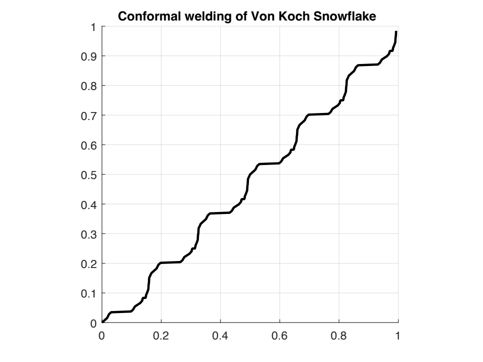

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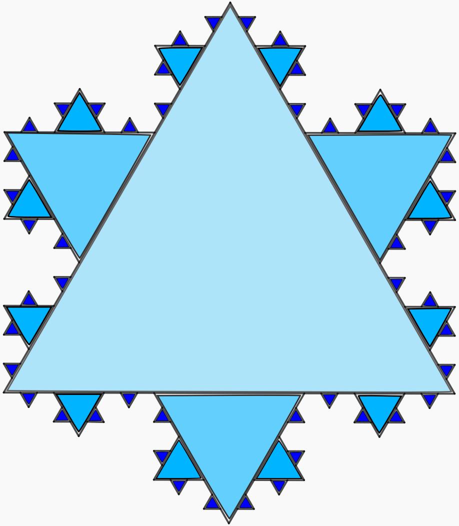

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If E is Borel,

$$\mathrm{Cap}(E) = \sup\{\mathrm{Cap}(K) \colon K \subset E \text{ compact}\}.$$

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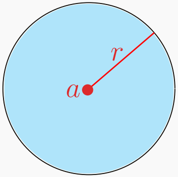

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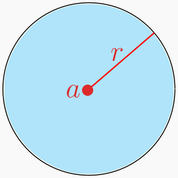

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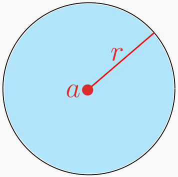

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$\mathrm{Cap}(E) = 0 \implies E$ has Hausdorff dim 0.

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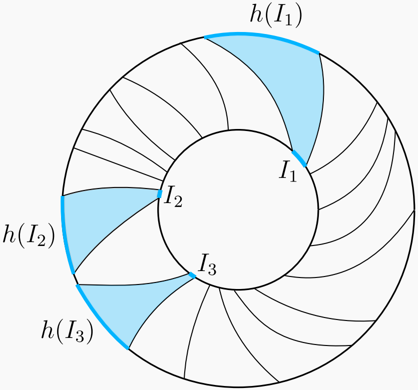

LOG-SINGULAR CIRCLE HOMEOMORPHISMS

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Every circle homeomorphism is the composition of two conformal welding homeomorphisms.

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Corollary (Vainio 1985): Weldings are not closed under composition.

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They encode partitions of intervals/arcs.

Ex: Dyadic partitions

$$\left\{ \left[a + \frac{j}{2^n}, a + (b-a)\frac{j+1}{2^n} \right] : 0 \leq j < 2^n \right\}.$$

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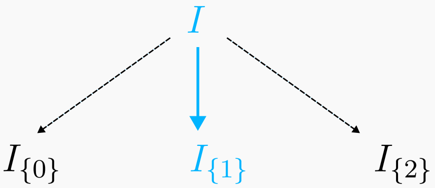

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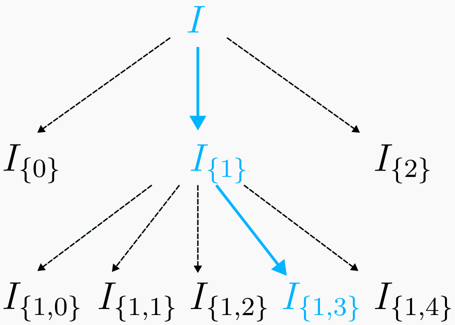

