

Homework 2

Due: June 16

Problem 1 (Compulsory). Let A, B, C be sets. Use truth tables to prove the distributive property

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Illustrate it with Venn diagrams (you may need to draw several to avoid an overly crowded diagram).

Problem 2. Let A, B, C be sets. Suppose A, B are disjoint, and A, C are disjoint. Suppose also that $A \cup B = A \cup C$. Show that $B = C$.

Problem 3 (Compulsory). Let $f : X \rightarrow Y$ be a function. Let $A, B \subseteq X$, and $C, D \subseteq Y$. Prove the following statements.

1. $f(A \cap B) \subseteq f(A) \cap f(B)$;
2. $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

Problem 4. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function.

1. Give an explicit example of f that is injective but not surjective;
2. Give an explicit example of f that is surjective but not injective;
3. Give an explicit example of f that is both injective and surjective (i.e. a bijection);
4. Give an explicit example of f that is neither injective nor surjective;

Problem 5 (Compulsory). Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be functions. Prove the following statements.

1. If both f and g are injective, then so is $g \circ f$;
2. If $g \circ f$ is injective, then so is f ;
3. If $g \circ f$ is surjective, then so is g .

Problem 6. Suppose $f : X \rightarrow X$ is injective, and satisfies $f \circ f = f$. Show that $f = \text{id}_X$.

Problem 7 (Compulsory). Let X be a set. Recall that for any subset $A \subseteq X$, we can define its characteristic function $\chi_A : X \rightarrow \{0, 1\}$ by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}.$$

Let A, B be two subsets of X . Treat X as the universal set, so that we can define A^c .

(1) Fill in the values of different characteristic functions in the table below.

	$x \in A - B$	$x \in A \cap B$	$x \in B - A$	$x \in X - (A \cup B)$
$\chi_A(x)$				
$\chi_B(x)$				
$\chi_{A \cup B}(x)$				
$\chi_{A \cap B}(x)$				
$\chi_{A^c}(x)$				

(2) Using the table in (1) or otherwise, show that

- (a) $\chi_{A \cap B}(x) = \chi_A(x)\chi_B(x)$ for any $x \in X$;
- (b) $\chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_{A \cap B}(x)$ for any $x \in X$.
- (c) $\chi_{A^c}(x) = 1 - \chi_A(x)$ for any $x \in X$.