Homework 3

Due: June 30

Problem 1 (Compulsory)**.** Find a bijection between $\mathbb{N}$ and the set of even natural numbers $\{2n \in \mathbb{N} \colon n \in \mathbb{N}\}$.

Problem 2 (Compulsory)**.** At an international conference of 100 people, 75 speak English, 60 speak Spanish, 45 speak French, and everyone present speaks at least one of these languages. Calculate the maximum and minimum possible number of people who can speak at least two languages. (Hint: argue why the maximum possible number occurs when no one speaks all three languages, and the minimum number occurs when every person speaking at least two actually speaks all three.)

Problem 3 (Compulsory)**.** Let A, B be finite nonempty subsets of $\mathbb{R}$

(1) Suppose $A \subseteq B$. Show that $\min B \leq \min A \leq \max A \leq \max B$.

(2) Define $-A = \{-a | a \in A\}$. Show that $\min(-A) = -\max A$ and $\max(-A) = -\min A$.

(3) Define $A_2 = \{a^2 | a \in A\}$. Is it always true that $\max A_2 = (\max A)^2$? Give a proof if yes, and a counterexample if no.

Problem 4. Let X and Y be nonempty finite sets. Prove that $|X| \geq |Y|$ if and only if there exists a surjective function $f : X \to Y$.

Problem 5. Suppose $f : \mathbb{N} \to X$ is an injection. Prove by contradiction that X is an infinite set. You may use the fact that the set of natural numbers $\mathbb{N}$ is an infinite set. Use this to show that any finite set is not denumerable.

Remark: the parts in the following two problems guide you through the steps to solve the corresponding problem, and many of them require only a short argument. Please read the questions closely. **You can do one of them in exchange for the final presentation.**

Problem 6. In this question, we will prove the following identity

$$n^2 - n = 2\sum_{i=1}^{n-1} i$$

for an integer $n \geq 2$ by counting the same set in two ways. Recall that $\mathbb{N}_n = \{1, 2, 3, \ldots, n\}$. Let $X = \mathbb{N}_n \times \mathbb{N}_n$.

(1) Determine the cardinality $|X|$ of X.

(2) Consider the following subset X_k of X for $k = 0, 1, \ldots, n-1$:

$$X_k = \{(a, b) \in X \,|\, |a - b| = k\}.$$

That is, $(a, b) \in X_k$ if and only if $a - b = k$ or $a - b = -k$. Show that

(i) $|X_0| = n$;

(ii) $|X_k| = 2(n - k)$ for $k = 1, \ldots, n-1$.

(3) Notice that $X = X_0 \cup X_1 \cup \cdots \cup X_{n-1}$ and $X_k \cap X_j = \varnothing$ when $k \neq j$. Use Part (2) above to show that

$$|X| = n + 2\sum_{k=1}^{n-1}(n - k).$$

(4) Notice that

$$\sum_{k=1}^{n-1}(n - k) \quad \text{and} \quad \sum_{i=1}^{n-1} i$$

are the same sum, simply in reverse order. Combine (1) and (3) to show that

$$n^2 - n = 2\sum_{i=1}^{n-1} i.$$

Problem 7. Suppose X and Y are disjoint, nonempty sets. Suppose $|X| = m$ and $|Y| = n$.

(1) Let t be a natural number satisfying $t \leq m$ and $t \leq n$. For each $i = 0, 1, \ldots, t$, we can define a subset $\mathcal{P}_t^i(X \cup Y) \subseteq \mathcal{P}_t(X \cup Y)$ as follows:

$$\mathcal{P}_t^i(X \cup Y) = \{C \in \mathcal{P}_t(X \cup Y) | C = A \cup B, \text{ where } A \in \mathcal{P}_i(X), B \in \mathcal{P}_{t-i}(Y)\}.$$

Show that

(i) $\mathcal{P}_t^i(X \cup Y) \cap \mathcal{P}_t^j(X \cup Y) = \varnothing$ when $i \neq j$;

(ii) $\mathcal{P}_t(X \cup Y) = \bigcup_{i=0}^{t} \mathcal{P}_t^i(X \cup Y)$.

Here, the notation on the right hand side means taking the union of

$$\mathcal{P}_t^0(X \cup Y), \mathcal{P}_t^1(X \cup Y), \ldots, \mathcal{P}_t^t(X \cup Y).$$

(2) Give a bijection between $\mathcal{P}_i(X) \times \mathcal{P}_{t-i}(Y)$ and $\mathcal{P}_t^i(X \cup Y)$. Use this to show that

$$|\mathcal{P}_t^i(X \cup Y)| = \binom{m}{i}\binom{n}{t-i}.$$

(3) Use (1) and (2) to show that for any natural numbers m, n and $t \leq m, n$, we have the identity

$$\binom{m+n}{t} = \sum_{i=0}^{t} \binom{m}{i}\binom{n}{t-i}.$$