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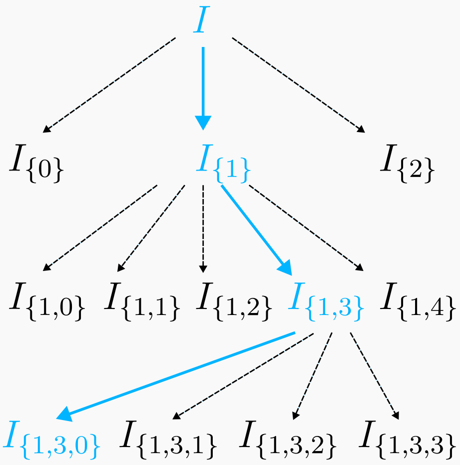
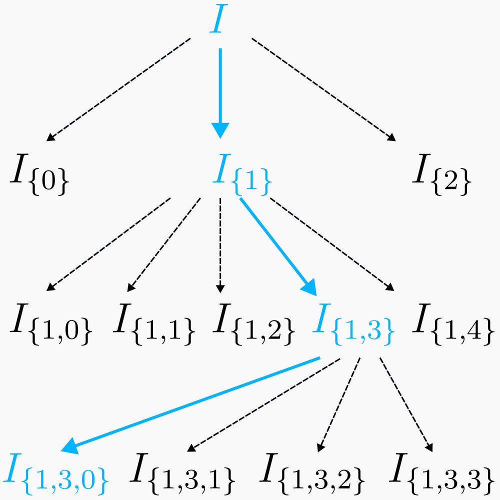


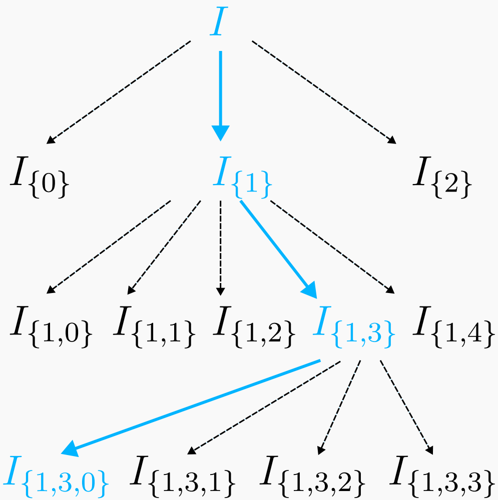


Addresses: $A_n = \{a_1, a_2, \ldots, a_n\}$ word of length n. Suppose $\{I_{A_n}\}$ are intervals.

- $A_n \neq \tilde{A}_n \Rightarrow I_{A_n}, I_{\tilde{A}_n}$ disjoint interior.
- $n \leq m$ and $I_{A_n} \subset I_{\tilde{A}_m}$. Then $\tilde{A}_m = \{a_1, \ldots, a_n, \ldots\}$
- If $A_{n-1} = \{a_1, \cdots, a_{n-1}\}$, then the union of all $A_{n-1}(a)$ is I_{A_n}.

$A_n = \{a_1, \cdots, a_n\}$ is the **address** of I_{A_n},

$$I_{A_n} = \bigcap_{j=1}^{n} I_{\{a_1, \cdots, a_j\}}.$$



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$$E = \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} \left(\bigcup_{A_n \in \mathcal{A}_n} \bigcup_{j=1}^{L_{n+1}/2} I_{A_n(2j-1)} \right),$$

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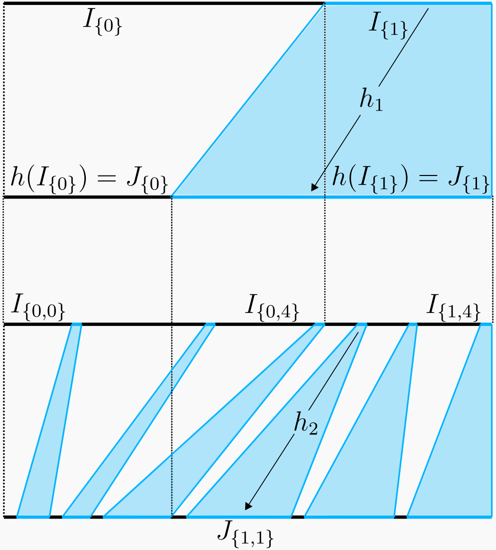

