

Homework 1

Due: June 4

*Please send me your homework via email alex.rodriguez@stonybrook.edu . If you choose to do two presentations you only need to submit the ones marked with **compulsory**. If you choose to do only one presentation then you need to submit all of them*

Problem 1 (Compulsory). Use truth tables to show that the statements in each question are equivalent:

1. $P \Rightarrow Q, (\neg P) \vee Q$;
2. $(P \vee Q) \Rightarrow P, Q \Rightarrow P$;
3. $(P \vee Q) \wedge R, (P \wedge R) \vee (Q \wedge R)$.

Problem 2. Is the statement in each question a contradiction or tautology? Justify your answers.

1. $P \Rightarrow (P \vee Q)$;
2. $(P \wedge Q) \Leftrightarrow (P \Rightarrow \neg Q)$.

Problem 3 (Compulsory). Write the following statements entirely in symbols. Construct useful denials of the following statements, first in English sentences, then entirely in symbols.

1. For any natural number n , $n + 3 > 0$ only if $n + 4 > 0$.
2. There exist a pair of real numbers x, y so that we have both $x + y > 0$ and $x^2 + y^2 \geq 4$.

Problem 4 (Compulsory). Let a, b, c be real numbers. Consider the following statement: if $abc = 0$ then $a = 0$ or $b = 0$.

1. Write down the converse, inverse, and contrapositive of the statement.
2. Among the four statements, which are true and which are false? You do not need to justify your answers.

Problem 5. Let a, x be real numbers. Consider the predicate $P(x) : x^2 + ax + a = 0$.

1. For what values of a is the statement $\exists x \in \mathbb{R}, P(x)$ a contradiction? Justify your answers. (Hint: you may want to review some basic facts about quadratic equations.)
2. Prove the following statement: For any $x, a \in \mathbb{R}$, if $x^2 + ax + a \neq 0$ then $x \neq 0$ or $a \neq 0$. (Hint: what is the contrapositive?)

Problem 6 (Compulsory). The following proof is flawed (since it proves an obviously wrong statement!). Identify the mistake(s) and explain why it does not work.

Claim. For any non-negative integer n , $2 \times n = 0$.

Proof. The base case $n = 0$ is obviously true. Now suppose by induction that $2 \times k = 0$ for some non-negative integer k . Then we can write $k + 1 = i + j$ where i, j are non-negative integers smaller than k . Thus

$$2 \times (k + 1) = 2 \times (i + j) = 2i + 2j = 0 + 0 = 0,$$

as desired, since $2i = 0$ and $2j = 0$ by inductive hypothesis. So by the induction principle, $2 \times n = 0$ for any non-negative integer n . $\square$

Problem 7 (Compulsory). Prove by induction that for any nonnegative integer n , $4^n + 5$ is divisible by 3.

Problem 8 (Compulsory). Prove by induction that for any integer $m \geq 10$, $m^3 \leq 2^m$. (Hint: $(k + 1)^3 = k^3 + 3k^2 + 3k + 1$.)

Problem 9. Let $\alpha = 2 + \sqrt{3}$.

(a) Calculate $\alpha + 1/\alpha$. Simplify it as much as possible.

(b) Prove by induction that for any integer $n \geq 0$,

$$\alpha^{2^n} + \frac{1}{\alpha^{2^n}}$$

is an integer. (Hint: at the inductive step, consider $(a^{2^k} + 1/a^{2^k})^2$.)

Problem 10. Show that for any natural number n ,

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$