MAT 342: Homework 4

INSTRUCTIONS: Please read carefully the following instructions for this assignment.

1. You have time to submit this assignment until August 5, you need to do by uploading it in **Gradescope**. Unless you can do it by writing it down in a tablet, use an app to scan it (like CamScanner, not just pictures) and send it in .pdf format.
2. When you submit your assignment, indicate what pages correspond to what problem in your submission.
3. This an assignment, so you can discuss these problems, but in case you do that, please make sure that the solutions that you send me are in your own words. This is our chance to correct any mathematical mistake that you do when writing.
4. You will get a grade from 0 to 9 depending on how many problems you attempt.

Problem 1. *Compute the following integrals using Cauchy's theorem, Cauchy's integral formula or Cauchy's integral formula for derivatives.*

$$\text{(a)} \int_{-\infty}^{\infty} \frac{dx}{x^2+1} \qquad \text{(b)} \int_{|z|=2} \frac{z^2}{z-1} dz \qquad \text{(c)} \int_{|z|=2} \frac{dz}{z^2-1} \qquad \text{(d)} \int_{|z|=2} \frac{\sin(z)}{z^4} dz$$

Problem 2. *Let f be an entire function, that is, f is holomorphic in $\mathbb{C}$.*

(a) If there is $n \in \mathbb{N}$, $C > 0$ and $R > 0$ so that $|f(z)| \leq C|z|^n$ for all $|z| > R$, show that f is a polynomial of degree at most n.

(b) If there are constants $C, M > 0$ so that $|f(z)|e^{-C|z|} \leq M$ for all $z \in \mathbb{C}$, show that $|f'(z)|e^{-C|z|} \leq CMe$.

Problem 3. *Let f be an entire function, that is, f is holomorphic in $\mathbb{C}$.*

(a) If $|f| \geq 1$, then f is constant.

(b) If $Ref \geq 0$, then f is constant.

(c) If $Imf \leq 1$, then f is constant.

(d) If Ref has no zeros, then f is constant.

Problem 4. *Let f be an entire function so that $|f'(z)| < |f(z)|$ for all $z \in \mathbb{C}$. What can be said about f?*

Problem 5. *Let f be an entire function so that $|f'(z)| \leq |z|$ for all $z \in \mathbb{C}$. What can be said about f?*

Problem 6. *Consider the domain* $\Omega = \mathbb{C} \setminus (-\infty, 1]$ *and the determination of* $\sqrt{z^2-1}$ *in* Ω *that is positive in* $(1, \infty)$*. If* γ *is the arc from the circle* $|z-1| = \sqrt{2}$ *that goes from* i *to* $-i$ *through* $\{\mathrm{Re}(z) > 0\}$*, compute the integral*

$$\int_\gamma \frac{dz}{\sqrt{z^2-1}}.$$

(Hint: Remember Problem 4 from HW3.)

Problem 7. *Let* P *be a holomorphic polynomial of degree* n*. Prove that* P *can have at most* n *zeros counting with multiplicity. If all the zeros of* P *are contained in the disk* $\mathbb{D}(0, R)$*, show that*

$$\int_{|z|=R} \frac{P'(z)}{P(z)} dz = 2\pi i n.$$

Problem 8. *Let* $h\colon \mathbb{R} \to \mathbb{R}$ *be a continuous function with compact support, that is, there exists a compact set* $K \subset \mathbb{R}$ *so that* $h(x) = 0$ *if* $x \notin K$*. Consider,*

$$H(z) = \int_{\mathbb{R}} h(t) e^{-itz} dt,$$

its Fourier transform. Prove that H *is an entire function with exponential growth, that is, there exist* $A, C > 0$ *so that* $|H(z)| \leq Ce^{A|Imz|}$*.*

Problem 9. *Remember that* $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$*, and how in lecture (see the notes) we proved that*

$$\int_{-\infty}^{\infty} e^{-(-x+ia)^2} = \sqrt{\pi}$$

for all $a > 0$*. Use this to show that for* $n \in \mathbb{Z}$*,*

$$\int_{-\infty}^{\infty} e^{-x^2/2} \cos(nx) dx = \sqrt{2\pi} e^{-n^2/2}.$$