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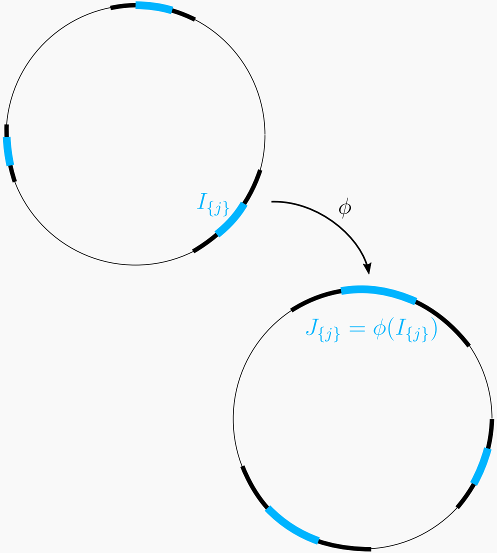


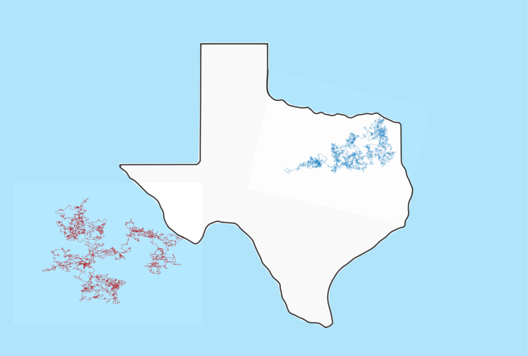
OPEN PROBLEMS

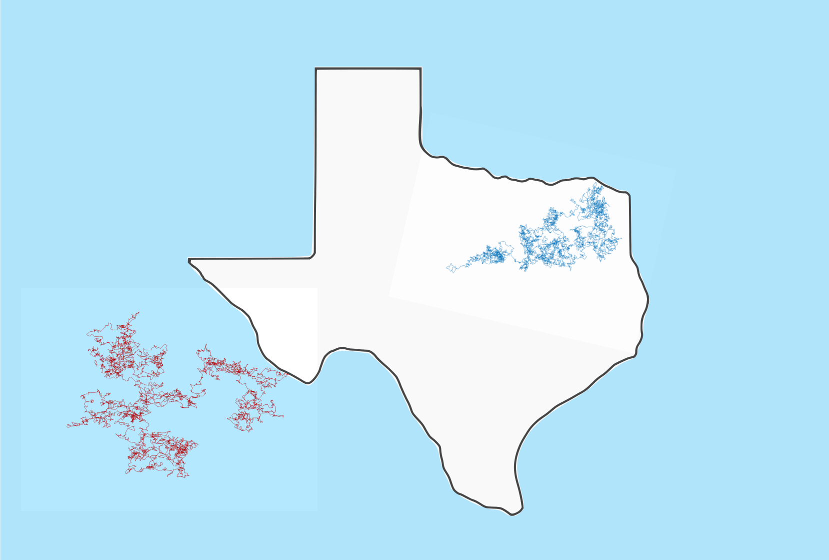
Characterize α-singular curves.

How bad can a curve with bi-Hölder welding be?

Is the composition of two bi-Hölder weldings a welding?

Is the set of weldings Borel?





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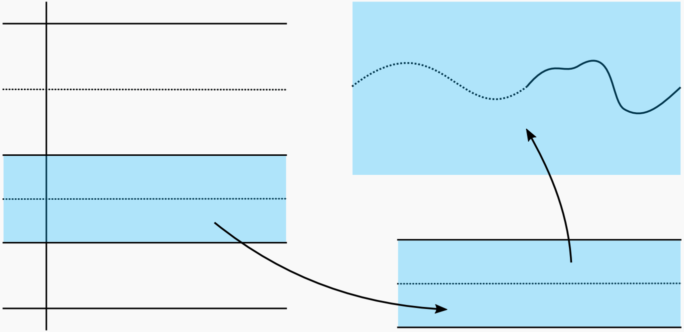

Argument from Peter Lin's thesis. Originally proved by Oikawa.