MAT 342: Practice final

INSTRUCTIONS: This is a practice final for the in person final exam for MAT 342 that will take place on August 14. During the review session on August 12 we will discuss these problems. Remember that you will have 2:30hrs to complete the final exam. The length of the problems in this exam are supposed to match what you will find on August 14. The recommendation is to attempt the first two problems first.

Problem 1. *Answer the following questions, justifying the different steps.*

(a) *Given* $u_\lambda(x,y) = e^{-x}(\lambda x \sin(y) - y\cos(y))$, *find for which* λ *it is possible to obtain* v_λ *so that* $f_\lambda = u_\lambda + iv_\lambda$ *is entire. Find the function* f_λ *for such values of* λ.

(b) *Find all holomorphic functions for which* $f(i/n) = 1/n^4$ *for all* $n \in \mathbb{N}$.

(c) *Let* $f \in \mathcal{H}(\mathbb{C} \setminus \{0\})$ *be such that* $|f(z)| \leq |\ln|z||$ *for all* $z \neq 0$. *Prove that* $f \equiv 0$.

(d) *Study the convergence and find the sum of* $\sum_{n\geq 0}(n+1)z^n$.

Problem 2. *Let* $f(z) = \log(z)/(z^3+1)$, *where* $\log(z)$ *denotes the determination of the logarithm at* $\Omega = \mathbb{C} \setminus [0,\infty)$ *so that* $\log(-1) = \pi i$.

(a) *Find the power series of* $\log(z)$ *around* $z = -1$. *What is the radius of convergence?*

(b) *Classify the singularities of* f *in* Ω *and compute the residues of* f *at such points. Find the negative part of the Laurent series at* $z = -1$.

(c) *Integrate* f *in a convenient domain to show that*

$$\int_0^\infty \frac{dx}{1+x^3} = \frac{2\pi}{3\sqrt{3}}.$$

Justify rigorously all steps, including the integrability of the integral. In case it is useful, $e^{\pi i} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$.

Problem 3. *Let* $\{a_n\}_{n\in\mathbb{N}}$ *be a strictly decreasing sequence of real numbers* $a_n \in (0,1)$ *that converges to* 0. *Let* $f \in \mathcal{H}(\mathbb{D})$. *Prove that,*

(a) *If* $f(a_n) \in \mathbb{R}$ *for all* $n \in \mathbb{N}$, *then* $f(\bar{z}) = \overline{f(z)}$ *for all* $z \in \mathbb{D}$.

(b) *If we also have* $f(a_{2n}) = f(a_{2n+1})$ *for all* n, *then* f *is constant.*